Required for: MATH40002 Analysis I

Based on the lectures of Richard Thomas and Steven Sivek, Imperial College London

1 Logic

Logic features heavily in the analysis module and you should be able to write proofs that use fluent, unambiguous and correct logic.

2 Numbers

2.1 Rational Numbers

2.1.1 Definitions

$$\begin{aligned}
\mathbb{N} &:= \{0, 1, 2, 3, ...\} \\
\mathbb{Z} &:= \{..., -3, -2, -1, 0, 1, 2, 3, ...\} \\
\mathbb{Q} &:= \{cl(p, q) \mid (p, q) \in \mathbb{Z} \times \mathbb{N}\}
\end{aligned}$$

where $cl(p, q) = \{(r, s) \mid (p, q) \sim (r, s)\}$ by the equivalence relation $(p, q) \sim (r, s) \iff ps = qr$.

2.1.2 Definition

Write $cl(p, q)$ as $\frac{p}{q}$. The class contains a distinguished element (p_0, q_0) which is "in lowest terms" such that $\forall n > 1,\ n \nmid p_0 \ \vee \ n \nmid q_0$.

2.1.3 Definitions

$$\begin{aligned}
\frac{p}{q} + \frac{r}{s} &:= \frac{ps + qr}{qs}. \\
\frac{p}{q} - \frac{r}{s} &:= \frac{ps - qr}{qs}. \\
\frac{p}{q} \cdot \frac{r}{s} &:= \frac{pr}{qs}. \\
\frac{p}{q} \Big/ \frac{r}{s} &:= \frac{ps}{qr}, \quad r \neq 0. \\
\frac{p}{q} \leq \frac{r}{s} &\iff ps \leq qr \quad (\text{since } q, s \in \mathbb{N}).
\end{aligned}$$

2.1.4 Axioms

The definitions above satisfy all the following properties.

A1 $\forall x, y \in \mathbb{Q},\ x + y = y + x$ (commutativity of addition).

A2 $\forall x, y, z \in \mathbb{Q},\ (x + y) + z = x + (y + z)$ (associativity of addition).

A3 $\exists\, 0 \in \mathbb{Q}$ such that $x + 0 = x\ \forall x \in \mathbb{Q}$ (additive identity).

A4 $\forall x \in \mathbb{Q},\ \exists -x \in \mathbb{Q}$ such that $x + (-x) = 0$ (additive inverse).

M1 $\forall x, y \in \mathbb{Q},\ x \cdot y = y \cdot x$ (commutativity of multiplication).

M2 $\forall x, y, z \in \mathbb{Q},\ (x \cdot y) \cdot z = x \cdot (y \cdot z)$ (associativity of multiplication).

M3 $\exists 1 \in \mathbb{Q}$ such that $x \cdot 1 = x \; \forall x \in \mathbb{Q}$ (multiplicative identity).

M4 $\forall x \in \mathbb{Q}, \; \exists x^{-1} \in \mathbb{Q}$ such that $x \cdot x^{-1} = 1$ (multiplicative inverse).

D $\forall x, y, z \in \mathbb{Q}, \; (x + y) \cdot z = x \cdot y + x \cdot z$ (distributivity).

O1 $\forall x \in \mathbb{Q}, \; x > 0 \vee x = 0 \vee x > 0$ (trichotomy axiom).

O2 $\forall x, y \in \mathbb{Q}, \; x > 0 \wedge y > 0 \implies x + y > 0.$

O3 $\forall x, y \in \mathbb{Q}, \; x > 0 \wedge y > 0 \implies xy > 0.$

O4 $\forall x \in \mathbb{Q}, \; \exists n \in \mathbb{N}, \; n > x$ (Archimedean property).

2.2 Decimals

2.2.1 Definition

We define the finite decimal

$$a_0.a_1a_2...a_i := a_0 + \frac{a_1}{10} + \frac{a_2}{100} + ... + \frac{a_i}{10^i} \in \mathbb{Q}$$

where $a_0 \in \mathbb{N}$ and $a_{n>0} \in \{0, ..., 9\}$.

2.2.2 Definition

We define the eventually periodic decimal

$$a_0.a_1a_2...a_i\overline{a_{i+1}a_{i+2}...a_j} := a_0 + \frac{a_1}{10} + \frac{a_2}{100} + ... + \frac{a_i}{10^i} + \left(\frac{a_{i+1}a_{i+2}...a_j}{10^j}\right)\left(\frac{1}{1 - 10^{i-j}}\right)$$

where $a_0 \in \mathbb{N}$ and $a_{n>0} \in \{0, ..., 9\}$.

2.2.3 Theorem

There is an eventually periodic decimal expansion $x = a_0.a_1a_2...a_i\overline{a_{i+1}a_{i+2}...a_j}$ for every $x \in \mathbb{Q}$ ($a_0 \in \mathbb{N}$ and $a_{n>0} \in \{0, ..., 9\}$).

2.2.4 First Definition for $\mathbb{R}$: Decimals

We now have a definition for $\mathbb{R}$, which is

$$\mathbb{R} := \Big\{a_0.a_1a_2... \mid a_0 \in \mathbb{Z}, a_{n>0} \in \{0, ..., 9\}, \; \forall N \; \exists i \geq N, \; a_i \neq 9\Big\}. \tag{1}$$

2.2.5 Theorem

$\forall x, y \in \mathbb{Q}$,

1. $\exists z \in \mathbb{Q}, \; x < z < y.$
2. $\exists z \in \mathbb{R} \setminus \mathbb{Q}, \; x < z < y.$

2.3 Countability

2.3.1 Definition

A set S is countable $\iff \exists$ a bijection $S \longrightarrow \mathbb{N}$.

2.3.2 Theorem

$S \subset \mathbb{N}$ is infinite $\implies S$ is countable.

2.3.3 Theorem

$\mathbb{Z}$ is countable.

2.3.4 Theorem

$\mathbb{Q}$ is countable.

2.3.5 Theorem

$\mathbb{R}$ is not countable (uncountable).

2.4 The Completeness Axiom

2.4.1 Definition

$S \subset \mathbb{R}$ $(S \neq \emptyset)$ is bounded above if $\exists\, M \in \mathbb{R},\ s \leq M\ \forall\, s \in S$.
$S \subset \mathbb{R}$ $(S \neq \emptyset)$ is bounded below if $\exists\, M \in \mathbb{R},\ s \geq M\ \forall\, s \in S$.
S is bounded if it is bounded above <u>and</u> below.

2.4.2 Definition

For a set $S \subset \mathbb{R}$ which is bounded above, the supremum (least upper bound) of S, $\sup S$, is an upper bound such that $M < \sup S \implies M$ is not an upper bound.

For a set $S \subset \mathbb{R}$ which is bounded below, the infimum (greatest lower bound) of S, $\inf S$, is a lower bound such that $M > \inf S \implies M$ is not a lower bound.

2.4.3 Second Definition for $\mathbb{R}$: The Completeness Axiom

The axioms of the rational numbers, together with the axiom below, gives a second definition for $\mathbb{R}$.

$$\text{For a set } S \subset \mathbb{R}\ (S \neq \emptyset) \text{ which is bounded above, } \sup S \text{ exists and is in } \mathbb{R}. \tag{2}$$

The completeness axiom can either be used as a construction of $\mathbb{R}$, or proved to be a property of another construction, for example definition (1) on the previous page, or a third definition given below.

2.5 Dedekind Cuts

2.5.1 Definition

A set $S \subset \mathbb{Q}$ is a Dedekind cut if

i S is bounded above but has no maximum.

ii $s \in S \wedge s > t \in \mathbb{Q} \implies t \in S$ (S is a left semi-infinite interval).

We can think of Dedekind cuts as assigning every real number $r \in \mathbb{R}$ to a semi-infinite subset of $\mathbb{Q}$,

$$S_r := (-\infty, r) \cap \mathbb{Q}$$

(note that S_r contains no real numbers).

2.5.2 Third Definition for $\mathbb{R}$: Dedekind Cuts

$$\mathbb{R} := \Big\{\text{Dedekind cuts} \subset \mathbb{Q}\Big\} \tag{3}$$

2.6 Triangle Inequalities

2.6.1 Theorem

$\forall\, a, b, c \in \mathbb{R}$,

1. $|a + b| \leq |a| + |b|$.
2. $|a + b| \geq |a| - |b|$.
3. $|a + b| \geq |b| - |a|$.
4. $|a - b| \geq \big||a| - |b|\big|$.
5. $|a| \leq |b| + |a - b|$.
6. $|a| \geq |b| - |a - b|$.
7. $|a - b| \leq |a - c| + |b - c|$.

3 Sequences

3.0.1 Definition

A sequence is a function $a : \mathbb{N} \longrightarrow \mathbb{R}$.
We will let a_n denote $a(n)$ and $(a_n)_{n\geq 1}$ or simply (a_n) denote the sequence.

3.1 Convergence of Sequences

3.1.1 Definition: Convergence and Divergence

(a_n) converges $\iff \exists\, a \in \mathbb{R},\ a_n \to a$ as $n \to \infty \iff$

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \varepsilon.$$

(a_n) diverges $\iff \nexists$ such $a \iff$

$$\forall\, a \in \mathbb{R},\ \exists\, \varepsilon > 0 \text{ such that } \forall\, N \in \mathbb{N},\ \exists\, n \geq N \text{ such that } |a_n - a| \geq \varepsilon.$$

3.1.2 Definition

$a_n \to \infty \iff$

$$\forall\, A > 0,\ \exists\, N \in \mathbb{N} \text{ such that } n \geq N \implies a_n > A.$$

3.1.3 Note

$a_n \to a \in \mathbb{C} \iff \operatorname{Re} a_n \to \operatorname{Re} a$ and $\operatorname{Im} a_n \to \operatorname{Im} a$.

3.1.4 Theorem

$a_n \to a$ and $a_n \to b \implies a = b$.

3.1.5 Theorem

(a_n) is convergent $\implies (a_n)$ is bounded.

3.1.6 Theorem

If $a_n \to a$ and $b_n \to b$,

1. $a_n + b_n \to a + b$.
2. $a_n b_n \to ab$.
3. $\frac{a_n}{b_n} \to \frac{a}{b}$ (if $b \neq 0$).

3.1.7 Theorem

If (a_n) is bounded above and monotonically increasing then $a_n \to \sup\{a_i \mid i \in \mathbb{N}\} := a$, written $a_n \uparrow a$.

3.1.8 Theorem

$a_n \to a$ and $b_n \to b$ and $a_n \leq b_n\ \forall\, n \implies a \leq b$.

3.1.9 Theorem

$\left|\frac{a_{n+1}}{a_n}\right| \to L < 1 \implies a_n \to 0$.

3.2 Cauchy Sequences

3.2.1 Definition

(a_n) is a Cauchy sequence $\iff$

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } m, n \geq N \implies |a_m - a_n| < \varepsilon.$$

3.2.2 Theorem

$(a_n) \subset \mathbb{R}$ is Cauchy $\iff$ (a_n) is convergent.

3.3 Subsequences

3.3.1 Definition

A subsequence of (a_n) is a sequence $(a_{n(i)})$ where $n : \mathbb{N} \longrightarrow \mathbb{N}$ is strictly increasing $(n(i) < n(i+1)\ \forall\, i)$.

3.3.2 Theorem: Bolzano-Weierstrass

Every bounded sequence of real numbers has a convergent subsequence.

3.3.3 Theorem

$a_n \to a \implies$ all subsequences $a_{n(i)} \to a$.

4 Series

4.0.1 Definition

For a sequence $(a_n)_{n\geq 1}$, there is a sequence of partial sums $(S_n)_{n\geq 1}$ where

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ...$$

and the infinite series is written

$$\sum_{n=1}^{\infty} a_n.$$

4.1 Convergence of Series

A series is convergent (to $A \in \mathbb{R}$) if and only if the sequence of partial sums converges to A:

$$\sum_{n=1}^{\infty} a_n = A \iff S_n \to A.$$

(and divergent if $\nexists$ such A).

4.1.1 Theorem

$$\sum_{n=1}^{\infty} a_n \text{ is convergent } \implies a_n \to 0.$$

4.1.2 Theorem

For a sequence $(a_n)_{n\geq 1}$ where $a_n \geq 0\ \forall\, n \iff S_n$ is monotonically non-decreasing:

1. $\sum_{n=1}^{\infty} a_n$ is convergent $\iff$ (S_n) are bounded above.
2. $\sum_{n=1}^{\infty} a_n$ is divergent (to ∞) $\iff$ (S_n) are unbounded.

4.1.3 Corollary: Comparison Test

Suppose $0 \leq a_n \leq b_n\ \forall\, n$

1. $\sum_{n=1}^{\infty} b_n$ is convergent $\implies \sum_{n=1}^{\infty} a_n$ is convergent (and $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$).
2. $\sum_{n=1}^{\infty} a_n$ is divergent to $\infty \implies \sum_{n=1}^{\infty} b_n$ is divergent to ∞.

4.1.4 Theorem: Algebra of Series Limits

$$\sum a_n \text{ and } \sum b_n \text{ converge } \implies \sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) = \lambda \sum_{n=1}^{\infty} a_n + \mu \sum_{n=1}^{\infty} b_n.$$

4.2 Absolute Convergence

4.2.1 Definition

For a sequence $(a_n)_{n\geq 1} \in \mathbb{C}$, $\sum_{n=1}^{\infty} a_n$ is absolutely convergent $\iff \sum_{n=1}^{\infty} |a_n|$ is convergent.

4.2.2 Theorem

For a sequence $(a_n)_{n\geq 1} \in \mathbb{C}$, $\sum a_n$ is absolutely convergent $\implies \sum a_n$ is convergent.

4.3 Further Tests for Convergence

4.3.1 Theorem: Comparison Test II

$a_n \leq b_n \leq c_n \ \forall n$ and $\sum a_n$ and $\sum c_n$ are both convergent $\iff \sum b_n$ is convergent.

4.3.2 Lemma

$\sum_{n\geq 1} a_n$ is convergent $\iff \sum_{n\geq N} a_n$ is convergent for any N.

4.3.3 Theorem: Comparison Test III

$\frac{a_n}{b_n} \to L \in \mathbb{R}$ and $\sum b_n$ is absolutely convergent $\implies \sum a_n$ is convergent.

4.3.4 Definition

$(a_n)_{n\geq 1}$ is alternating if $a_{2n} \geq 0$ and $a_{2n+1} \leq 0$ (or vice versa).

4.3.5 Theorem: Alternating Series Test

$(a_n)_{n\geq 1}$ is alternating and $|a_n| \to 0 \implies \sum a_n$ converges.

4.3.6 Theorem: Ratio Test

For a sequence $(a_n)_{n\geq 1}$, $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1 \implies \sum a_n$ is absolutely convergent.

4.3.7 Theorem: Root Test

For a sequence $(a_n)_{n\geq 1}$, $|a_n|^{\frac{1}{n}} \to r < 1 \implies \sum a_n$ is absolutely convergent.

4.4 Rearrangement of Series

4.4.1 Definition

Given a bijection $n : \mathbb{N} \longrightarrow \mathbb{N}$ and a sequence $(a_n)_{n\geq 1}$, the sequence of $b_i := a_{n(i)}$, $(b_n)_{n\geq 1}$ is a reordering of (a_n).

4.4.2 Theorem

$$\begin{aligned} &\sum a_n \text{ is absolutely convergent} \\ \iff &\sum_{a_n\geq 0}^{\infty} a_n = A^+ \text{ and } \sum_{a_n<0}^{\infty} a_n = A^- \\ \implies &\sum a_n = A^+ + A^- = \sum b_n \end{aligned}$$

where (b_n) is any reordering of (a_n).

4.5 Power Series

4.5.1 Theorem: Radius of Convergence

For the series $\sum a_n z^n$ where (a_n) is a sequence and $z, a_n \in \mathbb{C} \ \forall n$, $\exists R \in [0,\infty) \cup \{\infty\}$ such that

1. $|z| < R \implies \sum a_n z^n$ is absolutely convergent.
2. $|z| > R \implies \sum a_n z^n$ is divergent.

4.5.2 Corollary

Suppose, for a sequence (a_n), that $\left|\frac{a_{n+1}}{a_n}\right| \to L \in [0,\infty) \cup \{\infty\}$ as $n \to \infty$. The radius of convergence of the power series $\sum a_n z^n$ is $\frac{1}{L}$.

4.5.3 Definition

The Cauchy Product of the series $\sum a_n$ and $\sum b_n$ is $\sum c_n$ where $c_n := \sum_{i=0}^{n} a_{n-i} b_i$.

4.5.4 Theorem

$\sum a_n$ and $\sum b_n$ are absolutely convergent $\implies \sum |c_n| \to \left(\sum a_n\right)\left(\sum b_n\right)$.

4.5.5 Corollary

$\sum a_n z^n$ and $\sum b_n z^n$ have radii of convergence R_a and R_b $\implies$ $\sum c_n z^n$ has radius of convergence $R_c \geq \min\{R_a, R_b\}$.

4.6 Exponential Power Series

4.6.1 Definition: Exponential Series

Where $z \in \mathbb{C}$, define

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

4.6.2 Lemma

$E(z)$ is absolutely convergent $\forall\, z \in \mathbb{C}$.

4.6.3 Theorem

$E(z)E(w) = E(z+w)$.

4.6.4 Corollary

$\frac{1}{E(z)} = E(-z)$.

4.6.5 Corollary

$E(0) = 1$.

4.6.6 Definition

$$e := E(1).$$

4.6.7 Theorem

$E(n) = e^n$ for $n \in \mathbb{N}$.

4.6.8 Theorem

$E(q) = e^q$ for $q \in \mathbb{Q}$.

4.6.9 Theorem

$x \longrightarrow E(x)$ is a bijection $\mathbb{R} \longrightarrow (0,\infty)$.

4.6.10 Definition

$$\begin{aligned} \log : (0,\infty) &\longrightarrow \mathbb{R} \\ x &\longmapsto \log x \end{aligned}$$

where $\log x$ is such that $E(\log x) = x$ (log is the inverse of E).

4.6.11 Theorem

Properties of log such as $\log xy = \log x + \log y$ (which follows from 4.6.3) follow from the corresponding properties of E.

4.6.12 Definition

$$a^x := E(x \log a)$$

for $x \in \mathbb{R},\ a \in (0, \infty)$.

4.6.13 Definition

$$\cos\theta := \operatorname{Re} E(i\theta), \qquad \sin\theta := \operatorname{Im} E(i\theta).$$

5 Continuity

5.1 Prerequisite: Limits

For a countable sequence (a_n) (a function $\mathbb{N} \longrightarrow \mathbb{R}$), 3.1.1 defines $\lim_{n\to\infty} a_n$. For a function $f : \mathbb{R} \longrightarrow \mathbb{R}$, can we define $\lim_{x\to a} f(x)$?

5.1.1 Definition

For a function f, $\lim_{x\to a} f(x) = l \iff f(x) \to l$ as $x \to a \iff$

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - l| < \varepsilon.$$

5.1.2 Theorem

$f(x) \to b$ and $f(x) \to c$ as $x \to a \implies b = c$ (provided the unique limit exists).

5.2 Continuity

5.2.1 Definition

A function $f : S \longrightarrow \mathbb{R}$ is continuous at $a \iff \lim_{x\to a} f(x) = f(a) \iff$

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \varepsilon.$$

f is (pointwise) continuous on $S \subseteq \mathbb{R} \iff$ it is continuous at all $a \in S$, and we may write 'f is continuous'.

5.2.2 Theorem

$$\begin{aligned} E : \mathbb{C} &\longrightarrow \mathbb{C} \\ z &\longmapsto \sum_{n=0}^{\infty} \frac{z^n}{n!} \end{aligned}$$

is continuous on $\mathbb{C}$.

5.2.3 Theorem

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $a \in \mathbb{R} \iff f(a_n) \to f(a)\ \forall$ sequences where $a_n \to a$.

5.2.4 Theorem

$f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a and $g : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at $f(a) \implies g \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous at a.

5.3 The Intermediate Value Theorem

5.3.1 Theorem: Intermediate Value Theorem

$f : [a, b] \longrightarrow \mathbb{R}$ is continuous $\implies \forall\, d \in [f(a), f(b)]\ \exists\, c \in [a, b]$ such that $d = f(c)$.

5.3.2 Corollary (Application)

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0$ where $n \geq 1,\ a_n \neq 0$. n is odd $\implies p(x)$ has a root.

5.3.3 Corollary (Application)

$f : [0,1] \longrightarrow [0,1]$ is continuous $\implies f$ has a fixed point ($\exists\, x \in [0,1],\ f(x) = x$).

5.4 The Extreme Value Theorem

5.4.1 Definition

For a function $f : S \longrightarrow \mathbb{R}$ ($S \subset \mathbb{R}$),
f is bounded above $\iff \exists\, M \in \mathbb{R},\ f(x) \leq M\ \forall\, x \in S$.
f is bounded below $\iff \exists\, M \in \mathbb{R},\ f(x) \geq M\ \forall\, x \in S$.
f is bounded if it is bounded above <u>and</u> below.

5.4.2 Theorem

$f : [a,b] \longrightarrow \mathbb{R}$ is continuous $\implies f$ is bounded.

5.4.3 Theorem: Extreme Value Theorem

Any continuous $f : [a,b] \longrightarrow \mathbb{R}$ is bounded and attains its bounds, i.e. $\exists\, c, d \in [a,b]$,

$$f(c) \leq f(x) \leq f(d)\ \forall x \in [a,b]$$

$\Big(f(c) = \inf_{x \in [a,b]} f(x)$ and $f(d) = \sup_{x \in [a,b]} f(x)\Big)$.

5.4.4 Corollary

For a continuous function $f : [a,b] \longrightarrow \mathbb{R}$, $\exists\, c, d \in [a,b]$ such that the image $f([a,b])$ is the closed interval $[f(c), f(d)]$.

5.4.5 Theorem

Let $f : [a,b] \longrightarrow \mathbb{R}$ (or indeed $f : \mathbb{R} \longrightarrow \mathbb{R}$, excluding the third statement) be continuous.

$$f \text{ is strictly monotonic} \iff f \text{ is injective} \iff f \text{ is a bijection } [a,b] \longrightarrow [f(a), f(b)].$$

5.4.6 Theorem

Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and injective. $f^{-1} : f(\mathbb{R}) \longrightarrow \mathbb{R}$ is continuous (where $f(\mathbb{R})$ is the image of f).

5.5 Open, Closed and Compact Sets

5.5.1 Definition

A set $S \subset \mathbb{R}$ is open $\iff$

$$\forall\, s \in S,\ \exists\, \delta > 0 \text{ such that } (x - \delta, x + \delta) \subset S.$$

5.5.2 Definition

A set $S \subset \mathbb{R}$ is closed $\iff$

$$\forall \text{ sequences } (s_n) \subseteq S,\ s_n \to s \implies s \in S.$$

S is compact if, additionally, it is bounded.

5.5.3 Note

The use of 'open' and 'closed' in these definitions is misleading. The definitions are not antonymous as the words are in English. For example:

- $(0,1]$ is neither open nor closed.
- $\mathbb{R}$ is both open and closed.
- $\emptyset$ is both open and closed.

5.5.4 Theorem

The open interval (a, b) is an open set.

5.5.5 Theorem

1. For a set of open subsets of $\mathbb{R}$, $\{S_i\}$, $S = \bigcup_i S_i$ is also an open set.
2. For a finite set of open subsets of $\mathbb{R}$, $\{S_i\}$, $S = \bigcap_i S_i$ is also an open set.

5.5.6 Theorem

The closed interval $[a, b]$ is compact.

5.5.7 Theorem

1. For a finite set of closed subsets of $\mathbb{R}$, $\{S_i\}$, $S = \bigcup_i S_i$ is also a closed set.
2. For a set of closed subsets of $\mathbb{R}$, $\{S_i\}$, $S = \bigcap_i S_i$ is also a closed set.

5.5.8 Theorem

$S \subset \mathbb{R}$ is open $\iff$ $\mathbb{R} \setminus S$ is closed.

5.6 Uniform Continuity and Convergence

5.6.1 Definition

A function $f : S \longrightarrow \mathbb{R}$ is uniformly continuous $\iff$

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } \forall\, x, y \in S,\ |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

5.6.2 Theorem

$f : S \longrightarrow \mathbb{R}$ is uniformly continuous $\implies$ f is continuous.

5.6.3 Theorem

$f : S \longrightarrow \mathbb{R}$ is continuous <u>and</u> S is compact $\implies$ f is uniformly continuous.

5.6.4 Definition

Let $f_1, f_2, f_3, ... : S \longrightarrow \mathbb{R}$ be a sequence of functions defined on $S \subset \mathbb{R}$. (f_n) converges pointwise to $f : S \longrightarrow \mathbb{R}$ $\iff$

$$\forall\, x \in S,\ \forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } n \geq N \implies |f_n(x) - f(x)| < \varepsilon$$

and (f_n) converges uniformly to $f : S \longrightarrow \mathbb{R}$ $\iff$

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N} \text{ such that } \forall\, x \in S,\ n \geq N \implies |f_n(x) - f(x)| < \varepsilon.$$

5.6.5 Theorem

$(f_n) : S \longrightarrow \mathbb{R}$ are all uniformly continuous and converge uniformly to $f : S \longrightarrow \mathbb{R}$ $\implies$ f is uniformly continuous.

5.6.6 Theorem

$(f_n) : S \longrightarrow \mathbb{R}$ are all continuous (not necessarily uniformly) and converge uniformly to $f : S \longrightarrow \mathbb{R}$ $\implies$ f is continuous.

5.6.7 Definition

$$\sum_{i=1}^{\infty} f_i \text{ converges} \iff \text{the sequence of partial sums } S_n(x) = \sum_{i=1}^{n} f_i(x) \text{ converges}$$

and the series converges uniformly $\iff$ the sequence of partial sums converges uniformly.

5.6.8 Theorem: Weierstrass M-Test

Let $f_1, f_2, f_3, ... : S \longrightarrow \mathbb{R}$ be a sequence of continuous functions and let $M_1, M_2, M_3, ...$ be such that $\forall\, i,\ \forall\, x \in S,\ |f_i(x)| \le M_i$.

$$\sum_{i=1}^{\infty} M_i \text{ converges } \implies \sum_{i=1}^{\infty} f_i(x) \text{ converges uniformly to } g : S \longrightarrow \mathbb{R}$$

and g is continuous.

6 Differentiation

6.0.1 Definition

A function $f : S \longrightarrow \mathbb{R}$ is differentiable at $a \in \mathbb{R} \iff$

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \left(\text{or } \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \right) \quad \text{exists}$$

which is equivalent to

$$\forall\, \varepsilon > 0,\ \exists\, \delta > 0 \text{ such that } |x - a| < \delta \implies \left| \tfrac{f(x) - f(a)}{x - a} - f'(a) \right| < \varepsilon$$

(for some limit $f'(a)$). We write the derivative function of f as f', defined wherever the derivative exists. f is differentiable on $S \subseteq \mathbb{R} \iff f$ is differentiable at all $a \in S$, and we may write 'f is differentiable'.

6.0.2 Theorem

$f(x) = x^n$ has derivative $f'(x) = nx^{n-1}$ for $n \ge 0$.

6.0.3 Theorem

$f(x) = e^x$ has derivative $f'(x) = e^x$.

6.0.4 Theorem

f is differentiable at $a \implies f$ is continuous at a.

6.1 Properties of Derivatives

6.1.1 Theorem: Linearity

f, g are differentiable at $a \implies h = \lambda f + \mu g$ is differentiable at a, and $h'(a) = \lambda f'(a) + \mu g'(a)$.

6.1.2 Theorem: Product Rule

f, g are differentiable at $a \implies h = fg$ is differentiable at a, and $h'(a) = f'(a)g(a) + f(a)g'(a)$.

6.1.3 Theorem: Quotient Rule

f, g are differentiable at $a \implies h = \frac{f}{g}$ is differentiable at a, and $h'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$.

6.1.4 Theorem: Chain Rule

f, g are differentiable at $a \implies h = f \circ g$ is differentiable, and $h'(a) = f' \circ g(a) g'(a)$.

6.2 The Mean Value Theorem

6.2.1 Definition

The function $f : S \longrightarrow \mathbb{R}$ has a local minimum at $x \in S \iff$

$$\exists\, \delta > 0 \text{ such that } |x - y| < \delta \implies f(x) \le f(y)$$

and has a local maximum at $x \in S \iff$

$$\exists\, \delta > 0 \text{ such that } |x - y| < \delta \implies f(x) \ge f(y).$$

6.2.2 Theorem

$f : [a, b] \longrightarrow \mathbb{R}$ has a local minimum or maximum at $x \in (a, b)$ and f is differentiable at $x \implies f'(x) = 0$. It is important to use the specific interval notation in this result because if $x = a$ or b, the definitions allow for a local minimum or maximum which may not have $f'(x) = 0$ necessarily. Also note that a function may satisfy $f'(x) = 0$ for some x, but not have a local minimum or maximum at x, e.g. $f(x) = x^3$.

6.2.3 Theorem: Rolle's Theorem

For a function f which is continuous on $[a, b]$ and differentiable on (a, b), $f(a) = f(b) \implies \exists\, c \in (a, b),\ f'(c) = 0$.

6.2.4 Theorem: Mean Value Theorem

For a function f which is continuous on $[a, b]$ and differentiable on (a, b), $\exists\, c \in (a, b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

6.2.5 Theorem

Let f be continuous on $[a, b]$ and differentiable on (a, b).

1. $f'(x) \geq [>]\, 0\ \forall\, x \in (a, b) \implies f$ is monotonically [strictly] increasing on $[a, b]$.
2. $f'(x) \leq [<]\, 0\ \forall\, x \in (a, b) \implies f$ is monotonically [strictly] decreasing on $[a, b]$.
3. Corollary of 1 and 2: $f'(x) = 0\ \forall\, x \in (a, b) \implies f$ is constant on $[a, b]$.
4. Corollary of 3: let g also be continuous on $[a, b]$ and differentiable on (a, b). $f'(x) = g'(x)\ \forall\, x \in (a, b) \implies \exists\, c \in \mathbb{R}$ such that $f = g + c$.

6.3 L'Hôpital's Rule

6.3.1 Theorem: L'Hôpital's Rule

Let f and g be defined and differentiable (with $g'(x) \neq 0$) on an interval containing a, except possibly at a. If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ and $\lim_{x\to a}(f'(x)/g'(x))$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

The rule also applies where $f, g \to \infty$ as $x \to a$, or where a is ∞ or $-\infty$.

6.3.2 Corollary

Assuming $f''(x)$ exists on the neighbourhood of a and is continuous at a, it is equal to

$$\lim_{h\to 0} \frac{f(a + h) - 2f(a) + f(a - h)}{h^2}.$$

6.4 Higher Derivatives

Assuming they exist, the first, second, ..., n^{th} derivatives of f will be written $f', f'', ..., f^{(n)}$. Notice the parentheses; f^n denotes n iterations of f, not a derivative. Derivatives may also be written $\frac{df}{dx}, \frac{d^2 f}{dx^2}, ..., \frac{d^n f}{dx^n}$, where $\frac{df}{dx} = \frac{d}{dx}[f]$, indicating that f is being differentiated, and higher derivatives in this notation represent iterations of differentiation.

6.4.1 Theorem: Taylor's Theorem

Suppose for a function $f : [p, q] \longrightarrow \mathbb{R}$ that $f^{(i)}$ is continuous $\forall\, i \leq n$ and that $f^{(n+1)}$ is defined on (p, q). Let $a \in [p, q]$. For any $x \in [p, q],\ x \neq a,\ \exists\, t$ between x and a such that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x - a)^n + R_n$$

where $R_n = \frac{f^{(n+1)}(t)}{(n+1)!}(x - a)^{n+1}$.

6.4.2 Definition: Taylor Series

Suppose now that $f^{(n)}(a)$ exists $\forall\, n \geq 0$. The Taylor series for f about a is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n + ...$$

6.5 Second Derivatives and Convexity

6.5.1 Theorem

Suppose $f'(a) = 0$.

1. $f''(a) > 0 \implies f$ has a local minimum at $x = a$.
2. $f''(a) < 0 \implies f$ has a local maximum at $x = a$.

6.5.2 Definition

$f : [a,b] \longrightarrow \mathbb{R}$ is convex $\iff \forall\, c < t < d \in [a,b]$,

$$f(c) + \frac{f(d) - f(c)}{d - c}(t - c) \geq f(t),$$

which is equivalent to saying that the set $S = \big\{(x,y) \mid x \in [a,b],\ y \geq f(x)\big\}$ is a geometrically convex subset of $\mathbb{R}^2$; any line segment between two points within S lies entirely within S.

6.5.3 Theorem

$f : [a,b] \longrightarrow \mathbb{R}$ is convex $\iff f''(x) \geq 0\ \forall\, x \in (a,b)$ (assuming f'' is continuous).

6.6 Limits of Differentiable Functions

6.6.1 Theorem

Let $f_n : [a,b] \longrightarrow \mathbb{R}$ be a sequence of differentiable functions and suppose $\exists\, c \in [a,b]$ such that $\lim_{n\to\infty} f_n(c)$ exists. If the sequence (f_n') converges uniformly, then (f_n) converges uniformly to a function f, and $\lim_{n\to\infty} f_n'(x) = f'(x)$.

6.6.2 Theorem: Differentiation of Power Series

$f(x) = \sum_{n=0}^{\infty} a_n x^n$ has a continuous derivative on $(-R, R)$ (where $R > 0$ is the radius of convergence) and $f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}\ \forall\, |x| < R$.

7 Integration

The definite integral

$$\int_a^b f(x)\ dx$$

is intended to find the area under the curve f between a and b. In calculus and applications, this was derived by the limit of Riemann sums. Here we will use Darboux integration, but the two are ultimately equivalent.

7.1 The Darboux Sum

7.1.1 Definition

A partition of $[a,b]$ is a sequence $x_0, ..., x_k$ such that $a = x_0 < x_1 < ... < x_{k-1} < x_k = b$.

7.1.2 Definition

Let $f : [a,b] \longrightarrow \mathbb{R}$ be bounded, let $P = \{x_0, ..., x_k\}$ be a partition of $[a,b]$ and let

$$m_i = \inf_{x_i \leq t \leq x_{i+1}} f(t), \qquad M_i = \sup_{x_i \leq t \leq x_{i+1}} f(t).$$

The lower and upper Darboux sums of f with respect to P are

$$L(f,P) := \sum_{i=0}^{k-1} m_i (x_{i+1} - x_i), \qquad U(f,P) := \sum_{i=0}^{k-1} M_i (x_{i+1} - x_i).$$

7.1.3 Lemma

For a bounded function $f : [a,b] \longrightarrow \mathbb{R}$ and any partition P of $[a,b]$,

$$L(f,P) \leq U(f,P).$$

7.1.4 Definition

A partition Q is a refinement of $P \iff$ every point of P belongs to Q, written $P \preccurlyeq Q$, and a proper refinement $\iff$ additionally, $Q \neq P$, written $P \prec Q$. R is a common refinement of P and $Q \iff P \preccurlyeq R \wedge Q \preccurlyeq R$. Note that some may write $\prec$ in general, and state a proper refinement explicitly using $\precneqq$, just as some use $\subset$ in general for a subset, and state a proper subset explicitly using $\subsetneq$.

7.1.5 Theorem

$P \prec Q \implies$

$$L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P).$$

7.1.6 Theorem

For a bounded function $f : [a,b] \longrightarrow \mathbb{R}$ and any two partitions P, Q of $[a,b]$,

$$L(f,P) \leq U(f,Q).$$

7.2 The Darboux Integral

7.2.1 Definition

Theorem 7.1.6 allows us to make the following definitions. The lower and upper Darboux integrals of f on $[a,b]$ are

$$\underline{\int_a^b} f(x)\ dx := \sup_P L(f,P), \qquad \overline{\int_a^b} f(x)\ dx := \inf_P U(f,P).$$

7.2.2 Lemma

Assuming $f : [a,b] \longrightarrow \mathbb{R}$ is bounded,

$$\underline{\int_a^b} f(x)\ dx \leq \overline{\int_a^b} f(x)\ dx.$$

7.2.3 Definition

f is Darboux integrable on $[a,b] \iff$ the upper and lower Darboux sums are equal, in which case,

$$\int_a^b f(x)\ dx := \underline{\int_a^b} f(x)\ dx = \overline{\int_a^b} f(x)\ dx.$$

7.2.4 Theorem

A bounded function $f : [a,b] \longrightarrow \mathbb{R}$ is Darboux integrable $\iff$

$$\forall\, \varepsilon > 0,\ \exists \text{ a partition } P \text{ of } [a,b],\ U(f,P) - L(f,P) < \varepsilon.$$

7.2.5 Corollary

Let $f : [a,b] \longrightarrow \mathbb{R}$ be a bounded function and (P_n) be a sequence of partitions of $[a,b]$ such that $\lim_{n\to\infty} \big(U(f,P_n) - L(f,P_n)\big) = 0$. f is Darboux integrable and

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n).$$

7.2.6 Theorem

$f : [a,b] \longrightarrow \mathbb{R}$ is continuous $\implies f$ is Darboux integrable.

7.3 Basic Properties

By the remark at the beginning of this section, Darboux integration will now be referred to as simply integration (and Darboux integrable as integrable).

7.3.1 Theorem

Let $f, g : [a, b] \longrightarrow \mathbb{R}$ be integrable.

1. $f(x) \leq g(x)\ \forall\, x \in [a, b] \implies \displaystyle\int_a^b f(x)\ dx \leq \int_a^b g(x)\ dx.$
2. Lemmas for 3: $\int_a^b cf(x)\ dx = d\int_a^b f(x)\ dx\ \forall\, c \in \mathbb{R}$; $\int_a^b f(x) + g(x)\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx.$
3. $\displaystyle\int_a^b cf(x) + dg(x)\ dx = c\int_a^b f(x)\ dx + d\int_a^b g(x)\ dx\ \forall\, c, d \in \mathbb{R}.$
4. $\forall\, c \in (a, b),\ f$ is integrable on $[a, c]$ and $[c, b]$ and $\displaystyle\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx.$

7.3.2 Theorem

Let $f : [a, b] \longrightarrow \mathbb{R}$ be integrable and let $m \leq f(x) \leq M\ \forall\, x \in [a, b]$. Let $g : [m, M] \longrightarrow \mathbb{R}$ be continuous.

$$h = g \circ f$$

is also integrable on $[a, b]$.

7.3.3 Corollary

Let $f : [a, b] \longrightarrow \mathbb{R}$ be integrable. $|f|$ is also integrable on $[a, b]$, and

$$\left|\int_a^b f(x)\ dx\right| \leq \int_a^b |f(x)|\ dx$$

which is analogous to the triangle inequality.

7.3.4 Corollary

$f, g : [a, b] \longrightarrow \mathbb{R}$ are integrable $\implies fg : [a, b] \longrightarrow \mathbb{R}$ is integrable (the product, not the composition).

7.4 The Fundamental Theorem of Calculus

7.4.1 Theorem: First Version

Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous and define $F : [a, b] \longrightarrow \mathbb{R}$ as

$$F(x) = \int_a^x f(t)\ dt.$$

F is continuous on $[a, b]$ and differentiable on (a, b) and $F'(x) = f(x)\ \forall\, x \in (a, b)$.

7.4.2 Theorem: Second Version

Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous and have continuous derivative on (a, b).

$$\int_a^b f'(x)\ dx = f(b) - f(a).$$

7.4.3 Theorem

Let $n \in \mathbb{Z}$.

1. $n \neq -1$ (and if $n < 0$, $0 \notin [a, b]$) $\implies \int_a^b x^n\ dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}.$
2. $\int_a^b \frac{1}{x}\ dx = \log b - \log a.$

7.4.4 Theorem: Integral Mean Value Theorem

For a continuous $f : [a, b] \longrightarrow \mathbb{R}$, $\exists\, c \in (a, b)$,

$$\int_a^b f(x)\ dx = f(c)(b - a).$$

7.4.5 Theorem: Integration by Parts

For two continuous functions $f, g : [a, b] \longrightarrow \mathbb{R}$ with continuous first derivatives,

$$\int_a^b f(x)g'(x)\ dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)\ dx.$$

7.4.6 Theorem: Integration by Substitution

For a continuous $f : [a, b] \longrightarrow \mathbb{R}$ and a function $\phi : [c, d] \longrightarrow [a, b]$ with continuous first derivative on (c, d),

$$\int_{\phi(c)}^{\phi(d)} f(x)\ dx = \int_c^d f(\phi(t))\phi'(t)\ dt.$$

7.5 Limits of Integrable Functions

7.5.1 Theorem

Just as theorem 6.6.1 states the commutativity of uniform convergence and differentiation, the same is true for integration: let $f_n : [a, b] \longrightarrow \mathbb{R}$ be a sequence of integrable functions which converge uniformly to f. f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx.$$

7.6 Improper Integrals

7.6.1 Definition

Suppose $f : (a, b] \longrightarrow \mathbb{R}$ is integrable on all subintervals $[c, b] \subset (a, b]$. The improper integral over $(a, b]$ is defined

$$\int_a^b f(x)\ dx = \lim_{c\downarrow a} \int_c^b f(x)\ dx$$

if the limit exists. Similarly, if $f[a, b) \longrightarrow \mathbb{R}$ is integrable on all subintervals $[a, c] \subset [a, b)$, we define

$$\int_a^b f(x)\ dx = \lim_{c\uparrow b} \int_a^c f(x)\ dx$$

if the limit exists.

7.6.2 Definition

If the integral $\int_a^b f\ dx$ is unbounded at some point $c \in (a, b)$, or if $a = -\infty$ and $b = \infty$, we can simply define the improper integral

$$\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$$

(and in the second case the choice of c is irrelevant, done only in order to apply the previous definition).

7.7 Lebesgue's Criterion for Integrability

Several results so far have outlined conditions which imply integrability, and there are many examples of discontinuous, bounded integrable functions. This section is about exactly which functions are and are not integrable.

7.7.1 Definition

An open cover of $S \subset \mathbb{R}$ is a set of open intervals $\{U_\alpha = (a_\alpha, b_\alpha)\}$ such that

$$S \subset \bigcup_\alpha U_\alpha.$$

7.7.2 Lemma

Let $\{U_\alpha = (a_\alpha, b_\alpha)\}$ be an open cover of $S \subset \mathbb{R}$.

1. $\{U_\alpha\}$ has a countable subcover ($\exists$ a countable set of $U_i \in \{U_\alpha\}$ which is also an open cover of S).
2. S is compact $\implies$ $\{U_\alpha\}$ has a finite subcover.

7.7.3 Definition

$S \subset \mathbb{R}$ has (outer) measure zero $\iff$ $\forall\, \varepsilon > 0$, $\exists$ a finite or countable open cover $\{U_\alpha = (a_\alpha, b_\alpha)\}$ of S such that

$$\sum_\alpha (b_\alpha - a_\alpha) < \varepsilon.$$

7.7.4 Corollary

A single point has measure zero and a countable union of sets of measure zero has measure zero.

7.7.5 Theorem

Any set containing an interval of the form $[a, b]$ $(a < b)$ does not have measure zero.

7.7.6 Definition

For the purposes of results that follow, we define the 'jump' of $f : [a, b] \longrightarrow \mathbb{R}$ at x as

$$j_f(x) = \inf_{\delta > 0} \left(\sup_{|x-y|<\delta} f(y) - \inf_{|x-y|<\delta} f(y) \right)$$

For any $x \in [a, b]$ and fixed $\delta > 0$, $\sup_{|x-y|<\delta} f(y) \geq f(x) \geq \inf_{|x-y|<\delta} f(y)$, so $j_f(x) \geq 0\ \forall\, x \in [a, b]$ with equality if and only if f is continuous at x.

7.7.7 Definition

We can now define the set of discontinuities of $f : [a, b] \longrightarrow \mathbb{R}$ by

$$D(f) = \Big\{ \xi \in [a, b] \mid f \text{ is not continuous at } \xi \Big\} = \Big\{ \xi \in [a, b] \mid j_f(\xi) > 0 \Big\}$$

and also, for $c > 0$, define

$$D_c(f) = \Big\{ \xi \in [a, b] \mid j_f(\xi) \geq c \Big\}.$$

7.7.8 Theorem

$f : [a, b] \longrightarrow \mathbb{R}$ is integrable $\implies$ $D_c(f)$ has measure zero $\forall\, c > 0$ (requires 7.7.4).

7.7.9 Corollary

Noting that $D(f) = \bigcup_{n\in\mathbb{N}} D_{\frac{1}{n}}(f)$, $f : [a, b] \longrightarrow \mathbb{R}$ is integrable $\implies$ $D(f)$ has measure zero by 7.7.4.

7.7.10 Lemma

For $f : [a, b] \longrightarrow \mathbb{R}$, each set $D_c(f)$ is compact.

7.7.11 Theorem

f is bounded and $D(f)$ has measure zero $\implies$ f is integrable.

7.7.12 Theorem: Lebesgue's Criterion for Integrability

The combination of 7.7.9 ($\implies$) and 7.7.11 ($\impliedby$) gives Lebesgue's criterion:
A bounded function $f : [a, b] \longrightarrow \mathbb{R}$ is integrable $\iff$

$$D(f) = \Big\{ \xi \in [a, b] \mid f \text{ is not continuous at } \xi \Big\}$$

has measure zero.