Analysis 1 - Concise Notes

MATH40001

Term 1 Content

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Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

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Contents

1 Logic

lowkey nothing to write here

2 Numbers

2.1 Rational numbers

Recall $\mathbb{Q} := (p.q) \in F \times \mathbb{N}/$, where is the equivalence relation.

$$(p_1, q_1)(p_2, q_2) \iff p_1 q_2 = p_2 q_1$$

We write the equivalence class of (p,q) as p/q. Each equivalence class has a distinguished element (p', q') suc h that $\nexists n \in \mathbb{N}$ with $n > 1$ and $n|p', n|q'$. We say p'/q' is int "in lowest terns"

Axiom 2.1

1. addition is commutative
2. multiplication is commutatitve
3. addition is associative
4. multiplication is associative
5. multiplication is distributive over addition
6. additive identity 0
7. multiplicative identity 1
8. additive inverse
9. multiplicative inverse

Axiom 2.2 *Order axioms*

10. for each $x \in \mathbb{Q}$ **precisely one of (a),(b),(c) holds:**
    $(a), x > 0$ or $(b), x = 0$, or $(c) - x > 0$ (*Tricohotomy axiom*)
11. $x > 0, y > 0 \longrightarrow x + y > 0 \forall$ in $\mathbb{Q}$
12. same as above but for multiplication
13. $\forall x \in \mathbb{Q} \exists n \in \mathbb{N}$ such that $n > x$ *Archimedean axiom*

1.2 Decimals

Finite deimals For $a_0 \in \mathbb{Z}$ and $a_i \in 0, 1, 2, \ldots, 9$ we **define** the finite decimal $a_0.a_1 a_2 \ldots a_i$ as follows. If $a_0 \geq 0$ then $a_0.a_1 a_2 \ldots a_i$ is set to be $a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \ldots + \frac{a_n}{10^n} \in \mathbb{Q}$ For $a_0 < 0$ we set $a_0.a_1 a_2 \ldots a_i$ to be $-(|a_0|.a_1 a_2 \ldots a_i)$. Putting $a_j := 0$ for $j > 1$ this is special case of an eventually periodic decimal For rational numbers with eventually preriodic decimals, for now we will take it as a definition.
$a_0.a_1 \ldots a_i a_{i+1} a_{i\bar{+}2} \ldots a_j$
to be the *rational number*
$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \ldots + \frac{a_n}{10^n} + (\frac{a_{i+1} a_{i+2} \ldots a_j}{10^j}(\frac{1}{1 - 10^{i-j}}))$ Thus eventually periodic decimal expansion gives a **rational number**. Conversely, periodic decimals give **all** the rational numbers.

Theorem 1.1.

Any $x \in \mathbb{Q}$ is equal to an eventually periodic decimal expansion:
$x = a_0.a_1 \ldots a_i a_{i+1} a_{i\bar{+}2} \ldots a_j$
But not all eventually periodic decimals give *different* rational numbers.
Proposition 2.14 If $x \in \mathbb{Q}$ has two different decimal expansions then they are of the form
$x = a_0.a_1 a_2 \ldots a_n \bar{9} = a_0.a_1 a_2 \ldots (a_n + 1)$ with $a_n \in 0, 1, 2, \ldots, 8$
Arbitrary Decimals So this gives us an obvious way to define the real numbers: as the set of decimal expansions which do not end in $\bar{9}$, $\mathbb{R} := a_1.a_2 a_3 \ldots : a_0 \in \mathbb{Z}, a_{i \geq 1} \in 0, 1, 2 \ldots, 9, \nexists N$ such that $a_i = 9 \forall i \geq N$ With some work one can then define +,-,×,÷¡ on $\mathbb{R}$ and chech they satisfy the Axioms 2.1 and 2.2
Theorem 2.8 gives us a way to produce many explicit irrational numbers like.

1.3 Countability

Definition 1.3.1.

A set is *Countable* if and only if there exists a bijection $f : \mathbb{N} \longrightarrow S$ **Proposition 2.16** Suppose S$\subset$ $\mathbb{N}$ is infinite. Then S is countable.
Proposition 2.17 $\mathbb{Z}$ is countable

Theorem 1.2.

Q is countable

Theorem 1.3.

R is uncountable

1.4 The completeness Axiom

$\emptyset \neq S \subset \mathbb{R}$ is *bounded above* is and only if $\exists M \in \mathbb{R}$ such that $\forall x \in S, x \leq M$ Such M is called an *upper bound* for S
bounded below is just the opposite innit
S is bounded if and only if S is bounded above and below

Definition 1.4.1.

Suppose $\emptyset \neq S \subset \mathbb{R}$ is bounded above. We say $x \in \mathbb{R}$ is *least upper bound* of S or **supremum** of S if and only if

- x is an uppose bound for S
- $x \leq y$ for any y which is an upper bound fort S

Theorem 1.4.

The completeness Axiom of $\mathbb{R}$
Suppose that $S \subset R$ is nonempty and bounded above, Then S has a supremum
Proposition 2.34 There exists $0 < x \in \mathbb{R}$ such that $x^2 = 3$
Proposition 2.38 Suppose $\emptyset \neq S \subset \mathbb{R}$ and y is an upperbound for S. Then $y = supS \iff \forall \epsilon > 0 \exists S : s > y - \epsilon$

1.5 Alternative approach: Dedekind cuts

Definition 1.5.1.

We say a non-empty subset $S \subset \mathbb{Q}$ is a *Dedekind cut* if it satisfies the following

i If $s \in \S$ amd $s > t \in \mathbb{Q}$ then $t \in S$(S os a semi-infinite interval to the left)

ii S is bounded above but has no maximum.

Definition 1.5.2.

$\mathbb{R}$:=Dedekind cuts $S \subset \mathbb{Q}$

1.6 Triangle inequalities

Theorem 1.5.

For all $a, b \in \mathbb{R}$ we have $|a + b| \leq |a| + |b|$

2 Sequence

Definition 2.0.1.

A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$

2.1 Convergence of Sequences

Definition 2.1.1.

Definition 2.1.2. Convergence
We say that $a_n \to a$ as $n \to \infty$ if and only if

$$\forall \epsilon > 0 N \in \mathbb{N} \text{ such that } \forall n \geq N, |a_n - a| < \epsilon$$

Definition 2.1.3.

We say a_n *diverges* if and only if it does not converge.

$$\forall a \epsilon > 0 \text{ such that } \forall N \in, \geq N \text{ such that } |a_n - a| \geq \epsilon$$

The definition for convergence is basically the same for complex numbes too Turns out convergence for complex numbers is equivalent to the real part converging to the real part and the imaginary part converging to the imaginary part/

Theorem 2.1.

The Uniqueness of Limits
Limits are unique. If $a_n \to a$ and $a_n \to b$ then $a = b$
Proposition 3.16 If (a_n) is convergent then it is bounded

Theorem 2.2.

Algebra of limits If $a_n \to a$ and $b_n \to b$, then:

1. $a_n + b_n \to a + b$
2. $a_n b_b \to ab$
3. $\frac{a_n}{b_n} \to \frac{a}{b}$ if $b \neq 0$

Theorem 2.3.

If (a_n) is bounded above and monotonically increasing then a_n converges to $a := supa_i : i \in \mathbb{N}$ Cauchy Sequences

Definition 2.1.4.

$(a_n)_{n \geq 1}$ is called a Cauchy sequence if and only if $\forall \epsilon > 0 N \mathbb{N}$ such that $\forall n, m \geq N, |a_n - a_m| < \epsilon$ **Proposition 3.25** If $a_n \to a$ then (a_n) is Cauchy **Lemma 3.27**
If (a_n) is a Cauchy sequence then (a_n) is bounded.

Theorem 2.4.

If (a_n) is a Cauchy sequene of real numbers then a_n converges.
There for a sequence is Cauchy if and only if it is convergent and vice versa

2.2 Subsequences

Definition 2.2.1.

A *A subsequence* of (a_n) is a new sequence $b_i = a_{n(i)} \forall i \in \mathbb{N}$ where $n(1) < n(2) < \cdots < n(i) < \cdots \forall i$

Theorem 2.5. Bolzano-Weierstrass

If (a_n) is a bounded sequence of real numbers then it has a convergent subsequence
Proposition3.39
If $a_n \to a$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$
$Bolzano - Weierstrass \iff CauchyTheorem$

Definition 2.2.2.

We say $a_n \to +\infty$ if and only if $\forall R > 0 N \in \mathbb{N}$ such that $a_n > R \forall n \geq N$

3 Series

Definition 3.0.1.

An (infinite) series is an expression $\sum a_{n}{}_{n=1}^{\infty}$ where $((a_i)_i)$ is a sequence.

3.1 Convergence of Series

Definition 3.1.1.

We say that the series $\sum a_n$ converges to $A \in \mathbb{R}$ if and only if the sequence of partial sums converges to A

Theorem 3.1.

$\sum_{n=0}^{\infty}$ is convergent $\longrightarrow a_n \to 0$ **Proposition 4.6** Suppose $a_n \geq 0 \forall n$. Then the following facts are true:

1. $\Sigma_{n=1}^{\infty}$ converges if and only if (s_n) is bounded above
2. Similarly $\Sigma_{n=1}^{\infty}$ diverges to $+\infty$ if and only if (s_n) is unbounded

Theorem 3.2.

Comparison test If $0 \leq a_n \leq b_n$ and Σb_n converges, then Σa_n converges

Theorem 3.3.

Algebra of limits for series
If $\Sigma a_n, \Sigma b_n$ are convergent then so is $\Sigma(\lambda a_n + \mu b_n)$. What it converges to is obvious

3.2 Absolute convergence

Definition 3.2.1.

For $a_n \in \mathbb{R}$ or , we say the series $\Sigma_{n=1}^{\infty} a_n$ is *absolutely convergent* if and only if the series $\Sigma_{n=1}^{\infty} |a_n|$ is convergent

Theorem 3.4.

Let $(a_n)_{n\geq}$ be a real or complex sequence.
If $\Sigma_{n=1}^{\infty} a_n$ is absolutely convergent, then it is convergent.

3.3 Test for convergence

Theorem 3.5.

Comparison II
Suppose $c_n \leq a_n \leq b_n \forall n$ and $\Sigma c_n, \Sigma b_n$ are both convergent.
Then Σa_n is convergnet

Theorem 3.6.

Comparison III If $\frac{a_n}{b_n} \to L \in \mathbb{R}$ and Σb_n is absolutely convergent, then Σa_n is absolutely convergent.

Theorem 3.7.

Alternating Series Test
Suppose a_n is alternating with $|a_n| \downarrow 0$. Then Σa_n converges.

Theorem 3.8.

Ratio Test
If a_n is a sequence such that $|\frac{a_{n+1}}{a_n} \to r < 1$, then Σa_n is absolutely convergent

Theorem 3.9.

Root Test
If $|a_n|^{1/n} \to r < 1$, them Σa_n is absolutely convergent

Theorem 3.10.

Σa_n is absolutely convergent $\Leftrightarrow$ (1) + (2) $\Rightarrow$ (3) + (4) where,

(1) $_{a_n \geq 0} a_n$ is convergent to A

(2) $_{a_n < 0} a_n$ is convergent to B

(3) $\Sigma a_n = A + B$

(4) $\Sigma b_m = A + B$ where (b_m) is any rearrangement of (a_n)

3.4 Power Series

Theorem 3.11.

Radius of Convergence Fix a real or complex series (a_n) and consider the series $\Sigma a_n z^n$ for $z \in$ Then $\mathbb{R} \in [0, \infty]$ such that

- $|z| < R \longrightarrow \Sigma a_n z^n$ is absolutely convergent
- and if not it's divergent

Definition 3.4.1.

Given series $\Sigma a_n, \Sigma b_n$ their *Cauchy Product* is the series Σc_n where $c_n := \Sigma_{i=0}^{n} a_i b_{n-i}$

Theorem 3.12.

Cauchy Product If two serieses are absolutely convergent then their Cauchy product is absolutely congergent to the product of the series'

3.5 Exponential Power Series

Definition 3.5.1.

For any $z \in$ set $E(z) := \Sigma_{n=0}^{\infty} \frac{z^n}{n!}$ E(z) is convergent $\forall z \in$ **Proposition 4.43** $E(z) + E(w) = E(z + w)$

Definition 3.5.2.

$e := E(1) = \Sigma \frac{1}{n!} \in (0, \infty)$
Corollary 4.45 $E(n) = e^n forn \in \mathbb{N}$
Proposition 4.46
$E(q) = e^q$ for $q \in \mathbb{Q}$
Proposition 4.47 $E(x)$ has the following properties for $x \in \mathbb{R}$

1. $E(x) > 0 \forall x \in \mathbb{R}$
2. $x \geq 0 \longrightarrow E(x) \geq 1$ and $x > 0 \longrightarrow E(x) > 1$
3. $E(x)$ *is strictly increasing* for $x \in \mathbb{R}$
4. $|E(x) - 1| \leq \frac{|x|}{1-|x|}$ for $|x| < 1$
5. $x \mapsto E(x)$ is a continuous bijection $\mathbb{R} \tilde{\rightarrow} (0, \infty)$

Point 5 enables us to define an inverse function from $(0, \infty) \tilde{\rightarrow} \mathbb{R}$

4 Continuity

4.1 Limits

Definition 4.1.1.

Fix a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and points $a, b \in \mathbb{R}$
We say that $f(x) \rightarrow b$ as $x \rightarrow a$ if and only if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } 0 < |x - a| < \delta \longrightarrow |f(x) - b| < \epsilon$$

Definition 4.1.2.

Fix a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a point $a \in \mathbb{R}$. We say that f **continuous** as a if and only if $lim_{x \rightarrow a} f(x) = f(a)$

Theorem 4.1.

$f : \mathbb{R} \rightarrow \mathbb{R}$ is contiunous at $a \in \mathbb{R}$ if and only if

$$\forall \epsilon > 0 \exists \delta > 0 \text{ such that } |x - a| < \delta \longrightarrow |f(x) - f(a)| < \epsilon$$

4.2 Continuity

Definition 4.2.1.

We say that f is continuous on $\mathbb{R}$ if it is continuous at all a
Proposition 5.10 $E :\to$ defined by $E(z) := \Sigma_{n=0}^{\infty} \frac{z^n}{n!}$ is continuous on

Theorem 4.2.

$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R} \iff f(x_n) \to f(a) \forall$ sequences $x_n \to a$