Math40002: Analysis I

except for minor edits, lectures notes by Richard Thomas

Obvious is the most dangerous word in mathematics.

- E.T. Bell

The irreducible price of learning is realizing that you do not know.

- James Baldwin

We will build on MATH40001 "Introduction to University Mathematics", practising the language, logic and rigour of pure mathematics.

We will learn to formulate rigorous definitions and proofs, forming a solid foundation for future courses in pure and applied mathematics.

We will have infinite fun.

Syllabus

Real numbers: Review of rational and real numbers. (Un)countability. Triangle inequalities. Suprema and infima.

Limits of sequences: definitions, techniques, results and examples. Tests for convergence. Cauchy sequences. Bolzano-Weierstrass theorem.

Summing infinite series: definitions, results and examples. Tests for convergence. Manipulation of convergent series.

Continuity: definition of continuous functions.

Books - on Leganto (accessible on Blackboard)

Martin Liebeck *A Concise Introduction to Pure Mathematics.*
Mary Hart, *Guide to Analysis.*

KG Binmore, *Mathematical Analysis, A Straightforward Approach.*
David Brannan, *A first course in mathematical analysis.*
Steven Lay, *Analysis: with an introduction to proof.*
Stephen Abbott, *Understanding analysis.*

Assessment

Assessed Quizzes released Monday on weeks 2,3, and 5 and on Thursday weeks 6 and 7. These quizzes are multiple choice and completed via Blackboard. They will be due within 72 hours and you will be given 90 minutes to work on them. In total they are worth – 5%.
Mid-module test in Week 4 of the module (Week 8 of the term) – 5%.
January test – 10%.
May exam – 70%. (Exercise: how much is next term's coursework worth?)

Discussion

We will be using Piazza for discussion. The system is highly catered to getting you help fast and efficiently from classmates, the GTAs, and the instructors. Rather than emailing questions to the teaching staff, I encourage you to post your questions on Piazza. If you have any problems or feedback for the developers, email team@piazza.com.

Find our class signup link at: `https://piazza.com/imperial.ac.uk/fall2020/hy202010`

Contents

1 Logic

Mathematics is less related to accounting than it is to philosophy.

- Leonard Adleman

Exercise 1.1. At which stage does this argument go wrong?

Suppose $x = 2$.

1. $\implies x - 2 = 0$
2. $\implies x^2 - 2x = 0$
3. $\implies x(x - 2) = 0$
4. $\implies x = 0$ or $x = 2$.
5. Nowhere; the argument is correct. ✓ $x = 2$ *does* imply $x = 0$ or 2

Exercise 1.2. "A unless B" is the same logical statement as

1. $A \iff B$
2. $\overline{A} \iff \overline{B}$
3. $A \implies B$
4. $A \implies \overline{B}$
5. $\overline{A} \implies B$ ✓
6. $\overline{A} \implies \overline{B}$
7. None of these; something else.
8. More than one of these.

It says $\overline{B} \implies A$; equivalently $\overline{A} \implies B$. Think of "We'll go out unless it rains".

Exercise 1.3.

"*Find two real numbers x which satisfy the equation $x^2 - 3x + 2 = 0$.*"
Student solution:

$$\begin{aligned} & x^2 - 3x + 2 = 0 \\ \implies & (x-1)(x-2) = 0 \\ \implies & x = 1 \text{ or } x = 2. \end{aligned}$$

How many marks would this get in an exam?

> 1. Two marks – completely solved the problem.
> 2. One mark – partially solved the problem.
> 3. No marks – failed to solve the problem. ✓ Showed $x \notin \{1, 2\} \implies x^2 - 3x + 2 \neq 0$.

> **Exercise 1.4.** Is this a correct proof that $3|n^2 \implies 3|n$?
> If $3|n$ then $n = 3m$ for some $m \in \mathbb{N}$ so $n^2 = 3(3m^2)$ is divisible by 3.
>
> 1. Yes.
> 2. No. ✓ Showed $\Longleftarrow$ instead of $\implies$
> 3. Uh?

Correct proof. By dividing any n by 3 and taking remainders we know it can be written as $3q$, $3q + 1$ or $3q + 2$ for some $q \in \mathbb{Z}$.
(Proof: set $q = \lfloor \frac{n}{3} \rfloor := \max\{Q \colon n - 3Q \geq 0\}$ using the Archimedean axiom and show it works – exercise!)
If $n = 3q + 1$ or $n = 3q + 2$ then squaring gives $n^2 = 3N + 1$ for some $N \in \mathbb{Z}$. Thus $3 \nmid n^2$ ※ So $n = 3q$.

> **Exercise 1.5.** What does $x \in \bigcup_{n=1}^{\infty} S_n$ mean?
>
> 1. $x \in S_n$ for some $n \in \mathbb{N}$ ✓
> 2. Either $x \in S_n$ for some $n \in \mathbb{N}$ or $x \in S_\infty$
> 3. Either $x \in S_n$ for some $n \in \mathbb{N}$ or $x \in \lim_{n\to\infty} S_n$
> 4. Other

Conclusion: you need to practice your logic; please do so. Almost everything we do in this course relies on it.

Furthermore, the reason employers will offer you obscene amounts of money at the end of your course is because they expect you to have a better grasp of logic and problem solving than other graduates. To achieve this you need the basics to be solid. If you find them boring, grit your teeth and think of the cash. If you make small mistakes (like confusing $\implies$ with $\Longleftarrow$) at this stage, you won't be able to solve much harder problems later on.

2 Numbers

If you spent less time asking what's examinable and more time trying to understand new maths, you would get far more marks. Worry about what's examinable in April; for now just try to think and solve and learn.

- All your lecturers, annually

2.1 Rational numbers

Recall $\mathbb{N} := \{1, 2, 3, \ldots\}$ and $\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with $+, \times, >$.

Recall $\mathbb{Q} := \{(p, q) \in \mathbb{Z} \times \mathbb{N}\}/\sim$, where $\sim$ is the equivalence relation

$$(p_1, q_1) \sim (p_2, q_2) \iff p_1 q_2 = p_2 q_1.$$

We write the equivalence class of (p, q) as p/q or $\frac{p}{q}$. Each equivalence class has a distinguished element (p', q') such that $\nexists\, n \in \mathbb{N}$ with $n > 1$ and $n|p'$, $n|q'$. We say $\frac{p'}{q'}$ is "in lowest terms". We define

$$\begin{aligned}
\frac{p_1}{q_1} + \frac{p_2}{q_2} &:= \frac{p_1 q_2 + p_2 q_1}{q_1 q_2}, \\
\frac{p_1}{q_1} - \frac{p_2}{q_2} &:= \frac{p_1 q_2 - p_2 q_1}{q_1 q_2}, \\
\frac{p_1}{q_1} \times \frac{p_2}{q_2} &:= \frac{p_1 p_2}{q_1 q_2}, \\
\frac{p_1}{q_1} \div \frac{p_2}{q_2} &:= \frac{p_1 q_2}{q_1 p_2}, \quad p_2 \neq 0, \\
\frac{p_1}{q_1} \leq \frac{p_2}{q_2} &\iff p_1 q_2 \leq p_2 q_1.
\end{aligned}$$

These satisfy certain properties that we list next. They are sufficiently strong that you can deduce everything about $\mathbb{Q}$ just from these properties, i.e. you can treat them as axioms if you wish.

Axiom 2.1.

1. $a + b = b + a \quad \forall a, b \in \mathbb{Q}$ (+ is *commutative*)
2. $a \times b = b \times a \quad \forall a, b \in \mathbb{Q}$
3. $a + (b + c) = (a + b) + c$ (+ is *associative*)
4. $a \times (b \times c) = (a \times b) \times c$
5. $a \times (b + c) = (a \times b) + (a \times c)$ (× is *distributive* over +)

6. $\exists\, 0 \in \mathbb{Q}\colon a + 0 = a \quad \forall a \in \mathbb{Q}$
7. $\exists\, 1 \in \mathbb{Q}\colon 0 \neq 1, a \times 1 = a \quad \forall a \in \mathbb{Q}$
8. $\forall a \in \mathbb{Q},\ \exists(-a) \in \mathbb{Q}$ such that $a + (-a) = 0$
9. $\forall a \in \mathbb{Q}\backslash\{0\}\ \exists a^{-1} \in \mathbb{Q}$ such that $a \times (a^{-1}) = 1$

Axiom 2.2 (Order axioms)**.**

10. for each $x \in \mathbb{Q}$ **precisely one** of (a), (b), (c) holds:

$$\text{(a) } x > 0 \quad \text{or} \quad \text{(b) } x = 0 \quad \text{or} \quad \text{(c) } -x > 0 \qquad \text{(Trichotomy axiom)}$$

11. $x > 0,\ y > 0 \implies x + y > 0 \quad \forall x, y \in \mathbb{Q}$
12. $x > 0,\ y > 0 \implies xy > 0 \quad \forall x, y \in \mathbb{Q}$
13. $\forall x \in \mathbb{Q}\ \exists n \in \mathbb{N}$ such that $n > x$ (Archimedean axiom)

Notation: $a - b := a + (-b)$, and $a/b := a \times (b^{-1})$, while $a > b$ ($a < b$) is defined to mean $a - b > 0$ (respectively $-(a - b) > 0$).

Exercise 2.3. $x > y > z \implies x > z$.

Just write down what the LHS means:

$$\begin{aligned}
x > y > z \quad &\iff \quad \big[x - y > 0 \text{ and } y - z > 0\big] \\
&\overset{11}{\implies} \quad (x - y) + (y - z) > 0 \\
&\overset{3}{\iff} \quad x + ((-y) + y) - z > 0 \\
&\overset{8}{\iff} \quad x + 0 - z > 0 \\
&\overset{6}{\iff} \quad x - z > 0 \\
&\iff \quad x > z.
\end{aligned}$$

Exercise 2.4. Fix $a \in \mathbb{Q}$. For each $x \in \mathbb{Q}$ exactly one of the following holds

$$\text{(a) } x > a \quad \text{or} \quad \text{(b) } x = a \quad \text{or} \quad \text{(c) } x < a.$$

The real numbers $\mathbb{R}$ satisfy the exact same axioms, plus one more – the **completeness axiom** – designed to fix the problem that $\mathbb{Q}$ has holes. For instance,

Proposition 2.5. *There is no $x \in \mathbb{Q}$ such that $x^2 = 3$.*

Proof. Suppose $x = p/q$ in lowest terms satisfies $x^2 = 3 \iff p^2 = 3q^2$.

Thus $3|p^2$ so $3|p$.

(Proof: recall exercise 1.4.)

So writing $p = 3n$ we find $q^2 = 3n^2$ so $3|q$ as well as $3|p$, contradicting the assumption that p/q is in lowest terms. □

2.2 Decimals

Finite decimals

For $a_0 \in \mathbb{Z}$ and and $a_i \in \{0, 1, \dots, 9\}$ we **define** the finite decimal $a_0.a_1 \dots a_i$ as follows. If $a_0 \geq 0$ then $a_0.a_1 \dots a_i$ is set to be

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \cdots + \frac{a_i}{10^i} \ \in \ \mathbb{Q}.$$

For $a_0 < 0$ we set $a_0.a_1 \dots a_i$ to be $-\big(|a_0|.a_1 \dots a_i\big)$. Putting $a_j := 0$ for $j > i$ this is a special case of an eventually periodic decimal.

Eventually periodic decimals

At school you became happy with the idea that

$$\begin{aligned} 0.\overline{3} &= 0.3333\dots \\ &= 0.3 + 0.03 + 0.003 + 0.0003 + \dots \\ &= \frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + \dots \\ &= \frac{3}{10}\left(1 + \frac{1}{10} + \left(\frac{1}{10}\right)^2 + \left(\frac{1}{10}\right)^3 + \dots\right) \\ &\overset{?}{=} \frac{3}{10} \cdot \frac{1}{1 - \frac{1}{10}} = \frac{3}{10}\frac{10}{9} = \frac{1}{3}. \end{aligned}$$

Now I'll grant you we can certainly justify

$$1 + x + x^2 + \cdots + x^n = \frac{1 - x^{n+1}}{1 - x}, \qquad x \neq 1,$$

by multiplying both sides by $1 - x$. But it will be many lectures (see Example 4.1) before we know how to conclude that

$$\begin{aligned} 1 + x + x^2 + \cdots + x^n + \dots &= \lim_{n\to\infty} \frac{1 - x^{n+1}}{1 - x} \\ &= \frac{1}{1 - x}, \quad -1 < x < 1, \end{aligned}$$

to justify the $\stackrel{?}{=}$ above. So for now we simply take it as a **definition**: for $a_0 \in \mathbb{N}$, $a_{i>0} \in \{0, 1, \dots, 9\}$ we define

$$a_0.a_1 \dots a_i\overline{a_{i+1}a_{i+2} \dots a_j}$$

to be the *rational number*

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \dots + \frac{a_i}{10^i} + \left(\frac{a_{i+1}a_{i+2} \dots a_j}{10^j}\right)\left(\frac{1}{1 - 10^{i-j}}\right) \tag{2.6}$$

motivated by the "fact" (which we've yet to prove) that the last fraction equals $1 + 10^{-(j-i)} + 10^{-2(j-i)} + \dots$. (There's a bit of checking that (2.6) is well-defined: if I consider the repetition to begin one period later, show I get the same rational number; similarly if I consider the period to be a multiple of $i - j$ show I get the same rational number.)[1]

> **Exercise 2.7.** Consider two eventually periodic decimals differing in only one place:
>
> $$a = a_0.a_1a_2 \dots a_{n-1}a_na_{n+1} \dots, \qquad b = a_0.a_1a_2 \dots a_{n-1}b_na_{n+1} \dots$$
>
> Show using our definition (2.6) that $a < b$ if and only if $a_n < b_n$.

Thus any eventually periodic decimal expansion gives a **rational number** (2.6). Conversely, periodic decimals give **all** the rational numbers. Before proving this let's do an example.

Let's try to write $\frac{25}{11}$ as a decimal,

$$\frac{25}{11} = a_0.a_1a_2a_3 \dots$$

To find $a_0 = \left\lfloor \frac{25}{11} \right\rfloor$ we divide 11 into 25:

$$25 = \mathbf{2} \times 11 + 3 \implies a_0 = 2. \tag{$*$}$$

To find a_1 we take $\frac{25}{11} - 2 = 0.a_1a_2a_3 \dots$ (i.e. $\frac{3}{11}$, where 3 is the remainder in $(*)$) then multiply by 10 and take integer part, i.e. $a_1 = \left\lfloor \frac{30}{11} \right\rfloor$:

$$30 = \mathbf{2} \times 11 + 8 \implies a_1 = 2.$$

Similarly multiplying the remainder 8 by 10 and dividing 11 in gives $a_2 = \left\lfloor \frac{80}{11} \right\rfloor$:

$$80 = \mathbf{7} \times 11 + 3 \implies a_2 = 7.$$

[1] E.g. show that our definition makes $0.\overline{3}$ and $0.3\overline{3}$ and $0.\overline{33}$ all the same number.

Notice we've got remainder 3 again, just as in $(*)$. So from now on everything repeats, because the remainder always determines the next step (we mutltiply it by 10 then divide 11 into the result). So $a_3 = \left\lfloor \frac{30}{11} \right\rfloor$ is just the same as a_1. And a_4 is the same as a_2. Etc. The result is the eventually periodic decimal

$$2.\overline{27}.$$

(Beware we've only shown that this decimal approximates $\frac{25}{11}$; we then need to prove they're equal. But this is clear by our definition (2.6).)

Since there are only finitely many possible remainders (i.e. nonnegative integers < 11) it was inevitable this periodicity would happen eventually. The general case is as follows.

Theorem 2.8

Any $x \in \mathbb{Q}$ is equal to an eventually periodic decimal expansion: $x = a_0.a_1 \ldots a_i\overline{a_{i+1}a_{i+2} \ldots a_j}$ $\;(a_0 \in \mathbb{Z},\; a_\ell \in \{0, 1, \ldots, 9\}$ for $\ell \geq 1)$.

Proof. Without loss of generality we take $x \geq 0$. It will be convenient to temporarily use a notation $\{x\} := x - \lfloor x \rfloor \in [0, 1)$ for the non-integer part of x.

To write x as a decimal we let $a_0 := \lfloor x \rfloor$ and $e_0 := \{x\}$, so

$$x \;=\; a_0 + e_0, \qquad a_0 \in \mathbb{N},\; e_0 \in [0, 1) \tag{2.9}$$

is the sum of an integer and a small "error". Now repeat for $10e_0 \in [0, 10)$, setting $a_1 := \lfloor 10e_0 \rfloor$ and error $e_1 := \{10e_0\}$:

$$10e_0 \;=\; a_1 + e_1, \qquad a_1 \in \{0, 1, \ldots, 9\},\; e_1 \in [0, 1).$$

Inductively, given $a_i \in \{0, 1, \ldots, 9\}$ and $e_i \in [0, 1)$ for $i < k$ we set $a_k := \lfloor 10e_{k-1} \rfloor$ and the error $e_k := \{10e_{k-1}\}$, so

$$10e_{k-1} \;=\; a_k + e_k, \qquad a_k \in \{0, 1, \ldots, 9\},\; e_k \in [0, 1). \tag{2.10}$$

Plugging each equation into the former gives

$$x \;=\; a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \cdots + \frac{a_k}{10^k} \;+\; \frac{e_k}{10^k}\,, \qquad e_k \in [0, 1). \tag{2.11}$$

Now remember $x = p/q$ $(p, q \in \mathbb{N})$ is rational! So $q\times$(2.9) tells us

$$p \;=\; qa_0 + r_0,$$

where $r_0 := qe_0 = p - qa_0 \in [0, q)$ is therefore an integer (in fact the remainder when we divide q into p). Inductively $q\times$(2.10) shows that $r_k := qe_k$ is an integer in $[0, q)$.

So the remainders $r_k \in \{0, 1, \dots, q-1\}$ lie in a finite set, so after a while they must repeat: $r_j = r_i$ for some $j > i$. Therefore the $e_k = r_k/q$ also repeat: $e_j = e_i$, so in the construction (2.10) we see the a_k repeat as well: $a_{j+1} = a_{i+1}$. Inductively then, $a_{\ell+j-i} = a_\ell$ for every $\ell \geq i+1$, and we have produced a periodic decimal expansion

$$a_0.a_1a_2\dots a_i\overline{a_{i+1}a_{i+2}\dots a_j} \tag{2.12}$$

that really *ought* to be x. (It gets closer and closer to x; some might say it converges to x, but we've not defined convergence yet! That's what the rest of the course is about. So for now we have to prove it's x using our definition (2.6).) Let's check it using our convention (2.6). It says (2.12) is the rational number

$$a_0 + \frac{a_1}{10} + \frac{a_2}{10^2} + \dots + \frac{a_i}{10^i} \;+\; \frac{a_{i+1}a_{i+2}\dots a_j}{10^j}\frac{1}{1-10^{i-j}}\,.$$

Comparing with (2.11) we see we just need to show that

$$\frac{e_i}{10^i} \;=\; \frac{a_{i+1}a_{i+2}\dots a_j}{10^j}\frac{1}{1-10^{i-j}}\,.$$

Multiplying by $(1-10^{i-j})$ and using the periodicity $e_i = e_j$ this is equivalent to

$$10^{-i}e_i - 10^{-j}e_j \;=\; 10^{-j}(a_{i+1}a_{i+2}\dots a_j). \tag{2.13}$$

But adding the equalities $10^{1-k}e_{k-1} - 10^{-k}e_k = \frac{a_k}{10^k}$ of (2.10) for $k = i+1, i+2, \dots, j$ gives

$$10^{-i}e_i - 10^{-j}e_j \;=\; \frac{a_{i+1}}{10^{i+1}} + \dots + \frac{a_j}{10^j},$$

which is precisely (2.13), as required. □

However, not all eventually periodic decimals give *different* rational numbers: by (2.6),

$$0.\overline{9} \;=\; \left(\frac{9}{10}\right)\left(\frac{1}{1-10^{-1}}\right) \;=\; 1.$$

Proposition 2.14. *If $x \in \mathbb{Q}$ has two different decimal expansions then they are of the form*

$$\begin{aligned} x \;&=\; a_0.a_1a_2\dots a_n\overline{9} \\ &=\; a_0.a_1a_2\dots(a_n+1) \quad \text{with } a_n \in \{0,1,\dots,8\}. \end{aligned}$$

Proof. Suppose the two expansions are:

$$\begin{aligned} x &= a_0.a_1a_2\dots a_{n-1}a_na_{n+1}\dots \\ &= a_0.a_1a_2\dots a_{n-1}b_nb_{n+1}\dots \end{aligned}$$

with $a_n < b_n$ without loss of generality. Then by Exercise 2.7 (and some easier Exercises like $0 \le 0.c_1c_2\dots \le 1$ with equality if and only if all c_i are 0 or all c_i are 9)

$$\begin{aligned} x &= a_0.a_1a_2\dots a_na_{n+1}\dots \\ &\le a_0.a_1a_2\dots a_n999\dots \\ &= a_0.a_1a_2\dots(a_n+1)000\dots \\ &\le a_0.a_1a_2\dots b_nb_{n+1}\dots \\ &= x. \end{aligned}$$

Therefore the $\le$s must have been $=$s and the proposition follows. □

Arbitrary decimals: the real numbers

So this gives us an obvious (but ugly!) way to define the real numbers: as the set of decimal expansions which do not end in $\overline{9}$,

$$\mathbb{R} := \left\{ a_0.a_1a_2\dots \ : \ a_0 \in \mathbb{Z},\ a_{i\ge1} \in \{0,1,\dots,9\},\ \nexists N \text{ such that } a_i = 9\ \forall i \ge N \right\}$$

With some work one can then define $+$, $-$ $\times$, $\div$ $<$ on $\mathbb{R}$ and check they satisfy the Axioms 2.1 and 2.2.

Theorem 2.8 gives us a way to produce *many* explicit irrational numbers like

$$x = 0.1010010001\dots \notin \mathbb{Q}.$$

Exercise 2.15. $\forall x, y \in \mathbb{R}$ with $x < y$ show

1. $\exists z \in \mathbb{Q} : x < z < y$, and
2. $\exists z \notin \mathbb{Q} : x < z < y$.

Do both by (a) decimal expansions, and (b) by using only the axioms. (Hint: use Archimedean axiom to find $n \in \mathbb{N}$ to magnify the difference $y - z$ to be ≥ 1.)

In fact there is a way to make precise the fact that there are many more irrational or real numbers than rational numbers.

2.3 Countability

Definition. A set S is *countable* if and only if there exists a bijection $f\colon \mathbb{N} \to S$.

This means I can put the elements of S into a list:

$$S = \{s_1, s_2, s_3, s_4, s_5, \dots\}$$

with no repeats (all s_i distinct). Here $s_n := f(n)$.

Note: A countable set is always infinite!

Since the even number $2\mathbb{N} \subset \mathbb{N}$ you might think there are less of them. But $\mathbb{N} \overset{\times 4}{\hookrightarrow} 2\mathbb{N}$ so maybe there are less? Really they're the same size, in the following sense.

Proposition 2.16. *Suppose $S \subset \mathbb{N}$ is infinite. Then S is countable.*

Proof. We just list the elements of S in order of size. Formally, we define $f\colon \mathbb{N} \to S$ inductively (recursively?) as follows:[2]

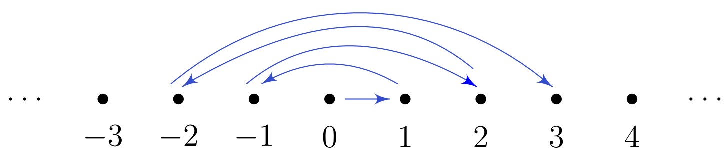
- $f(1) = \min S$,
- Assume $f(1), \dots, f(n-1)$ is defined already. Since S is infinite the set $S \backslash \{f(1), \dots, f(n-1)\}$ is nonempty, and all $s \in S$ are ≥ 0, so we may define

$$f(n) \;:=\; \min\Big(S\backslash\{f(1), \dots, f(n-1)\}\Big).$$

This function is injective since $f(1) < f(2) < f(3) < \dots$. If f were not surjective, then $\exists$ smallest $s \in S\backslash \mathrm{im}\,(f)$. Since $s \neq \min S$ (because $f(1) = \min S$) we know $\exists s' \in S$ such that $s' < s$ – picking the largest such, then $s' = f(n)$ and by our rule $s = f(n+1)$. □

Proposition 2.17. *$\mathbb{Z}$ is countable.*

Proof. We list them as $\mathbb{Z} = \{0, 1, -1, 2, -1, 3, -3, \dots\}$.



[2]Since S is infinite you should worry about taking min. Since it's nonempty pick an element $n_0 \in S$, then $S \cap \{1, 2, \dots, n_0\}$ is *finite and nonempty* so does have a minimum m say. Now check $m = \min S$.

Formally, define a bijection $f : \mathbb{N} \to \mathbb{Z}$ by declaring, for $k \geq 1$,

$$\begin{cases} f(2k-1) & := & -(k-1), \\ f(2k) & := & k. \end{cases}$$ □

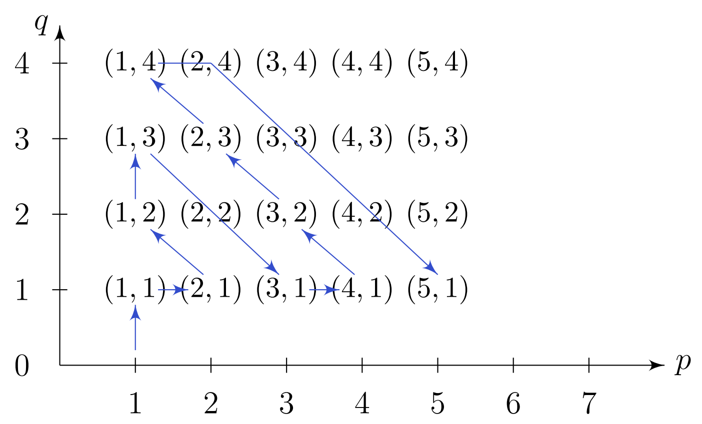
> **Exercise 2.18.** Show that f is indeed bijective.
> Show similarly than A, B countable $\implies A \cup B$ countable.
> Show similarly that A finite, B countable $\implies A \cup B$ countable ("Hilbert's infinite hotel").

Remarkably, $\mathbb{Q}$ is also countable.

Theorem 2.19

$\mathbb{Q}$ is countable.

Proof. First let's show $\mathbb{Q}_{>0}$ is countable. The usual proof is a bit sketchy (but informative!): arrange the pairs $(p, q) \in \mathbb{N}^2$ in a square:



Now list the pairs according to the path shown, *missing out pairs which aren't in lowest terms*. It's impossible to write down an explicit formula.

A slicker proof is to define the injection

$$f\colon \mathbb{Q}_{>0} \longrightarrow \mathbb{N}, \qquad f(m/n) \ := \ 2^m 3^n$$

where $m, n \geq 1$ and m/n is in lowest terms. (MATH40001 Exercise: give a careful proof this is an injection. What if we'd used $f(m/n) = 2^m(2n-1)$?) So f defines a bijection between $\mathbb{Q}_{>0}$ and an infinite subset of $\mathbb{N}$, which in turn is countable by Proposition 2.16. Therefore $\mathbb{Q}_{>0}$ is also countable, giving a bijection $F\colon \mathbb{N} \to \mathbb{Q}_{>0}$.

To finish off we define $g\colon \mathbb{N} \longrightarrow \mathbb{Q}$ by

$$g(1) := 0 \quad \text{and} \quad \begin{cases} g(2k) & := F(k), \\ g(2k+1) & := -F(k). \end{cases}$$

That is, if $q_1, q_2, \dots$ ($q_i := F(i)$) is our list of elements of $\mathbb{Q}_{>0}$ then our new list is $0, q_1, -q_1, q_2, -q_2, \dots$. □

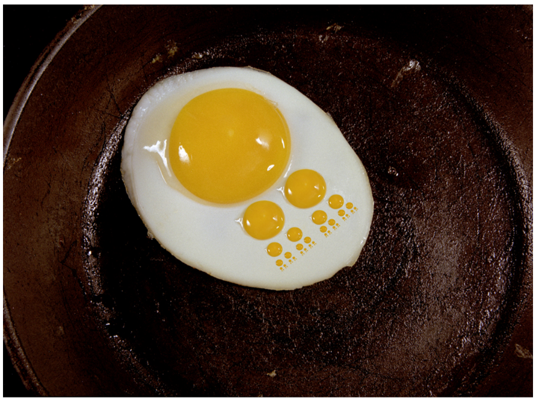
Exercise 2.20. We showed that we can list the positive rational numbers $\mathbb{Q}_{>0} = \{q_1, q_2, \dots\}$. Show this cannot be done in order of size, i.e. with $q_1 < q_2 < \dots$.

Next we see that $\mathbb{R}$ is genuinely bigger than $\mathbb{Q}$.

Theorem 2.21

$\mathbb{R}$ is uncountable.

Proof. (Cantor's Diagonal Argument)



We suppose for a contradiction that you can "list" all the real numbers. We write this as follows, using decimal expansions with no $\overline{9}$s:

$$\begin{aligned} x_1 &= a_1.a_{11}\,a_{12}\,a_{13}\,a_{14}\dots \\ x_2 &= a_2.a_{21}\,a_{22}\,a_{23}\,a_{24}\dots \\ x_3 &= a_3.a_{31}\,a_{32}\,a_{33}\,a_{34}\dots \\ x_4 &= a_4.a_{41}\,a_{42}\,a_{43}\,a_{44}\dots \\ &\vdots \\ x_m &= a_m.a_{m1}\,a_{m2}\,a_{m3}\,a_{m4}\dots \\ &\vdots \end{aligned}$$

As usual $a_1, a_2, a_3 \in \mathbb{Z}$ and $a_{11}, a_{12}, \ldots, a_{ij} \in \{0, 1, 2, \ldots, 9\}$.

Now we can produce a real number $x := 0.b_1\, b_2 \ldots b_n \ldots$ not on the list:

1. Pick $b_1 \in \{0, 1, \ldots, 8\}$ such that $b_1 \neq a_{11}$,
2. Pick $b_2 \in \{0, 1, \ldots, 8\}$ such that $b_2 \neq a_{22}$,

   $\vdots$

n. Pick $b_n \in \{0, 1, \ldots, 8\}$ such that $b_n \neq a_{nn}$

   $\vdots$

Since we don't allow 9 we don't end up with a decimal ending in $\overline{9}$.

Then $\forall i \geq 1$, we see $x \neq x_i$ because it differs in the ith decimal place x_{ii}. Therefore we have found an $x \in \mathbb{R}$ not on the list. □

There's a set of numbers in between $\mathbb{Q}$ and $\mathbb{R}$ called the *algebraic numbers*: those $x \in \mathbb{R}$ which satisfy a polynomial equation $p(x) = 0$, where p has integer coefficients. E.g. any rational number $x = p/q$ satisfies an equation $p(x) := qx - p = 0$. And $x = \sqrt[n]{m}$ satisfies $p(x) := x^n - m = 0$.

On the exercise sheet you'll prove that the set of algebraic numbers is also *countable*. Therefore (by Exercise 2.18) the set of *transcendental numbers* – those which are not algebraic – is uncountable. It turns out that e and π are transcendental.

2.4 The Completeness Axiom

Exercise 2.22. Show if a subset $S \subset \mathbb{R}$ has a maximum (i.e. an element $\max S \in S$ such that $\max S \geq s \ \forall s \in S$) then it is *unique*.

Show if $\max S$ exists then $-S := \{-s : s \in S\}$ has a minimum, $\min(-S) = -\max S$.

Exercise 2.23. What is the maximum of the interval $(0, 1)$?

1. 0
2. 0.5
3. $0.\overline{9}$.
4. 1
5. Something else.
6. More than one of these.
7. It has no maximum. ✓

Proof: if $x = \max(0, 1)$ then $x \in (0, 1) \implies x < 1$ so I claim $x' := \frac{1+x}{2}$ satisfies $x < x' < 1$, i.e. $x' \in (0, 1)$ is $> \max(0, 1)$ ※

Proof of claim: $x < 1 \implies x + x < 1 + x < 1 + 1 \implies x < \frac{1+x}{2} < 1$.

Clearly we think of 1 as being some kind of substitute for $\max(0, 1)$. We call it $\sup S$, the *supremum* of S.

Definition. $\emptyset \neq S \subset \mathbb{R}$ is *bounded above* if and only if

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in S, \ x \leq M.$$

Such an M is called an *upper bound* for S.

S is *bounded below* if and only if

$$\exists M \in \mathbb{R} \text{ such that } \forall x \in S, M \leq x.$$

Such an M is called a *lower bound*.

S is *bounded* if and only if S is *bounded above* and *below*.

So for instance $S = (0, 1)$ is bounded above by any $M \geq 1$.

Exercise 2.24. Show that S is bounded if and only if

$$\exists R > 0 \text{ such that } \forall x \in S,\ -R \le x \le R,$$

or equivalently

$$\exists R > 0 \text{ such that } \forall x \in S,\ |x| \le R.$$

Definition. Suppose $\emptyset \ne S \subset \mathbb{R}$ is bounded above. We say $x \in \mathbb{R}$ is a *least upper bound* of S or **supremum** of S if and only if

- x is an upper bound for S (i.e. $x \ge s\ \forall s \in S$), and
- $x \le y$ for any y which is an upper bound for S ($y \ge s\ \forall s \in S \implies x \le y$).

Remark 2.25. Once we are bounded above we can pick an upper bound, then we slide it leftwards until we first hit an element of S (so long as $S \ne \emptyset$).

It is common for people write things down $\sup S = +\infty$ if S is not bounded above and $\sup S = -\infty$ if S is empty.

We are being super careful in this class and will want $\sup S$ to be a real number if it exists, so we enforce that S is non-empty and bounded above in our definition.

Exercise 2.26. Prove such an x is *unique* if one exists. Therefore we can call it $\sup S$.

Exercise 2.27. Write down the definition *greatest lower bound* (or *infimum*) of S. We call this $\inf S$ when it exists.

Exercise 2.28. Suppose $\exists \sup S$. Then show that $\inf(-S)$ exists too, and equals

1. $\sup S$
2. $-\sup S$ ✓
3. $\inf S$
4. $-\inf S$
5. None of these

Proof: $\sup S$ is an upper bound for $S \implies \forall s \in S,\ \sup S \ge s \iff -\sup S \le -s$ so $-\sup S$ is a lower bound for $-S$.
If $-m > -\sup S$ is a greater lower bound then the same argument shows $m <$

$\sup S$ is a smaller upper bound for S ※

Example 2.29. $S = (0, 1)$. Then $T = \{y : y \text{ is an upper bound for } S\}$.

T has a minimum: $1 = \min T = \sup(0, 1)$. Similarly $0 = \inf(0, 1)$.

Exercise 2.30. $\sup S \in S \iff S$ has a maximum and $\max S = \sup S$.

Theorem 2.31: Completeness axiom of $\mathbb{R}$

Suppose that $S \subset \mathbb{R}$ is nonempty and bounded above. Then S has a supremum.

Remark 2.32. Recall "bounded above" rules out $\sup S = +\infty$, while "nonempty" rules out $\sup S = -\infty$.

Either we can work with this as an axiom (an act of faith) not worrying about whether anyone every created a set $\mathbb{R}$ satisfying both the completeness axiom and the previous Axioms 1-12. Or we can give a construction of $\mathbb{R}$ and show it has the property that any bounded above $\emptyset \neq S \subset \mathbb{R}$ has a supremum. Next we show our construction of $\mathbb{R}$ using decimal expansions has this property. (Later, in Section 2.5, we will give another construction using Dedekind cuts.)

Proof. Without loss of generality (replacing S by $S + a := \{s + a \colon s \in S\}$) we may assume $S \neq \emptyset$ has a positive element $0 \leq s \in S$. This will simplify things, because positive decimals behave better than negative decimals.

S is bounded above by $R \geq 0$, say. Set $N := \lceil R \rceil \in \mathbb{N}$. So we may replace S by $S \cap [0, N]$: both are nonempty with the same upper bounds (easy exercise), so one has a sup if and only if the other one does, and the two suprema are equal.

We will create the supremum $\sup S = a_0.a_1a_2 \ldots \geq 0$ digit by digit.

Leading integer. Write each $s \in S$ as a decimal $s_0.s_1s_2s_3 \ldots$ not ending in $\overline{9}$. Since $s \in [0, N]$ we see that $s_0 \in \{0, 1, \ldots, N\}$, a finite set. So the set of leading integers s_0 is finite, so has a maximum $a_0 \geq 0$.

First decimal place. So $S \cap [a_0, a_0 + 1)$ is nonempty and we may replace S by it (same easy exercise). All its elements are of the form $a_0.s_1s_2 \ldots$ with $s_1 \in \{0, 1, \ldots, 9\}$ – a finite set. Thus there is a maximum s_1 value; call it a_1.

Second decimal place. So can replace S by $S \cap \big[a_0.a_1,\, a_0.(a_1 + 1)\big)$. (If $a_1 = 9$ we mean $S \cap [a_0.9,\, a_0 + 1)$.) Every $s \in S$ has decimal expansion $a_0.a_1s_2s_3 \ldots$ with $s_2 \in \{0, 1, \ldots, 9\}$ – a finite set. Thus there is a maximum s_2 value; call it a_2.

nth decimal place. Assume I've defined $a_0, \ldots, a_{n-1}$ and shown that

$$S \cap \big[a_0.a_1 \ldots a_{n-1},\ a_0.a_1 \ldots (a_{n-1}+1)\big)$$

is nonempty and has the same upper bounds as the original S. Any element is $s = a_0.a_1 \ldots a_{n-1} s_n s_{n+1} \ldots$ with $s_n \in \{0, 1, \ldots, 9\}$ – a finite set. Thus there is a maximum s_n value; call it a_n.

Upper bound. We claim $a := a_0.a_1a_2 \ldots$ is an upper bound for S. By the construction of a_0, every element of $s_0.s_1s_2 \ldots$ of S has either

- $s_0 < a_0$ ($\Longrightarrow s < a$ and we're done), or
- $s_0 = a_0$.

In the second case, by the construction of a_1, either

- $s_1 < a_1$ ($\Longrightarrow s < a$ and we're done), or
- $s_1 = a_1$.

In the second case, by the construction of $a_2 \ldots$

Ok you get the idea. Either this process terminates ($\Longrightarrow s < a$) or it doesn't ($\Longrightarrow s = a$). Either way $s \leq a$ so a is an upper bound.

Least upper bound. If $b < a$ is a smaller upper bound for S, suppose their decimal expansions first differ in the nth place, i.e.

$$b \ = \ a_0.a_1a_2 \ldots a_{n-1} b_n \ldots \qquad \text{with } b_n < a_n.$$

But remember the construction of a_n above: there was an element $s \in S$ with $s = a_0.a_1a_2 \ldots a_{n-1}a_n \ldots$ so $s > b$ ※ So a is the *least* upper bound of S.

Finally note $a_0.a_1a_2 \ldots$ could end in $\overline{9}$ (in fact check it will indeed do so if the set is $(0,1)$. So to consider it as a real number according to our definition we should round up the 9s. □

> **Exercise 2.33.** Apply Theorem 2.31 to $-S$ to deduce if $\emptyset \neq S \subset \mathbb{R}$ is bounded below then S has an inf.

The completeness axiom means $\mathbb{R} \supset \mathbb{Q}$ fills in all the "holes". For example:

Proposition 2.34. *There exists $0 < x \in \mathbb{R}$ such that $x^2 = 3$. We call $x =: \sqrt{3}$.*

Proof. Since this is not true in $\mathbb{Q}$ we'd better use the completeness axiom! So we need a set $S \subset \mathbb{R}$ to apply it to. Set

$$S \ := \ \big\{0 < a \in \mathbb{R} \ : \ a^2 < 3\big\},$$

then we'd like to set $x := \sup S$ so we must check

- S is nonempty (easy: $1 \in S$), and
- S is bounded above (by 2, say): if $a \geq 2$ then $a^2 \geq 4$ so $a \notin S$.

To prove $x^2 = 3$ we need to show $x^2 \not< 3$ and $x^2 \not> 3$ by the trichotomy axiom.

• If $x^2 < 3$ then we *expect* $(x+\epsilon)^2 < 3$ for sufficiently small $\epsilon > 0$. (This would give a contradiction: $S \ni x + \epsilon > x = \sup S$.) So let's compute

$$(x+\epsilon)^2 \ = \ x^2 + \epsilon(2x+\epsilon) \ \leq \ x^2 + \epsilon(2 \times 2 + 1) \ = \ x^2 + 5\epsilon \ < \ 3$$

if $\epsilon \leq 1$ and $5\epsilon < 3 - x^2$. Therefore, if we set

$$\epsilon \ := \ \min\left(1, \frac{3-x^2}{10}\right),$$

then $\epsilon > 0$ (because $x^2 < 3$) and $(x+\epsilon)^2 < 3$. Thus $(x+\epsilon) \in S$ is bigger than $x = \sup S$ ※

• If $x^2 > 3$ then we *expect* $(x-\epsilon)^2 \geq 3$ for *all* sufficiently small $\epsilon > 0$. (Then $x - \epsilon$ would be an upper bound for S smaller than $\sup S$.) So let's compute

$$(x-\epsilon)^2 \ = \ x^2 - 2\epsilon x + \epsilon^2 \ \geq \ x^2 - 4\epsilon \ \geq \ 3$$

if $\epsilon \geq 0$ and $4\epsilon \leq x^2 - 3$ (where we have used $x \leq 2$ again). So if we set

$$\epsilon_0 \ := \ \frac{x^2-3}{4}$$

then $(x-\epsilon)^2 > 3$ for all $0 \leq \epsilon \leq \epsilon_0$. Therefore $\forall y \in [x-\epsilon_0, x]$ we have $y^2 \geq 3 \Longrightarrow y \notin S$. And $\forall y \in (x, \infty)$, $y \notin S$ (because $y > x = \sup S$). So $x - \epsilon_0$ is an upper bound for S, less than $x = \sup S$ ※ □

Once it's all written out, the proof looks like a blur of symbols and formulae. Reading it will do you little good. Trying to "learn" or remember it will do you even less good. You must come up with your own proof. When you do, the formulae will all make perfect sense to you.

Go home, close your lecture notes, and write out your own proof. Only look at my proof if you're stuck and you've struggled for more than 10 minutes.

You shouldn't be thinking about exams, but if you are, this is a typical type of proof you'll have to give in exams. But the exam will ask for a proof you've not seen before, so practise coming up with your own proofs.

Exercise 2.35. Show $\sqrt[3]{2}$ exists.

See Question sheet 2 for nth roots and even rational qth powers of positive real numbers. Next term, once you're experts on continuity, it will be easy to create xth powers of all positive real numbers $\forall x \in \mathbb{R}$.

Exercise 2.36. A student is trying to prove there exists $0 < x \in \mathbb{R}$ such that $x^2 = 2$. Since

$$S := \left\{0 < a \in \mathbb{R} : a^2 < 2\right\},$$

is nonempty ($1 \in S$) and bounded above by 2 (if $a \geq 2$ then $a^2 \geq 4$ so $a \notin S$) he sets $x := \sup S > 0$.
Next he gives this proof that $x^2 \not< 2$. Is any step wrong?

1. Suppose $x^2 < 2$, then we try to find $\epsilon > 0$ such that $(x+\epsilon)^2 < 2$.
2. Note $2 > (x+\epsilon)^2 = x^2 + 2\epsilon x + \epsilon^2 > x^2 + 2\epsilon x$ implies that $\frac{2-x^2}{2x} > \epsilon$.
3. So if we take $0 < \epsilon < \frac{2-x^2}{2x}$ then $(x+\epsilon)^2 < 2$. ✓
4. So $x + \epsilon \in S$ but $x + \epsilon > x = \sup S$ ※
5. Nothing wrong, full marks for the Buzzard.

(2) says $2 > (x+\epsilon)^2 \implies \epsilon < \frac{2-x^2}{2x}$ (correct) whereas
(3) says $2 > (x+\epsilon)^2 \impliedby \epsilon < \frac{2-x^2}{2x}$.

So (3) does not follow from (2)! and is in fact incorrect.

Though 2. is correct, it is not useful – we need an implication in the opposite direction! Notice in 2. we used $(x+\epsilon)^2 > x^2 + 2\epsilon x$, which is useless, we want $(x+\epsilon)^2$ to be LESS than something < 2, not more than something.

So instead say something like $(x+\epsilon)^2 = x^2 + 2\epsilon x + \epsilon^2 < x^2 + 2\epsilon \times 2 + \epsilon$ so long as $\epsilon \leq 1$ (because we know $x < 2$ because 1.5 is clearly an upper bound for S).

So if $\epsilon \in (0, 1]$ then $(x+\epsilon)^2 < x^2 + 5\epsilon \leq 2$ if $\epsilon \leq \frac{2-x^2}{5}$.

So if we set $\epsilon := \min\left\{1, \frac{2-x^2}{5}\right\}$ then we have proved that $(x+\epsilon)^2 < 2$ which means $x + \epsilon \in S$ but $x + \epsilon > x = \sup S$ ※

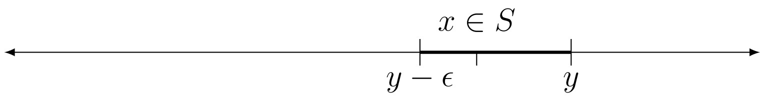
A key point here is that "if" means $\impliedby$!

Exercise 2.37. Let $S = \{x \in \mathbb{Z} : x^2 < 3\}$. Then S is nonempty and bounded above. What is $\sup S$?

1. 0
2. 1 ✓

> 3. 2
> 4. 3
> 5. $\sqrt{3}$
> 6. Something else
>
> $S = \{-1, 0, 1\}$ so $\sup S = \max S = 1$.

Proposition 2.38. *Suppose $\emptyset \neq S \subset \mathbb{R}$ and y is an upper bound for S. Then $y = \sup S \iff \forall \epsilon > 0\ \exists s \in S\colon s > y - \epsilon$.*



Think of $\epsilon > 0$ as *small*, then $y - \epsilon$ is a little bit smaller than y and the condition is just saying that $y - \epsilon$ is not an upper bound for S.

Proof. $\Longrightarrow$ direction: $y - \epsilon$ is less then the *least* upper bound y, so it is not an upper bound.
This means $\exists s \in S\colon s > y - \epsilon$.

$\Longleftarrow$ direction: given any $x < y$, set $\epsilon = y - x > 0$. Then $\exists s \in S\colon s > y - \epsilon = x$. So x is not an upper bound.
So y is the least upper bound. □

As an aside we note that the completeness axiom implies the Archimedean axiom: that if we fix $x \in \mathbb{R}$ then $\exists N \in \mathbb{N}$ such that $N \geq x$.

Suppose for a contradiction that no such N exists, i.e. $\mathbb{N}$ is bounded above by x. Since it is also nonempty, $\exists \sup \mathbb{N} =: y$. But $n \in \mathbb{N} \Longrightarrow (n+1) \in \mathbb{N}$, so

$$\forall n \in \mathbb{N},\ y \ \geq\ n+1 \iff y - 1 \ \geq\ n,$$

so $y - 1$ is a smaller upper bound for $\mathbb{N}$ ※

2.5 Alternative approach: Dedekind cuts

Intuition. Suppose we have a construction of $\mathbb{R}$, e.g. by decimals. Then to every real number $r \in \mathbb{R}$ we can associate a semi-infinite subset S_r of $\mathbb{Q}$,

$$S_r \ := \ (-\infty, r) \cap \mathbb{Q}.$$

That is S_r is the set of *all rational numbers* $< r$.

Exercise 2.39. Show $r_1 = r_2 \iff S_{r_1} = S_{r_2}$.
(Obviously $\Leftarrow$ is the important one!)

Slightly abusing the original notation, we will call the $S_r \subset \mathbb{Q}$ *Dedekind cuts*. The fantastic thing is, we only need to know about $\mathbb{Q}$ to define them. So now we forget all about $\mathbb{R}$ and pretend we only know about $\mathbb{Q}$.

Definition. We say a nonempty subset $S \subset \mathbb{Q}$ is a *Dedekind cut* if it satisfies (i) and (ii) below.

(i) If $s \in S$ and $s > t \in \mathbb{Q}$ then $t \in S$ (S is a semi-infinite interval to the left).

(ii) S is bounded above but has no maximum.

So using only this notion from $\mathbb{Q}$ we can *construct* $\mathbb{R}$, showing that $\mathbb{Q}$ "knows" about its "completion" $\mathbb{R}$.

Definition.

$$\mathbb{R} := \big\{\text{Dedekind cuts } S \subset \mathbb{Q}\big\}.$$

(I.e. we think of identifying S_r with $r \in \mathbb{R}$.)

Exercise 2.40. Check that we can identify $\mathbb{Q} \subset \mathbb{R}$ by taking $q \in \mathbb{Q}$ to the Dedekind cut $S_q := \{s \in \mathbb{Q} : s < q\}$.

We can generalise all the usual arithmetic operations that we already have on $\mathbb{Q}$ to our newly constructed $\mathbb{R}$; eg if $S \subset \mathbb{Q}$ and $T \subset \mathbb{Q}$ are Dedekind cuts, we define

$$S + T := \{s + t \colon s \in S,\ t \in T\} \subset \mathbb{Q}.$$

Exercise 2.41. Check this is a Dedekind cut (an element of $\mathbb{R}$!) and gives the usual $+$ on $\mathbb{Q}$: i.e. $S_{q_1} + S_{q_2} = S_{q_1+q_2}$.

Similarly we can define $<$ on $\mathbb{R}$ to be just $\subsetneq$ on Dedekind cuts:

$$S < T \iff S \subsetneq T.$$

Exercise 2.42. Show a set of real numbers $A \subset \mathbb{R}$ is bounded above iff A is a set of Dedekind cuts S all contained in some fixed interval $(-\infty, N)$ for some $N \in \mathbb{N}$.

So now proving Theorem 2.31 (i.e. verifying the completeness axiom) becomes rather easy:

Exercise 2.43. If A is a bounded above nonempty set of Dedekind cuts, define

$$\sup A \;:=\; \bigcup_{S\in A} S \;\subset\; \mathbb{Q}.$$

Show this is also a Dedekind cut (i.e. a real number!) and check it is the least upper bound of A.

For more details of this construction see for example **W. Rudin**, *"Principles of mathematical analysis"*, or the webpage http:\tinyurl.com/yjt5olv

2.6 Triangle inequalities

Theorem 2.44

For all $a, b \in \mathbb{R}$ we have

$$|a+b| \;\le\; |a|+|b|.$$

Proof. If $|a+b| > |a|+|b|$ then applying order Axiom 11 several times,

$$|a+b|^2 \;>\; (|a|+|b|)|a+b| \;>\; (|a|+|b|)^2 \;=\; |a|^2+2|a||b|+|b|^2.$$

But this contradicts

$$|a+b|^2 \;=\; (a+b)^2 \;=\; a^2+2ab+b^2 \;\le\; |a|^2+2|a||b|+|b|^2.$$ □

Exercise 2.45. Why is this called the triangle inequality?
Give a direct proof *without squaring* by first proving $x \le |x|$ by splitting into two cases.

There are many more which you will prove on Problem Sheet 2.

(a) $|x+y| \;\le\; |x|+|y|$
(b) $|x+y| \;\ge\; |x|-|y|$
(c) $|x+y| \;\ge\; |y|-|x|$
(d) $|x-y| \;\ge\; \big||x|-|y|\big|$
(e) $|x| \;\le\; |y|+|x-y|$
(f) $|x| \;\ge\; |y|-|x-y|$
(g) $|x-y| \;\le\; |x-z|+|y-z|$

Don't try to memorise them! Understand them, and then work each one out as you need it. They're all intuitively obvious if thought about in the right way.

E.g. (d) says that if x is close to y then $|x|$ is close to $|y|$ (possibly closer – if x, y have different signs).

E.g. (e) says the distance in $\mathbb{R}$ from 0 to x is no more than the distance if we go $0 \to y \to x$ via y.

E.g. (g) says the distance from x to y is no more than the distance if we go $x \to z \to y$ via z.

For instance, let's prove (g) from (a):

$$|x-y| \;=\; |(x-z)-(y-z)| \;\leq\; |x-z|+|-(y-z)|.$$

They are a crucial component of the rest of this course, as there's many things close to each other where we need to estimate distance between things.

Here are some different ways of saying a, b are close to each other; get used to them.

$$\begin{aligned} |a-b| \;<\; \epsilon &\iff b-\epsilon < a < b+\epsilon \\ &\iff a-\epsilon < b < a+\epsilon \\ &\iff a \in (b-\epsilon, b+\epsilon) \\ &\iff b \in (a-\epsilon, a+\epsilon) \\ &\implies \big||a|-|b|\big| < \epsilon. \end{aligned}$$

Exercise 2.46. Fix $a \in \mathbb{R}$. What does the statement

$$\forall \epsilon > 0,\; |x-a| < \epsilon \tag{$*$}$$

mean for the number x? I.e. which of the following is it equivalent to?

1. x is close to a
2. $x \in (a-\epsilon, a+\epsilon)$
3. $x = a$ ✓
4. $x = a+\epsilon$
5. $x = a-\epsilon$
6. More than one of these
7. None of these

Assume $x \neq a$. Take $\epsilon := \frac{1}{2}|x-a| > 0$. Then $(*)$ does not hold ※

3 Sequences

If you really understood this course, you won't need to do any revision.
Just relax and have a beer.

- Alessio Corti, 2014

A sequence $(a_n)_{n\geq 1}$ of real (or complex, etc.) numbers is an infinite list of numbers $a_1, a_2, a_3, \ldots$ all in $\mathbb{R}$ (or $\mathbb{C}$, etc.) Formally:

Definition. A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$.

Notation: We let $a_n \in \mathbb{R}$ denote $a(n)$ for $n \in \mathbb{N}$. The data $(a_n)_{n=1,2,\ldots}$ is equivalent to the function $a : \mathbb{N} \to \mathbb{R}$ because a function a is determined by its values a_n over all $n \in \mathbb{N}$.
We will denote a by $a_1, a_2, a_3, \ldots$ or $(a_n)_{n\in\mathbb{N}}$ or $(a_n)_{n\geq 1}$ or even just (a_n).

Remark 3.1. a_is could be repeated, so (a_n) is *not* equivalent to the set $\{a_n : n \in \mathbb{N}\} \subset \mathbb{R}$. E.g. $(a_n) = 1, 0, 1, 0, \ldots$ is different from $(b_n) = 1, 0, 0, 1, 0, 0, 1, \ldots$ This is why we use round brackets ( ) instead of { }.

We can describe a sequence in many ways,

- As a **list** $1, 0, 1, 0, \ldots,$
- Via a **closed formula**, like $a_n = \frac{1-(-1)^n}{2}$ for the sequence above,
- By a **recursion**, e.g. the Fibonacci sequence $F_1 = 1 = F_2$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ (so (F_n) is $1, 1, 2, 3, 5, 8, 13, \ldots$)
- By a summation, e.g. $a_n = \sum_{i=1}^n \frac{1}{i} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$. Such a sequence $a_n = \sum_{i=1}^n b_n$ is called a **series** and will be studied later in the course.

Exercise 3.2. Show any sequence (a_n) can be written as a series $a_n = \sum_{i=1}^n b_i$ for an appropriate choice of sequence (b_n).

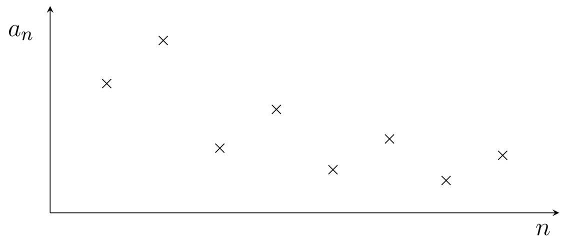
3.1 Convergence of Sequences

We want to *rigorously* define $a_n \to a \in \mathbb{R}$, or "a_n converges to a as $n \to \infty$" or "a is the limit of (a_n)". We will spend a while exploring various formulations before we choose our definitive definition.

Idea 1: a_n should get closer and closer to a. Not necessarily monotonically, e.g.

for:

$$a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{1}{2n} & n \text{ even} \end{cases} \qquad \text{we want } a_n \to 0.$$



Idea 2: But notice that $\frac{1}{n}$ also gets closer and closer to -73.6! So we want to say instead that a_n gets "*as close as we like to a*" or "*arbitrarily close to a*". We will measure this with $\epsilon > 0$: we say a_n gets to within ϵ of a by

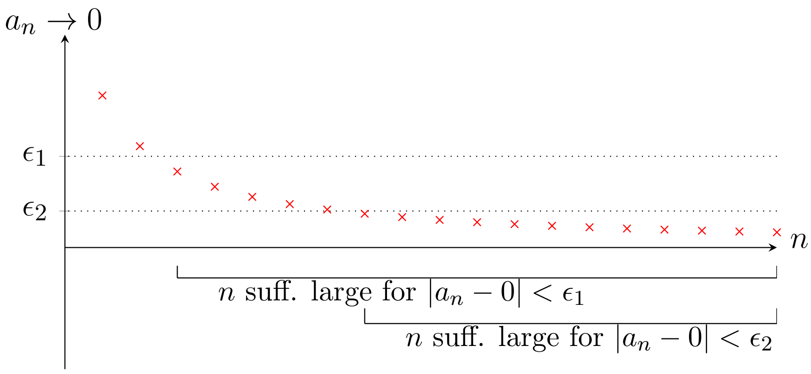
$$|a_n - a| < \epsilon \qquad \text{or} \qquad a_n \in (a - \epsilon, a + \epsilon).$$

We phrase "a_n gets *arbitrarily* close to a" by "*a_n gets to within ϵ of a for* ***any*** $\epsilon > 0$". This suggests the following definition.

> **Exercise 3.3.** Prof Buzzard tries to define $a_n \to a$ if and only if $\forall n$ sufficiently large, $|a_n - a|$ is *arbitrarily small.* When pushed he defines a real number $b \in \mathbb{R}$ to be arbitrarily small if it is smaller than any $\epsilon > 0$ i.e. $\forall \epsilon > 0,\ |b| < \epsilon$.
>
> Leaving aside what he means by "sufficiently large" for now, which of these sequences converges (to some $a \in \mathbb{R}$) according to his definition?
>
> 1. 0, 1, 0, 1, . . .
> 2. 1, 1, 1, 1, . . . ✓
> 3. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . .
> 4. $a_n = 2^{-n}$
> 5. More than one of these
> 6. None of these
>
> Notice his definition of b being "arbitrarily small" means $b = 0$. (Proof: if $b \neq 0$ then $\exists\, \epsilon := \frac{|b|}{2} > 0$ such that $|b| \not< \epsilon$ so b is not arbitrarily small.)
>
> So for the Buzzard, $a_n \to a$ if and only if $a_n = a$ for all n sufficiently large.

Idea 3: Prof B said that once n is large enough, $|a_n - a|$ is less than every $\epsilon > 0$, but that means it's zero, i.e. $a_n = a$. The problem he missed is that if we take smaller ϵ we will usually have to take bigger n to make $|a_n - a| < \epsilon$.

So we want to say that to get *arbitrarily close to the limit a* (i.e. $|a_n - a| < \epsilon$), we need to go sufficiently far down the sequence. If I change $\epsilon > 0$ to be smaller, I simply go further down the sequence to get within ϵ of a.



Don't be a Buzzard – there will not be a "n sufficiently large" that works for all ϵ at once! (Unless $a_n \equiv a$ eventually.)

That is, we want to *reverse* the order of specifying n and ϵ: only once we've seen how small ϵ is do we know how big to take n. If we chose a smaller ϵ we can then choose a larger n.

For *any* (fixed) $\epsilon > 0$ we want there to be an n sufficiently large such that $|a_n - a| < \epsilon$. So we change "$\exists n$ such that $\forall \epsilon > 0$" to "$\forall \epsilon > 0,\ \exists n$". *This allows n to depend on ϵ.*

Exercise 3.4. Professor Buzzard takes your point, and modifies his definition of $a_n \to a$ to

$$\forall \epsilon > 0\ \exists n \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon.$$

Which of these sequences converges to $a = 0$ according to his new definition?

1. 0, 1, 0, 1, . . . ✓
2. 1, 1, 1, 1, . . .
3. 1, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, . . . ✓
4. $a_n = 2^{-n}$ ✓
5. More than one of these ✓
6. None of these

Sequences 1, 3 and 4 all converge to 0 according to this definition, but we really don't want 1 to converge. We do want $|a_n - a| < \epsilon$ eventually, but we also want it to *stay there!*

Idea 4: So we measure "*eventually*" (or "sufficiently large") by a point $N \in \mathbb{N}$ beyond which ("$\forall n \geq N$") a_n **stays** within ϵ of a. That is

Definition (Convergence)

We say that $a_n \to a$ as $n \to \infty$ if and only if

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_n - a| < \epsilon.$$

Read this as follows:

However close ($\forall \epsilon > 0$) I want to get to the limit a, there's a point in the sequence ($\exists N \in \mathbb{N}$) beyond which ($n \geq N$) *all* a_n are indeed that close to the limit a ($|a_n - a| < \epsilon$).

Remark 3.5. N depends on ϵ! For a while we will sometimes denote it N_ϵ, as a reminder. We often write ($a_n \to a$ as $n \to \infty$) as just ($a_n \to a$) or ($\lim_{n\to\infty} a_n = a$).

Equivalently:

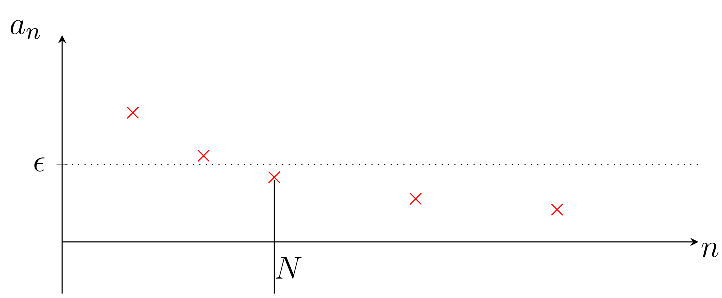
$$\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N} \text{ such that } \left[n \geq N_\epsilon \Longrightarrow |a_n - a| < \epsilon\right]$$

or equivalently

$$\forall \epsilon > 0,\ \exists N_\epsilon \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon\ \ \forall n \geq N_\epsilon.$$

Example 3.6. Prove $\frac{1}{n} \to 0$ as $n \to \infty$.

Rough working: Fix $\epsilon > 0$. I want to find $N_\epsilon \in \mathbb{N}$ such that $|a_n - a| = |\frac{1}{n} - 0| = \frac{1}{n} < \epsilon$ for all $n \geq N_\epsilon$.



Since this is equivalent to $n > \epsilon^{-1}$ then it is enough to take $N_\epsilon > \epsilon^{-1}$, which we know exists by the Archimedean axiom (e.g. $N_\epsilon = \lfloor \epsilon^{-1} \rfloor + 1$). So now the formal proof runs as follows:

Proof. Fix $\epsilon > 0$. Pick any $N_\epsilon \in \mathbb{N}$ such that $N_\epsilon > \frac{1}{\epsilon}$. Then $n \geq N_\epsilon \implies |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N_\epsilon} < \epsilon$. □

How to prove $a_n \to a$

$$\boxed{\forall \epsilon > 0\ \exists N_\epsilon \in \mathbb{N} \text{ such that } |a_n - a| < \epsilon\ \forall n \geq N_\epsilon}$$

(I) Fix $\epsilon > 0$.

(II) Calculate $|a_n - a|$.

(II′) Find a good estimate $|a_n - a| \leq b_n$.

(III) Try to solve $b_n < \epsilon$. $(*)$

(IV) Find $N_\epsilon \in \mathbb{N}$ such that $(*)$ holds whenever $n \geq N_\epsilon$.

(V) Put everything together into a logical proof (usually involves rewriting everything in reverse order - see examples below).

Notice you only have to do this for **one** $\epsilon > 0$, so long as it is arbitrary; that way you've done it for **any** $\epsilon > 0$.

The key point is to choose b_n so that solving $b_n < \epsilon$ is easier than solving $|a_n - a| < \epsilon$.

Example 3.7. Prove that $a_n = \dfrac{n+5}{n+1} \to 1$.

Rough working:

$$|a_n - 1| = \left|\frac{n+5}{n+1} - 1\right| = \frac{4}{n+1} < \frac{4}{n}.$$

This is $< \epsilon \iff n > 4/\epsilon$, so take $N \geq 4/\epsilon$.

Proof. Fix $\epsilon > 0$. Pick N such that $N \geq 4/\epsilon$. Then $\forall n \geq N$,

$$|a_n - 1| = \frac{4}{n+1} \leq \frac{4}{N+1} < \frac{4}{N} \leq \epsilon.$$ □

Example 3.8. Prove that $a_n = \dfrac{n+2}{|n-2|} \to 1$.

Rough working: We assume $n > 2$ so we can drop the absolute value, this is okay since we can always choose $N_\epsilon > 2$. We have

$$|a_n - 1| = \left|\frac{n+2}{n-2} - 1\right| = \frac{4}{n-2}.$$

We *want* $\frac{4}{n-2} < \epsilon$, so we want implications in the $\Longleftarrow$ direction
(i.e. $\frac{4}{n-2} < \epsilon \Longleftarrow n \geq N$)
not the $\Longrightarrow$ direction
(i.e. the fact that $\frac{4}{n-2} < \epsilon \Longrightarrow \frac{4}{n} < \epsilon$ *is of no use to us*).
[Notice the importance of the direction of implications!]

So we need something *bigger* than $\frac{4}{n-2}$, i.e. an estimate $\frac{4}{n-2} < b_n$ for which it is easier to solve $b_n < \epsilon$. So we make the denominator *smaller*.

To make $n - 2$ smaller, make 2 bigger! e.g. $2 < \frac{n}{2}$ for $n > 4$. Then $\frac{4}{n-2} < \frac{4}{n-n/2} = \frac{8}{n}$.

As well as $n > 4$ we also want $b_n = \frac{8}{n} < \epsilon \Longleftrightarrow n > \frac{8}{\epsilon}$. So take $N_\epsilon > \max(4, 8/\epsilon)$.
(Notice using $2 < n$ here would ruined everything.)

Proof. Fix $\epsilon > 0$. Choose $N_\epsilon \in \mathbb{N}$ such that $N_\epsilon > \max(4, 8/\epsilon)$. Then $n \geq N_\epsilon \Longrightarrow n > \frac{8}{\epsilon}$ $(*)$ and $n > 4$ $(\dagger)$

$$\Longrightarrow \left|\frac{n+2}{n-2} - 1\right| = \frac{4}{n-2} \overset{(\dagger)}{<} \frac{4}{n-n/2} = \frac{8}{n} \overset{(*)}{<} \epsilon. \qquad \square$$

Definition. We say that a_n *converges* if and only if $\exists a \in \mathbb{R}$ such that $a_n \to a$, i.e.

$$\exists a \text{ such that } \forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } n \geq N \Longrightarrow |a_n - a| < \epsilon.$$

Negating the above statement gives the following

Definition. We say a_n *diverges* if and only if it does not converge (to any $a \in \mathbb{R}$), i.e.

$$\forall a\ \exists \epsilon > 0 \text{ such that } \forall N \in \mathbb{N},\ \exists n \geq N \text{ such that } |a_n - a| \geq \epsilon.$$

Remark 3.9. Notice *diverge* does not mean $\to \pm\infty$, for instance we will prove later that $a_n = (-1)^n$ diverges.

Exercise 3.10. Fix a sequence of real numbers $(a_n)_{n\geq 1}$. Consider

$$\boxed{\forall n \geq 1\ \exists \epsilon > 0 \text{ such that } |a_n| < \epsilon}$$

This means?

1. $a_n \to 0$
2. $(a_n)_{n\geq 1}$ is bounded
3. Precisely nothing ✓
4. More than one of these
5. None of these

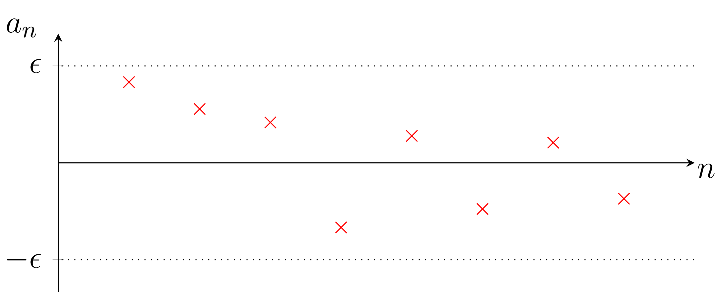
Proof. Fix any $n \in \mathbb{N}$. Take $\epsilon = |a_n| + 1$. □

Order of $\exists$, $\forall$ very important!

Exercise 3.11. What about

$$\boxed{\exists \epsilon > 0 \text{ such that } \forall n \geq 1,\ |a_n| < \epsilon}?$$

1. $a_n \to 0$
2. $(a_n)_{n\geq 1}$ is bounded ✓
3. Precisely nothing
4. More than one of these
5. None of these



It says $a_n \in (-\epsilon, \epsilon)\ \forall n \iff |a_n|$ is bounded by ϵ.

We can also define limits for *complex sequences*. Let $|z| := \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$.

Definition. $a_n \in \mathbb{C}$, $\forall n \geq 1$. We say $a_n \to a \in \mathbb{C}$ if and only if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \epsilon.$$

This definition is equivalent to $(\operatorname{Re} a_n) \to \operatorname{Re} a$ *and* $(\operatorname{Im} a_n) \to \operatorname{Im} a$ (see problem sheet 4!).

Example 3.12. Prove $a_n = \dfrac{e^{in}}{n^3 - n^2 - 6} \to 0$ as $n \to \infty$.

Rough working:

$$|a_n - 0| = \left|\frac{e^{in}}{n^3 - n^2 - 6}\right| = \left|\frac{1}{n^3 - n^2 - 6}\right|$$

which we would like to be $< b_n = \frac{1}{c_n}$ for some more manageable c_n smaller than $n^3 - n^2 - 6$, but not too small! (I.e. we still want $c_n \to \infty$ so $b_n \to 0$.) So let $c_n = n^3 - (\text{something bigger than}\, n^2 + 6)$.

We use $\frac{n^3}{2}$ to make the c_n simple. For $n \geq 4$, we have $\frac{n^3}{2} > n^2 + 6$. So for $n \geq 4$

$$\left|\frac{1}{n^3 - n^2 - 6}\right| < \frac{1}{n^3 - n^3/2} = \frac{2}{n^3} \leq \frac{2}{n},$$

which is $< \epsilon$ for $n > \frac{2}{\epsilon}$.

Proof. $\forall \epsilon > 0$ choose $N \geq \max\big(4, 2/\epsilon\big)$. Then $\forall n \geq N$,

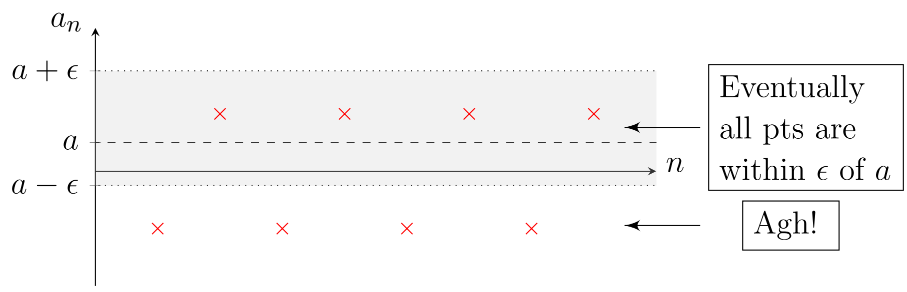
$$|a_n - 0| = \left|\frac{1}{n^3 - n^2 - 6}\right| < \frac{1}{n^3 - n^3/2} = \frac{2}{n^3} \leq \frac{2}{N^3} \leq \frac{2}{N} \leq \epsilon. \qquad \square$$

Example 3.13. Set $\delta = 10^{-1000000}$, $a_n = (-1)^n \delta$. Prove that a_n diverges, that is it does not converge (to any $a \in \mathbb{R}$).

Assume for contradiction that $a_n \to a$, i.e.

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \epsilon.$$

Rough working: Draw a picture! But don't make δ small in your picture, as then you won't see the contradiction. Magnify it to be big.



For small enough $\epsilon > 0$ (the picture shows that any $\epsilon \leq \delta$ will do), the fact that a is within ϵ of δ (a_{2n}) and $-\delta$ (a_{2n+1}) will be a contradiction.

Proof 1. Fix $a \in \mathbb{R}$. Take $\epsilon = \delta$.

Then if $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \epsilon$ this implies

1. $|a_{2N} - a| < \epsilon \iff a \in (\delta - \epsilon, \delta + \epsilon) \implies a > \delta - \epsilon = 0$, and
2. $|a_{2N+1} - a| < \epsilon \iff a \in (-\delta - \epsilon, -\delta + \epsilon) \implies a < -\delta + \epsilon = 0$ ※

So $a_n \not\to a$, but this holds $\forall a \in \mathbb{R}$, so a_n does not converge.

Or, *Proof 2*: Both $\pm\delta$ close to the limit a so must be close to each other by the triangle inequality:

$$|\delta - (-\delta)| \ \leq \ |\delta - a| + |a - (-\delta)| \ < \ \epsilon + \epsilon \ \implies \ 2\delta < 2\epsilon = 2\delta \ ※$$

So $a_n \not\to a$, but this holds $\forall a \in \mathbb{R}$, so a_n does not converge. $\square$

An alternative approach to that question is provided by the following.

Theorem 3.14: Uniqueness of Limits

Limits are unique. If $a_n \to a$ and $a_n \to b$, then $a = b$.

Idea: For n large, a_n is arbitrarily close to both a and b. So a arbitrarily close to $b \implies a = b$.

Proof 1.

1. $\forall \epsilon\ \exists N_a$ such that $\forall n \geq N_a,\ |a_n - a| < \epsilon$,
2. $\forall \epsilon\ \exists N_b$ such that $\forall n \geq N_b,\ |a_n - b| < \epsilon$.

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$, both 1 and 2 hold, so

$$|a-b| \ = \ |(a-a_n)+(a_n-b)| \ \leq \ |a-a_n|+|a_n-b| \ < \ 2\epsilon.$$

This is true $\forall\, \epsilon$, so in fact $|a-b| = 0$.

Proof of this last claim:
If not, set $\epsilon = \frac{1}{2}|a-b| > 0$ to get the contradiction $|a-b| < |a-b|$. □

Proof 2. By contradiction. Assume $a \neq b$ and again draw a *magnified* picture.

Eventually a_n is in *both* corridors. So if we choose ϵ sufficiently small so that the corridors don't overlap then we get a contradiction.

Set $\epsilon = \frac{|a-b|}{2} > 0$. Then $\exists N_a, N_b$ such that $\forall n \geq N_a, N_b$, we have

$$|a_n - a| \ < \ \epsilon \quad \text{and} \quad |a_n - b| \ < \ \epsilon.$$

Without loss of generality, $a > b$. Then $a_n > a - \epsilon$ and $a_n < b + \epsilon$

$$\implies b+\epsilon \ > \ a-\epsilon$$
$$\implies 2\epsilon \ > \ a-b = 2\epsilon \ ※$$

□

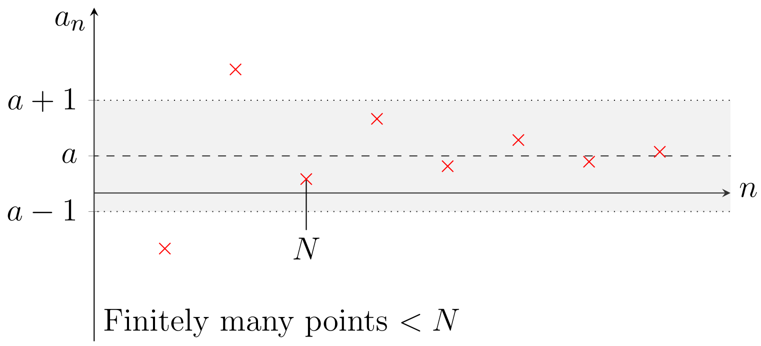
Exercise 3.15. Let a_n be defined by $a_1 = a_2 = 0$ and $a_n = \frac{1}{n-2}$ for $n > 2$. Show $a_n \to 0$.
Which step is incorrect in this student 's strategy?
Fix $\epsilon > 0$. We assume $n > 2$. Then

1. We want $|\frac{1}{n-2} - 0| = \frac{1}{n-2} < \epsilon$
2. $\implies n - 2 > 1/\epsilon$
3. $\implies n > 2 + 1/\epsilon$
4. $\implies n > 1/\epsilon \quad (*)$
5. So take $N > \max(1/\epsilon, 2)$, then
6. $\forall n \geq N$, $n > 1/\epsilon$ which is $(*)$
7. So $\frac{1}{n-2} \to 0$ ✓
8. More than one mistake
9. All correct

Although steps 2 and 4 cannot be reversed, they're not wrong (they're just not useful). But 7 IS wrong. It does not follow from 6 because $(*)$ does not imply the steps above it – it is implied by them. The implications are in the wrong direction.

Proposition 3.16. *If (a_n) is convergent, then it is bounded.*
$\big[$*I.e.* $a_n \to a \implies \exists A \in \mathbb{R}$ *such that* $|a_n| \leq A\ \forall n.\big]$

Proof. Fix $\epsilon = 1$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < 1 \implies |a_n| < 1 + |a|$.



Then $|a_n|$ is bounded $\forall n$ by $\max\big\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a| + 1\big\}$. □

Notice $a_n = \frac{1}{n-7}$ is not a counterexample! It is not a well defined sequence of real numbers because a_7 is either not defined or not real. Instead we could take

$$a_n = \begin{cases} \frac{1}{n-7} & n \neq 7, \\ 0 & n = 7. \end{cases}$$

This is then indeed bounded as $\forall n \in \mathbb{N}$ we have

$$-1 = a_6 \leq a_n \leq a_8 = 1.$$

Exercise 3.17. Give an example of a bounded sequence that is divergent.

Exercise 3.18. Let (a_n) be a bounded sequence. Let (b_n) be a sequence with $b_n = a_n$ for all $n \geq 100$. Prove that b_n is bounded.

Theorem 3.19: Algebra of limits

If $a_n \to a$ and $b_n \to b$ then:

1. $a_n + b_n \to a + b$,
2. $a_n b_n \to ab$,
3. $\frac{a_n}{b_n} \to \frac{a}{b}$ if $b \neq 0$.

Proof of 1. Fix any $\epsilon > 0$. Then

$$\exists N_a \in \mathbb{N} \text{ such that } \forall n \geq N_a,\ |a_n - a| < \frac{\epsilon}{2},$$
$$\exists N_b \in \mathbb{N} \text{ such that } \forall n \geq N_b,\ |b_n - b| < \frac{\epsilon}{2}.$$

Set $N = \max\{N_a, N_b\}$, so $\forall n \geq N$,

$$\begin{aligned} |(a_n + b_n) - (a + b)| &\leq |a_n - a| + |b_n - b| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

□

Rough working for 2: First a bit of a trick,

$$\begin{aligned} |a_n b_n - ab| &= |(a_n - a)b - a_n(b - b_n)| \\ &\leq |a_n - a||b| + |a_n||b_n - b|. \end{aligned}$$

We can easily make $|a_n - a||b| < \frac{\epsilon}{2}$ if we take $|a_n - a| < \frac{\epsilon}{2|b|}$.

But we **cannot** deduce $|b_n - b| < \frac{\epsilon}{2|a_n|}$ from $b_n \to b$ because in the definition, ϵ has to be independent of n.

Instead we bound $|a_n| < A$ by Proposition 3.16; then we can take $|b_n - b| < \frac{\epsilon}{2A}$.

Proof of 2. $a_n \to a \implies \exists\, A > 0$ such that $|a_n| < A\ \forall n \in \mathbb{N}$ by Proposition 3.16.

Fix $\epsilon > 0$. Then

$$\exists N_a \text{ such that } \forall n \geq N_a,\ |a_n - a| < \frac{\epsilon}{2(|b| + 1)},$$
$$\exists N_b \text{ such that } \forall n \geq N_b,\ |b_n - b| < \frac{\epsilon}{2A}.$$

(We added 1 to $2|b|$ to handle the case $|b| = 0$.)

Set $N = \max(N_a, N_b)$. Then $\forall n \geq N$,

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n - a||b| + |b_n - b||a_n| \\ &< \frac{\epsilon}{2}\frac{|b|}{|b| + 1} + A\frac{\epsilon}{2A} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Alternative trick-less proof of 2: Write $a_n = a + e_n$ and $b_n = b + f_n$ so that (easy exercise!) $e_n, f_n \to 0$. Then

$$\begin{aligned} |a_n b_n - ab| &= |(a + e_n)(b + f_n) - ab| = |af_n + be_n + e_n f_n| \\ &\leq |a||f_n| + |b||e_n| + |e_n||f_n|. \qquad (*) \end{aligned}$$

Now the idea is that if we make $|e_n|, |f_n| < \epsilon$, the last term is $< \epsilon^2$ which *should be* even smaller. In fact this only works if $\epsilon \leq 1$ so we need to ensure this.

So now fix $\epsilon > 0$ and set $\epsilon' := \min(\epsilon, 1)/(|a| + |b| + 1)$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$|e_n|,\ |f_n| \ < \ \epsilon' \ \stackrel{(*)}{\Longrightarrow} \ |a_n b_n - ab| \ < \ |a|\epsilon' + |b|\epsilon' + (\epsilon')^2.$$

Since $\epsilon' \leq 1$ we know $(\epsilon')^2 \leq \epsilon'$ so we get $|a_n b_n - ab| < \epsilon'(|a| + |b| + 1) \leq \epsilon$, so $a_n b_n \to ab$.

I deliberately missed out the rough working of how to choose ϵ'. Tonight **close your notes and write out your own proof of this result.** Do the rough working first, then write a concise, precise, logical proof. Don't be afraid to have several goes until the end result is undeniably a correct proof.

See exercise sheet for proof of (3). □

Remark 3.20. Now it's easier to handle things like $a_n = \dfrac{n^2 + 5}{n^3 - n + 6}$.

Dividing by n^3, we get

$$a_n \ = \ \frac{1/n + 5/n^3}{1 - 1/n^2 + 6/n^3}$$

Using the fact that $1/n \to 0$ as $n \to \infty$

(Recall proof: $\forall \epsilon > 0$ choose $N_\epsilon > 1/\epsilon$ so that

$$n \geq N_\epsilon \implies n > 1/\epsilon \implies 1/n < \epsilon)$$

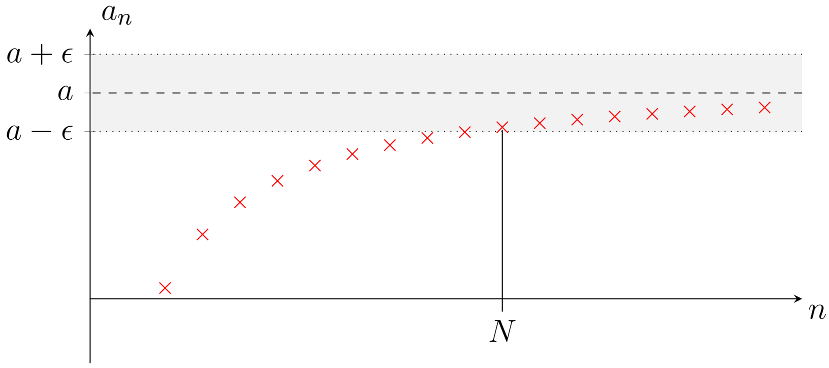
and the algebra of limits, we deduce

$$a_n \ \longrightarrow \ \frac{0 + 5 \times 0^3}{1 - 0^2 + 6 \times 0^3} \ = \ 0.$$

Theorem 3.21

If (a_n) is bounded above *and* monotonically increasing then a_n converges to $a := \sup\{a_i \colon i \in \mathbb{N}\}$. We write $a_n \uparrow a$.

Idea: Eventually we get in the ϵ-corridor around a (the shaded area) because $a - \epsilon$ is *not* an upper bound for $\{a_n \colon n \in \mathbb{N}\}$. We stay in there because a_n is monotonic and bounded above by a.



Proof. Set $a := \sup\{a_i : i \in \mathbb{N}\}$ and fix $\epsilon > 0$. Now $a - \epsilon$ is *not* an upper bound for $\{a_n : n \in \mathbb{N}\}$ (because a is the *smallest* upper bound), so $\exists N \in \mathbb{N}$ such that $a_N > a - \epsilon$. Monotonic so $\forall n \geq N$ we have

$$a \geq a_n \geq a_N > a - \epsilon \implies |a_n - a| < \epsilon.$$ □

Example 3.22. Suppose that (a_n) and (b_n) are sequences of real numbers such that $a_n \leq b_n \; \forall n$ and $a_n \to a, \; b_n \to b$. Prove that $a \leq b$.

Draw a picture! It will eventually lead you to a proof along the following lines.

Suppose for a contradiction that $a > b$, then set $\epsilon = \frac{a-b}{2} > 0$. Then:

$$\exists \, N_a \in \mathbb{N} \text{ such that } n \geq N \implies |a_n - a| < \epsilon \implies a_n > a - \epsilon = \frac{a+b}{2},$$

$$\text{and} \quad \exists \, N_b \in \mathbb{N} \text{ such that } n \geq N \implies |b_n - b| < \epsilon \implies b_n < b + \epsilon = \frac{a+b}{2}.$$

So for $n \geq \max(N_a, N_b)$ we have $b_n < \frac{a+b}{2} < a_n$ which contradicts $a_n \leq b_n$.

Example 3.23. Prove that if

$$\left|\frac{a_{n+1}}{a_n}\right| \to L < 1$$

then $a_n \to 0$.

Idea: $a_n \approx c \cdot L^n$ for $n \gg 0$, $L < 1 \implies a_n \to 0$.

Since $|a_{n+1}/a_n|$ is not exactly L, to turn this in to a proof, we must instead estimate/bound it by $|a_{n+1}/a_n| < L'$ for some $L' < 1$. Though we cannot take $L' = L$ we can take $L' = L + \epsilon$ (because $|a_{n+1}/a_n| \to L$). So we need $L + \epsilon < 1$, so let's take $\epsilon = \frac{1-L}{2}$.

Proof. Fix $\epsilon = \frac{1-L}{2}$. Then $\epsilon > 0$ because $L < 1$, so $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left|\frac{a_{n+1}}{a_n} - L\right| < \epsilon \implies \left|\frac{a_{n+1}}{a_n}\right| < L + \epsilon = \frac{1+L}{2} < 1.$$

Setting $L' := \frac{1+L}{2} < 1$ we find inductively that

$$\begin{aligned} |a_{N+k}| &\leq L'|a_{N+k-1}| \\ &\leq (L')^2|a_{N+k-2}| \\ &\leq \dots \\ &\leq (L')^k|a_N|. \end{aligned} \qquad (*)$$

[Exercise sheet: $\alpha^k \to 0$ as $k \to \infty$ if $|\alpha| < 1$.]

We apply this to $\alpha = L' < 1$. Fixing a new $\epsilon > 0$, $\exists M > 0$ such that $\forall k \geq M$,

$$(L')^k < \frac{\epsilon}{1 + |a_N|}. \qquad (**)$$

(We wanted to write $\frac{\epsilon}{|a_N|}$ but we have to beware the case $|a_N| = 0$.)

So by $(*)$ and $(**)$ we have

$$|a_{N+k}| < \frac{\epsilon}{1 + |a_N|}|a_N| < \epsilon \qquad \forall k \geq M.$$

Rewriting this:

$$\forall n \geq N + M, \quad |a_n| < \epsilon.$$ □

3.2 Cauchy Sequences

We're now world experts at proving a_n converges if we know what the limit is. Cauchy sequences gives us a way to prove convergence *without* knowing the limit.

Definition. $(a_n)_{n \geq 1}$ is called a *Cauchy* sequence if and only if

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n, m \geq N,\ |a_n - a_m| < \epsilon.$$

Remark 3.24. $m, n \geq N$ are arbitrary. It is not enough to say that $\forall \epsilon > 0\ \exists N \in \mathbb{N}$ such that $n \geq N \implies |a_n - a_{n+1}| < \epsilon$. See exercise sheet.

Proposition 3.25. *If $a_n \to a$ then (a_n) is Cauchy.*

Proof. $a_n \to a \implies \forall \epsilon > 0\ \exists N$ such that $n \geq N \implies |a_n - a| < \frac{\epsilon}{2}$. $(*)$

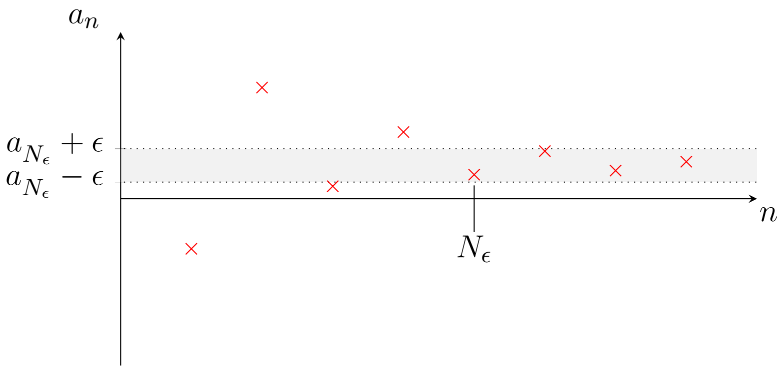
So $m \geq N \implies |a_m - a| < \frac{\epsilon}{2}$ $(\dagger)$.

Combining these, for $m, n \geq N$ we have

$$|a_n - a_m| \ \leq\ |a_n - a| + |a_m - a| \ <\ \underbrace{\epsilon/2}_{(*)} + \underbrace{\epsilon/2}_{(\dagger)} \ =\ \epsilon.$$ □

Next we want to prove the converse: Cauchy $\implies$ convergence.

We need a candidate for the limit a.



We will produce an auxiliary sequence which is *monotonic* (and bounded) $\implies$ convergent. Let $b_n := \sup\{a_i : i \geq n\}$. Then picture shows that $b_{N_\epsilon} \in (a_{N_\epsilon} - \epsilon, a_{N_\epsilon} + \epsilon]$ and b_ns are monotonically *decreasing* because $\{a_i : i \geq n+1\} \subseteq \{a_i : i \geq n\}$ so $b_{n+1} = \sup \leq \sup = b_n$.

So b_ns converge to $\inf\{b_n : n \in \mathbb{N}\}$. We will show that a_ns converge to same number, a, using the Cauchy condition.

Lemma 3.26. *(a_n) is Cauchy $\implies$ (a_n) is bounded.*

Proof. Pick $\epsilon = 1$, then $\exists N$ such that $\forall n, m \geq N$, $|a_n - a_m| < 1$.
In particular, taking $m = N$ gives $|a_n| < 1 + |a_N|$ $\forall n \geq N$, so

$$|a_n| \ \leq\ \max\Big\{|a_1|, |a_2|, \ldots |a_{N-1}|, 1 + |a_N|\Big\} \ \ \forall n \in \mathbb{N}.$$ □

Theorem 3.27

If (a_n) is a Cauchy sequence of real numbers then a_n converges.

Corollary 3.28. *(a_n) Cauchy $\iff$ (a_n) convergent.*

Exercise 3.29. Show this is not true in $\mathbb{Q}$: there exist Cauchy sequences (a_n) with $a_n \in \mathbb{Q}$ with no limit in $\mathbb{Q}$.

Proof. Since (a_n) is Cauchy, it is bounded by Lemma 3.26: $|a_n| \leq A$. So we can define $b_n := \sup\{a_i : i \geq n\}$.

Then $b_n \geq a_i \ \forall i \geq n$ so $b_n \geq a_i \ \forall i \geq n+1$ is an upper bound for $\{a_i : i \geq n+1\}$, so is $\geq \sup\{a_i : i \geq n+1\} = b_{n+1}$. So the sequence (b_n) is monotonically decreasing. And $b_n \geq a_n \geq -A$ shows it is also bounded below.

So we can define $a := \inf\{b_n : n \in \mathbb{N}\}$ and $b_n \downarrow a$. We claim that $a_n \to a$.

Fix $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ such that for all $n, m \geq N$,

$$|a_n - a_m| < \frac{\epsilon}{2} \iff a_n - \frac{\epsilon}{2} < a_m < a_n + \frac{\epsilon}{2}.$$

Fix $i \geq N$ and take the supremum over all $m \geq i$:

$$\begin{aligned}
&\implies a_n - \frac{\epsilon}{2} < \underbrace{\sup\{a_m : m \geq i\}}_{= b_i} \leq a_n + \frac{\epsilon}{2} \\
&\implies a_n - \frac{\epsilon}{2} \leq \underbrace{\inf\{b_i : i \geq N\}}_{= a} \leq a_n + \frac{\epsilon}{2} \\
&\iff |a - a_n| \leq \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

Since this holds for all $n \geq N$ it proves $a_n \to a$. □

In the proof we twice used:

Exercise 3.30. If $S \subseteq \mathbb{R}$ satisfies $x < M \ \ \forall x \in S$ then $\sup S \leq M$.

Example 3.31 (Decimals)**.** Suppose we didn't use decimals to construct $\mathbb{R}$ (e.g. if we used Dedekind cuts, or we just used the axioms without worrying about constructing the set).

Then using Cauchy sequences we can now make sense of the decimal $a_0.a_1a_2a_3\ldots$ as follows. (Here we fix $a_0 \in \mathbb{Z}$ and $a_1, a_2, a_3, \cdots \in \{0, 1, \ldots, 9\}$.)

Let $(A_n)_{n\geq 1}$ be the sequence of rational numbers defined by

$$A_n := a_0 + \frac{a_1}{10} + \cdots + \frac{a_n}{10^n}.$$

(A_n is the approximation to our decimal given by truncating at the nth place.)

> Exercise: for all $n, m \geq N$ we have $|A_n - A_m| < 10^{-N}$.
>
> Thus (A_n) is a Cauchy sequence: $\forall \epsilon > 0$ we can take $N > \epsilon^{-1}$ so that $10^N > \epsilon^{-1}$ so that $10^{-N} < \epsilon$.
>
> Thus it converges to a limit in $\mathbb{R}$. We call this limit $a_0.a_1a_2a_3\ldots$.

3.3 Subsequences

Definition. A *subsequence* of (a_n) is a new sequence $b_i = a_{n(i)}$ $\forall i \in \mathbb{N}$, where $n(1) < n(2) < \cdots < n(i) < \ldots$ $\forall i$.

Formally $n(\,\cdot\,)$ is a function $\mathbb{N} \to \mathbb{N}$ sending $i \mapsto n(i)$ which is strictly monotonically increasing. "Just go down the sequence faster, missing some terms out".

Exercise 3.32. Prove this implies $n(i) \geq i$ by induction.

Example 3.33. $a_n = (-1)^n$ has subsequences:

- $b_n = a_{2n}$, so $b_n = 1$ $\forall n \implies b_n \to 1$.
- $c_n = a_{2n+1}$, so $c_n = -1$ $\forall n \implies c_n \to -1$.
- $d_n = a_{3n}$, so $d_n = (-1)^n$ $(= a_n)$ doesn't converge.
- $e_n = a_{n+17}$, so $e_n = (-1)^{n+1}$ $(= -a_n)$ doesn't converge.

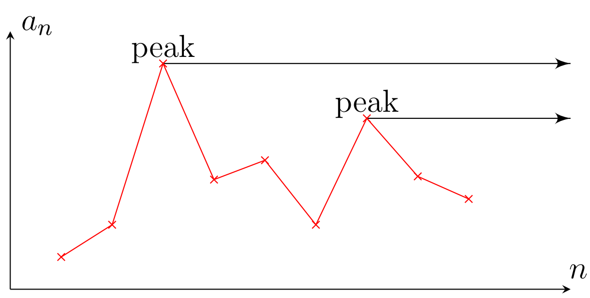
Next we work up to the following technical-sounding but vitally important:

Theorem 3.34: Bolzano-Weierstrass

If (a_n) is a *bounded* sequence of real numbers then it has a *convergent subsequence.*

Remark 3.35. Of course it will have *many* convergent subsequences, and they may converge to different limits; think of $a_n = (-1)^n$ for instance.

Cheap proof. Use "peak points" of (a_n):



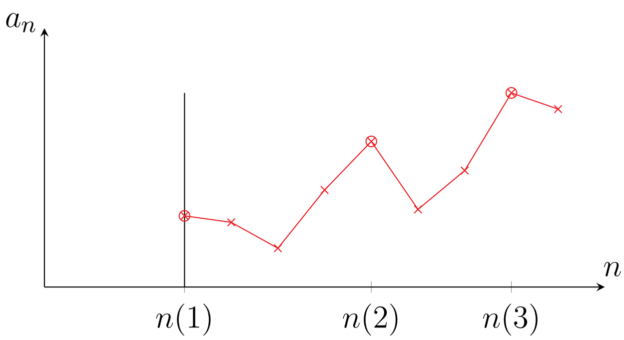
We say that a_j is a *peak point* if and only if $a_k < a_j$ $\forall k > j$. Either

1. (a_n) has a finite number of peak points, or

2. (a_n) has an infinite number of peak points.

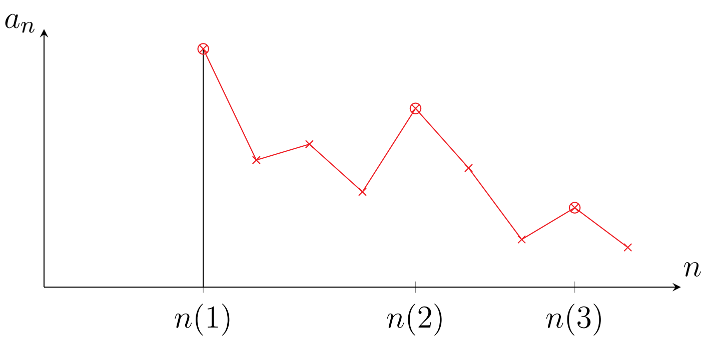
Case 1: We go beyond the (finitely many) peak points: pick $n(1) \geq \max(j_1, \ldots, j_k)$ where $a_{j_1}, \ldots, a_{j_k}$ are the peak points.

$a_{n(1)}$ is not a peak point $\implies \exists\, n(2) > n(1)$ such that $a_{n(2)} \geq a_{n(1)}$.
$a_{n(2)}$ not a peak point $\implies \exists\, n(3) > n(2)$ such that $a_{n(3)} \geq a_{n(2)}$.
Recursively no peak points beyond $n(1) \implies$ we get $n(i) > n(i-1) > \cdots > n(1)$ such that $a_{n(i)} \geq a_{n(i-1)}$ $\forall i$.



So $a_{n(i)}$ is a monotonically increasing subsequence of a_n.
$(a_n)_{n\geq 1}$ bounded $\implies$ $(a_{n(i)})_{i\geq 1}$ is bounded $\implies$ $a_{n(i)} \uparrow \sup\{a_{n(i)} : i \in \mathbb{N}\}$ by Theorem 3.21.

Case 2: There are infinitely many peak points, so we may call them $a_{n(1)}, a_{n(2)}, \ldots$ where $n(1) < n(2) < \ldots$



Now $n(i+1) > n(i)$ and $a_{n(i)}$ is a peak point $\implies$ $a_{n(i+1)} \leq a_{n(i)}$. Thus the subsequence $(a_{n(i)})_{i\geq 1}$ is monotonically decreasing and bounded $\implies$ convergent (to $\inf\{a_{n(i)} : i \in \mathbb{N}\}$). □

> **Exercise 3.36.** Give an example of an unbounded sequence with a convergent subsequence.

> **Exercise 3.37.** Given an example, with proof, of a sequence for which every subsequence is divergent.

> **Exercise 3.38.** Give an example of an unbounded sequence that has at least three convergent subsequences that converge to three different limits.

Proposition 3.39. *If $a_n \to a$ then any subsequence $a_{n(i)} \to a$ as $i \to \infty$.*

Proof. We are told

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |a_n - a| < \epsilon. \qquad (*)$$

But $\forall i \geq N$, then $n(i) \geq i \geq N$, so by $(*)$, $|a_{n(i)} - a| < \epsilon$. □

This gives us another proof that $(-1)^n$ is not convergent, because if $(-1)^n \to a$, then by Proposition 3.39, $(-1)^{2n} \to a$ and $(-1)^{2n+1} \to a \implies a = 1$ and $a = -1$ ※

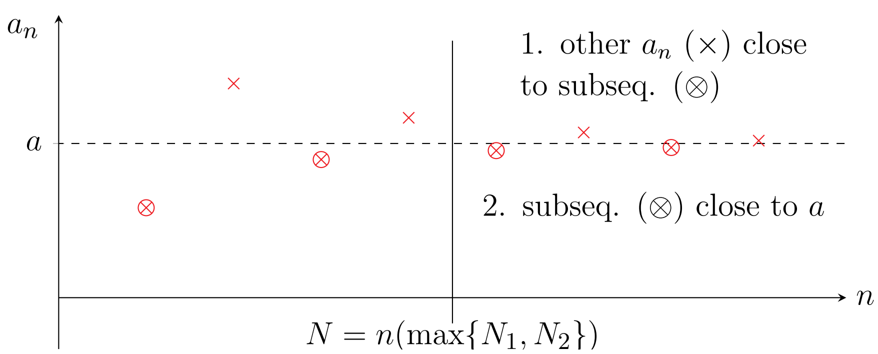
Bolzano-Weierstrass $\implies$ the Cauchy theorem

We also get another proof of "Cauchy $\implies$ convergence" using Bolzano-Weierstrass.

Proof #2 of Cauchy $\implies$ Convergence. We know from Lemma 3.26 that a_n is bounded (by $\max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |a_N| + 1\}$ remember). So by Bolzano-Weierstrass, $\exists$ a convergent subsequence $(a_{n(i)})_{i \geq 1}$ such that $a_{n(i)} \to a$ as $i \to \infty$ for some $a \in \mathbb{R}$. So fix $\epsilon > 0$. We have:

(1) $\exists N_1$ such that $\forall n, m \geq N_1$, $|a_n - a_m| < \epsilon$ (Cauchy)

(2) $\exists N_2$ such that $\forall i \geq N_2$, $|a_{n(i)} - a| < \epsilon$ (convergent subsequence)



Set $N = n(\max\{N_1, N_2\}) \geq \max\{N_1, N_2\} \geq N_1$. Then $\forall n \geq N$ we have

$$\begin{aligned} |a_n - a| &= |(a_n - a_N) + (a_N - a)| \\ &\leq |a_n - a_N| + |a_N - a| \\ &< \epsilon + \epsilon \ = \ 2\epsilon, \end{aligned}$$

the first $< \epsilon$ being by the Cauchy property (1) and the second $< \epsilon$ being from the convergence of the subsequence property (2) (since a_N is in the subsequence). □

Above, we used the following lemma.

Lemma 3.40. *Fix $c > 0$. Then $a_n \to a$ if and only if*

$$\boxed{\forall \epsilon > 0 \ \exists N_\epsilon \in \mathbb{N} \text{ such that } n \geq N_\epsilon \implies |a_n - a| < c\epsilon \qquad (*)}$$

Proof. $\implies$. Fix $\epsilon > 0$ and let $\epsilon' := c\epsilon$. Then by the definition of convergence applied to $\epsilon' > 0$ we find

$$\exists N \in \mathbb{N} : \ n \geq N \implies |a_n - a| < \epsilon',$$

which is $(*)$.

$\impliedby$. Fix $\epsilon > 0$. Set $\epsilon' = \epsilon/c > 0$. Then $(*)$ applied to $\epsilon' > 0$ implies

$$\exists N \in \mathbb{N} \text{ such that } n \geq N_\epsilon \implies |a_n - a| < c\epsilon' = \epsilon.$$

□

Warning. Do not let c depend on ϵ (nor N! or n).
E.g. if we let $c = \epsilon^{-1}$ then $(*)$ becomes $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall n \geq N$, $|a_n - a| < 1$. This is not a good definition of convergence; for instance it would say that the sequence $a_n = 1 \ \forall n$ converges to $\frac{3}{2}$!

Bolzano-Weierstrass $\impliedby$ the Cauchy theorem

We can also go the other way round: the Cauchy theorem $\implies$ Bolzano-Weierstrass.

Proof 2 of Bolzano-Weierstrass. Take a bounded sequence (a_n). We want to find a Cauchy subsequence, which will therefore be convergent.
Since $a_n \in [-R, R] \ \forall n$, repeatedly subdivide to make this interval smaller. Then either

1. $\exists$ infinite number of a_ns in $[-R, 0]$, or
2. $\exists$ infinite number of a_ns in $[0, R]$, (or both).

Pick one of these intervals with infinite number of a_ns; call it $[A_1, B_1]$ of length R.

Now subdivide again; call $[A_2, B_2]$ one of the intervals $[A_1, \frac{A_1+B_1}{2}]$ or $[\frac{A_1+B_1}{2}, B_1]$ with infinitely many a_ns in it, with length $R/2$. Etc.

Recursively we get a sequence of intervals $[A_n, B_n]$ of length $2^{1-n}R$ which are nested – i.e. $[A_{k+1}, B_{k+1}] \subseteq [A_k, B_k]$ – with each containing an infinite number of a_ns.

Now we use a *diagonal argument*.
Choose $n(1)$ so that $a_{n(1)} \in [A_1, B_1]$.
Choose $n(2) > n(1)$ so that $a_{n(2)} \in [A_2, B_2]$. (Recall there are infinitely many a_n in each $[A_k, B_k]$, so we can do this.)
Recursively choose $n(k+1) > n(k)$ so that $a_{n(k+1)} \in [A_{k+1}, B_{k+1}]$.

Claim: the subsequence $a_{n(i)}$ is convergent.

Fix $\epsilon > 0$. Take $N > \frac{2R}{\epsilon}$, so that $2^{1-N}R < 2N^{-1}R < \epsilon$. Then $\forall i, j \geq N$ we have $n(i) \geq i \geq N$ and $n(j) \geq j \geq N$, so $a_{n(i)}$, $a_{n(j)} \in [A_N, B_N]$ and

$$|a_{n(i)} - a_{n(j)}| \ < \ 2^{1-N}R \ < \ \epsilon.$$

Therefore $(a_{n(i)})$ is Cauchy and so convergent. □

Definition. We say $a_n \to +\infty$ if and only if

$$\forall R > 0 \ \exists N \in \mathbb{N} \text{ such that } a_n > R \ \forall n \geq N.$$

Remark 3.41. Recall this is not the same as (but it does imply) a_n being divergent!

Exercise 3.42. Suppose $a_n > 0 \ \forall n$. Show $a_n \to 0 \iff \frac{1}{a_n} \to +\infty$.

4 Series

Maths is not a spectator sport. How well you do comes down solely to the time you spend **doing** *maths.*

- Richard Thomas, annually

Definition. An (infinite) series is an expression

$$\sum_{n=1}^{\infty} a_n \quad \text{or} \quad a_1 + a_2 + a_3 + \dots\,,$$

where $(a_i)_{i\geq 1}$ is a sequence.

For now, it is **not** a real number. It is just a formal expression. We could write $\sum_{n=1}^{\infty} n$, for instance, without worrying about convergence (just as we write $a_n = n$ without worrying about convergence).

Partial sums

Given a sequence a_n we get a series (formal expression!) $\sum_{n=1}^{\infty} a_n$ and another sequence of **partial sums**

$$s_n \;:=\; \sum_{i=1}^{n} a_i. \qquad (*)$$

Recall in Exercise 3.2 you proved that a_n and s_n determine each other – they are equivalent information. In other words, the sequence (a_n) determines the sequence (s_n) by $(*)$, and conversely we can recover (a_n) from the (s_n) by

$$a_n \;=\; s_n - s_{n-1}.$$

4.1 Convergence of Series

Definition. We say that the series $\sum a_n$ "converges to $A \in \mathbb{R}$" if and only if the sequence of partial sums converges to A:

$$\sum_{n=1}^{\infty} a_n \;=\; A \iff s_n \longrightarrow A.$$

We often write A as $\sum_{n=1}^{\infty} a_n$. In other words, if $\sum_{n=1}^{\infty} a_n$ converges (to A) then we use the same notation to denote the real number A.

We can obviously let the sum be from $n = 0$, or over n even, or ...

Example 4.1. Consider $a_n = x^n,\ n \geq 0$, so that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} x^n$.

The partial sums are

$$s_n \;=\; \sum_{i=0}^{n} x^i \;=\; 1 + x + \cdots + x^n.$$

Therefore

$$x s_n \;=\; x + \cdots + x^n + x^{n+1},$$

so

$$s_n - x s_n \;=\; 1 - x^{n+1},$$

giving

$$s_n \;=\; \begin{cases} \dfrac{1 - x^{n+1}}{1 - x} & x \neq 1, \\ n + 1 & x = 1. \end{cases}$$

So for $|x| < 1$, we see that

$$s_n \;=\; \frac{1}{1-x} - \frac{x^{n+1}}{1-x} \;\longrightarrow\; \frac{1}{1-x} \quad \text{as } n \to \infty.$$

(Recall from the question sheet that $r^n \to 0$ if $|r| < 1$.)

So (s_n) is convergent and we can finally say $\displaystyle\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ if $|x| < 1$.

For $|x| \geq 1$, $a_n = x^n \not\to 0$ as $n \to \infty$. So $\sum a_n = \sum x^n$ is *not* a real number (does not converge) by the next result.

Theorem 4.2

$\sum_{n=0}^{\infty} a_n$ is convergent $\implies a_n \to 0$.

Proof. $s_n - s_{n-1} = a_n$. If $s_n \to A$ then $s_{n-1} \to A$ (exercise!). So by the algebra of limits a_n is convergent and $a_n \to A - A = 0$. □

Proof from first principles. Fix $\epsilon > 0$. Since $s_n \to A$,

$$\exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |s_n - A| < \epsilon$$

so that

$$|a_n| \;=\; |s_n - s_{n-1}| \;=\; |(s_n - A) - (s_{n-1} - A)| \;\leq\; |s_n - A| + |s_{n-1} - A|$$

which is $< \epsilon + \epsilon$ for $n-1 \geq N$. So $\forall n \geq N+1$, $|a_n| < 2\epsilon$. □

Remark 4.3. Converse is *not* true. E.g. $a_n = \frac{1}{n} \to 0$, but $\sum \frac{1}{n}$ is *not* convergent.

Example 4.4. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

Proof. Uses a slight trick. Arrange the partial sum as follows:

$$\begin{aligned} 1 + \frac{1}{2} + \frac{1}{3} + \dots &= 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \dots + \frac{1}{7}\right) \\ &\quad + \left(\frac{1}{8} + \dots + \frac{1}{15}\right) + \left(\frac{1}{16} + \dots + \frac{1}{31}\right) + \dots \end{aligned}$$

We can bound the kth bracketed term from below:

$$\left(\frac{1}{2^k} + \dots + \frac{1}{(2^{k+1}-1)}\right) \;>\; \frac{1}{2^{k+1}} + \dots + \frac{1}{2^{k+1}} \;=\; \frac{2^k}{2^{k+1}} \;=\; \frac{1}{2}\,.$$

In particular then

$$s_{2^{k+1}-1} \;>\; 1 + \underbrace{\frac{1}{2} + \dots + \frac{1}{2}}_{k \text{ terms}} \;=\; 1 + \frac{k}{2}$$

is arbitrarily large. But if s_n converged, it would be bounded: $|s_n| \leq C \;\; \forall n$. So we get the contradiction (to the Archimedean property) $1 + \frac{k}{2} \leq C \;\; \forall k \in \mathbb{N}$. □

Example 4.5. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent.

Proof. (Using a trick; we will give another proof soon.) First show $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent, using $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

$$\begin{aligned} s_n &= \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right) \\ &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \dots + \left(\tfrac{1}{n} - \tfrac{1}{n+1}\right) \\ &= 1 - \frac{1}{n+1} \;\longrightarrow\; 1 \;\text{ as } n \to \infty. \end{aligned}$$

Thus $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent to 1.

So now compare the partial sums σ_n of $\sum \frac{1}{n^2}$ to those of $\sum \frac{1}{n(n+1)} = 1$.

$$\begin{aligned} \sigma_n &= \sum_{i=1}^{n} \frac{1}{i^2} = 1 + \sum_{j=1}^{n-1} \frac{1}{(j+1)^2} \\ &\leq 1 + \sum_{j=1}^{n-1} \frac{1}{j(j+1)} \\ &= 1 + s_{n-1}. \end{aligned}$$

$s_{n-1} \uparrow 1$ because $\frac{1}{n(n+1)} > 0$. So $s_{n-1} < 1 \ \forall n \implies \sigma_n < 2 \implies$ bounded above monotonic increasing sequence $\implies \sigma_n$ is convergent $\implies \sum \frac{1}{n^2}$ is convergent. $\square$

Generalizing the method we used above gives the following proposition and theorem.

Proposition 4.6. *Suppose $a_n \geq 0 \ \forall n$ ($\iff s_n = \sum_{i=1}^{n} a_i$ is monotonically increasing), Then the following two facts are true:*

1. *$\sum_{n=1}^{\infty} a_n$ converges if and only if (s_n) is bounded above.*
2. *Similarly $\sum_{n=1}^{\infty} a_n$ diverges to $+\infty$ (i.e. $\forall M > 0 \ \exists N \in \mathbb{N}$ such that $s_n > M \ \forall n \geq N$) if and only if (s_n) is unbounded.*

Proof. Since (s_n) is monotonic increasing, we have by Proposition 3.16 and Theorem 3.21 that

$$s_n \text{ is bounded} \iff s_n \text{ is convergent.}$$

For the second statement, s_n is unbounded $\iff \forall M > 0 \ \exists N \in \mathbb{N}$ such that $s_N > M$. But s_N is monotonic, so this is $\iff \forall M > 0 \ \exists N \in \mathbb{N}$ such that $\forall n \geq N, \ s_n > M$. And this is the definition of $s_n \to +\infty$. $\square$

The theorem promised above is a very useful convergence test for positive series.

Theorem 4.7: Comparison test

If $0 \leq a_n \leq b_n$ and $\sum b_n$ converges, then $\sum a_n$ converges.
Moreover, $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Proof. Call the partial sums A_n, B_n respectively. Then

$$0 \le A_n \le B_n \le \lim_{n\to\infty} B_n = \sum_{i=1}^{\infty} b_i.$$

So A_n is bounded and monotonically increasing $\Longrightarrow$ convergent.

We are done since in previous exercise we have shown that $A_n \le B_n$ and $A_n \to A$, $B_n \to B$ implies that $A \le B$. □

Exercise 4.8 (Converse of Comparison Test.)**.** If $0 \le a_n \le b_n$ then $\sum a_n$ diverges to $+\infty \Longrightarrow \sum b_n$ diverges to $+\infty$.

Remark 4.9. So from $\sum \frac{1}{n^2}$ convergent (Example 4.5) we can now deduce $\sum \frac{1}{n^\alpha}$ convergent for $\alpha \ge 2$ by the Comparison Test. In fact we can improve on this.

Example 4.10. $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ is convergent for $\alpha > 1$.

Proof. (Cf. proof of divergence of $\sum \frac{1}{n}$ in Example 4.4.) Arrange the partial sums as follows:

$$1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \dots = 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) + \left(\frac{1}{4^\alpha} + \dots + \frac{1}{7^\alpha}\right) + \left(\frac{1}{8^\alpha} + \dots + \frac{1}{15^\alpha}\right) + \left(\frac{1}{16^\alpha} + \dots + \frac{1}{31^\alpha}\right) + \dots$$

Bound the kth bracketed term:

$$\left(\frac{1}{(2^k)^\alpha} + \dots + \frac{1}{(2^{k+1}-1)^\alpha}\right) \le \frac{1}{2^{k\alpha}} + \dots + \frac{1}{2^{k\alpha}} = \frac{2^k}{2^{k\alpha}} = \frac{1}{2^{k(\alpha-1)}}.$$

So any partial sum is less than some finite sum of these bracketed terms, i.e. for $n \le 2^{k+1} - 1$ we have

$$s_n < \sum_{i=0}^{k} \frac{1}{2^{i(\alpha-1)}} = \frac{1 - \frac{1}{2^{(k+1)(\alpha-1)}}}{1 - \frac{1}{2^{(\alpha-1)}}} \le \frac{1}{1 - \frac{1}{2^{\alpha-1}}}.$$

(It is here we used $\alpha > 1$, so $\left|\frac{1}{2^{\alpha-1}}\right| < 1$, so top and bottom of the central fraction are > 0.)

So partial sums are monotonic and bounded above $\Longrightarrow$ convergent. □

Theorem 4.11: Algebra of limits for series

If $\sum a_n$, $\sum b_n$ are convergent then so is $\sum(\lambda a_n + \mu b_n)$, to

$$\sum_{n=1}^{\infty}(\lambda a_n + \mu b_n) \;=\; \lambda\sum_{n=1}^{\infty} a_n \;+\; \mu\sum_{n=1}^{\infty} b_n.$$

Proof. Partial sum (to n terms) of LHS is

$$\sum_{i=1}^{n}(\lambda a_i + \mu b_i) \;=\; \lambda\sum_{i=1}^{n} a_i + \mu\sum_{i=1}^{n} b_i \;\longrightarrow\; \lambda\sum_{i=1}^{\infty} a_i + \mu\sum_{i=1}^{\infty} b_i$$

as $n \to \infty$ by the algebra of limits for sequences. So the partial sums converge. □

4.2 Absolute convergence

Definition. For $a_n \in \mathbb{R}$ or $\mathbb{C}$, we say the series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if and only if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Remark 4.12. It is possible for a series to be convergent (that is, its sequence of partial sums converges), but not absolutely convergent!

Example 4.13. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is *not* absolutely convergent (by Example 4.4), but it is convergent.

Rough Working:

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \ldots \;=\; (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \ldots$$

with kth bracket $\frac{1}{2k-1} - \frac{1}{2k} = \frac{1}{2k(2k-1)}$. This is positive and $\leq \frac{1}{2k(2k-2)} = \frac{1}{4k(k-1)}$. We saw this is convergent in Example 4.5. So cancellation between consecutive terms is enough to make series converge by comparison with $\sum \frac{1}{k(k-1)}$.

Proof. Fix $\epsilon > 0$. Then use 2 things

$$\sum \frac{1}{2k(2k-1)} \quad \text{is convergent to } L \text{ say} \tag{1}$$

$$\frac{(-1)^{n+1}}{n} \;\longrightarrow\; 0 \tag{2}$$

By (1) $\exists\, N_1$ such that $\forall n \geq N_1$, $\left|\sum_{k=1}^{n} \frac{1}{2k(2k-1)} - L\right| < \epsilon$.

By (2) $\exists\, N_2$ such that $\forall n \geq N_2$, $\left|\frac{(-1)^{n+1}}{n}\right| < \epsilon$.

Set $N = \max(N_1, N_2)$. Then $\forall n \geq N$, setting $j := \lfloor \frac{n}{2} \rfloor$ we have:

$$\begin{aligned} s_n &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots \left(\frac{1}{2j-1} - \frac{1}{2j}\right) + \delta \\ &= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2k(2k-1)} + \delta, \end{aligned}$$

where

$$\delta = \begin{cases} \frac{(-1)^{n+1}}{n} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases} \qquad \text{satisfies } |\delta| \leq \epsilon \text{ for } n \geq N_2 \text{ by (2).}$$

So

$$|s_n - L| \leq \left|\sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{2k(2k-1)} - L\right| + |\delta| < \epsilon + \epsilon$$

for all $n \geq 2N+1$ (so that $\lfloor \frac{n}{2} \rfloor \geq N \geq N_1$ and $n \geq N \geq N_2$) by (1) and (2). □

Theorem 4.14

Let $(a_n)_{n\geq 0}$ be a real or complex sequence.
If $\sum a_n$ is absolutely convergent, then it is convergent.

Proof. Let $s_n = \sum_{i=1}^{n} |a_i|$ and $\sigma_n = \sum_{i=1}^{n} a_i$ be the partial sums.

Fix $\epsilon > 0$. We're assuming that s_n converges, so it is Cauchy:

$$\exists N_\epsilon \text{ such that } n > m \geq N_\epsilon \implies |s_n - s_m| < \epsilon \iff |a_{m+1}| + \cdots + |a_n| < \epsilon,$$

i.e. the terms in the tail of the series contribute little to the sum. So by the triangle inequality,

$$|a_{m+1} + \cdots + a_n| < \epsilon \implies |\sigma_n - \sigma_m| < \epsilon$$

and (σ_n) is Cauchy, and so convergent. □

Example 4.15. For $z \in \mathbb{C}$ the power series $\sum_{n=1}^{\infty} z^n$ is absolutely convergent for $|z| < 1$ and divergent for $|z| \geq 1$.

Proof. $\sum_{n=1}^{\infty} z^n$ is absolutely convergent because in Example 4.1 we showed that $\sum_{n=1}^{\infty} |z|^n$ converges to $\frac{1}{1-|z|}$ for $|z| < 1$.
For $|z| \geq 1$, the individual terms z^n have $|z^n| \geq 1$, so $z^n \not\to 0$, so $\sum z^n$ is divergent by Theorem 4.2. □

4.3 Tests for convergence

We already met the first test:

Theorem 4.7: Comparison I

If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
Moreover, $0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$.

Recall proof: $s_n = \displaystyle\sum_{i=1}^{n} a_i$ is monotonic increasing and bounded above by $\displaystyle\sum_{i=1}^{\infty} b_i \in \mathbb{R}$.

Theorem 4.16: Comparison II: Sandwich Test

Suppose $c_n \leq a_n \leq b_n \ \forall n$ and $\sum c_n, \ \sum b_n$ are both convergent.
Then $\sum a_n$ is convergent.

Proof. We use the Cauchy criterion. $\forall \epsilon > 0 \ \exists N \in \mathbb{N}$ such that $\forall n > m > N$,

$$\left| \sum_{i=m+1}^{n} b_i \right| < \epsilon, \quad \left| \sum_{i=m+1}^{n} c_i \right| < \epsilon$$

since the partial sums of b_i, c_i are Cauchy. Therefore

$$-\epsilon < \sum_{i=m+1}^{n} c_i \leq \sum_{i=m+1}^{n} a_i \leq \sum_{i=m+1}^{n} b_i < \epsilon$$

which implies

$$\left| \sum_{i=1}^{n} a_i - \sum_{i=1}^{m} a_i \right| < \epsilon,$$

i.e. the partial sums $\sum_{i=1}^{n} a_i$ form a Cauchy sequence. $\square$

Theorem 4.17: Comparison III

If $\frac{a_n}{b_n} \to L \in \mathbb{R}$ and $\sum b_n$ is absolutely convergent, then $\sum a_n$ is absolutely convergent.

Remark 4.18. While writing $\frac{a_n}{b_n} \to L$ makes sense, writing $a_n \to Lb_n$ does not make sense (why)!

Proof. Set $L = \lim_{n\to\infty} \frac{a_n}{b_n}$. Pick $\epsilon = 1$, then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left|\frac{a_n}{b_n} - L\right| < 1 \implies \left|\frac{a_n}{b_n}\right| < |L| + 1 \implies |a_n| < (|L|+1)|b_n|.$$

So now by the comparison test $\sum_{n\geq N} |b_n|$ convergent $\implies \sum_{n\geq N} |a_n|$ convergent. By the next exercise this gives the result. $\square$

Exercise 4.19. Fix $N \in \mathbb{N}$. Then $\sum_{n\geq N} c_n$ is convergent if and only if $\sum_{n\geq 1} c_n$ is convergent.

We call a sequence a_n *alternating* if $a_{2n} \geq 0$ and $a_{2n+1} \leq 0$ $\forall n$ (or the opposite).

Theorem 4.20: Alternating Series Test

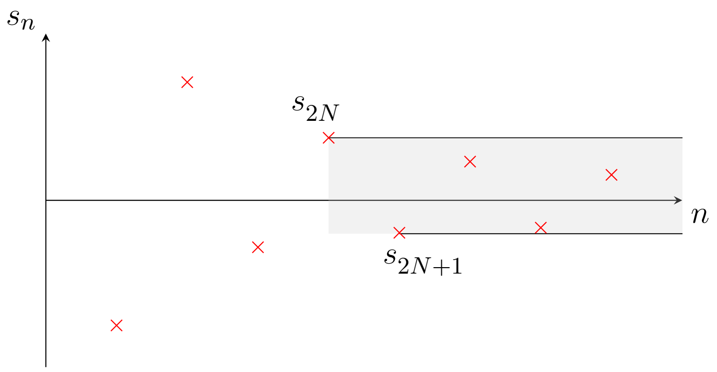
Suppose a_n is alternating with $|a_n| \downarrow 0$. Then $\sum a_n$ converges.

Proof. Without loss of generality write $a_n = (-1)^n b_n$ with $b_n := |a_n| \to 0$. Consider the partial sums $s_n = \sum_{i=1}^{n} (-1)^i b_i$.

We claim

(1) $s_i \leq s_{2n}$ $\forall i \geq 2n$,

(2) $s_i \geq s_{2n+1}$ $\forall i \geq 2n+1$.



Indeed if $i = 2j \geq 2n$ is even then

$$s_{2j} \;=\; s_{2n} + \underbrace{(-b_{2n+1} + b_{2n+2})}_{\leq 0} + \dots + \underbrace{(-b_{2j-1} + b_{2j})}_{\leq 0} \;\leq\; s_{2n}$$

by monotonicity, while if $i = 2j+1 > 2n$ is odd then $s_{2j+1} = s_{2j} - b_{2j+1} \leq s_{2j} \leq s_{2n}$.

Similarly if $i = 2j+1 \geq 2n+1$ is odd then

$$s_{2j+1} \;=\; s_{2n+1} + \underbrace{(b_{2n+2} - b_{2n+3})}_{\geq 0} + \dots + \underbrace{(b_{2j} - b_{2j+1})}_{\geq 0} \;\geq\; s_{2n+1},$$

while if $i = 2j+2 > 2n+1$ is even then $s_{2j+2} = s_{2j+1} + b_{2j+2} \geq s_{2j+1} \geq s_{2n+1}$.

The upshot is that $\forall n, m \geq 2N+1$,

$$s_{2N+1} \;\leq\; s_n,\, s_m \;\leq\; s_{2N},$$

and so

$$|s_n - s_m| \;\leq\; s_{2N} - s_{2N+1} \;=\; b_{2N+1}.$$

But $b_n \downarrow 0$ so $\forall \epsilon > 0\;\; \exists N \in \mathbb{N}$ such that $\forall n \geq N,\;\; b_n < \epsilon$. Thus (s_n) is Cauchy. □

Exercise 4.21. What do you think about the infinite sum

$$1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \frac{1}{10} - \dots ?$$

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$
4. Divergent to $-\infty$ ✓
5. Other

If we bracket the finite partial sums as

$$\left(1-\frac{1}{2}\right)-\frac{1}{3}+\left(\frac{1}{4}-\frac{1}{5}\right)-\frac{1}{6}+\left(\frac{1}{7}-\frac{1}{8}\right)-\frac{1}{9}+\left(\frac{1}{10}-\frac{1}{11}\right)-\dots$$

you can show the sum of the bracketed terms converges by the alternating series test where as the remaining terms add to something unboundedly negative. So you can show (exercise!) that the partial sum $\to -\infty$.

Alternatively bracket the partial sums differently as

$$1-\frac{1}{2}+\left(-\frac{1}{3}+\frac{1}{4}\right)-\frac{1}{5}+\left(-\frac{1}{6}+\frac{1}{7}\right)-\frac{1}{8}+\left(-\frac{1}{9}+\frac{1}{10}\right)-\frac{1}{11}+\dots$$

$$< 1-\frac{1}{2}-\frac{1}{5}-\frac{1}{8}-\frac{1}{11}-\dots < 1-\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots\right) \longrightarrow -\infty$$

and turn that into a proof (ex!).

Exercise 4.22. The alternating sequence $a_n = \begin{cases} \frac{1}{n^2}+\frac{1}{n} & n \text{ even,} \\ -\frac{1}{n^2} & n \text{ odd,} \end{cases}$ has sum $\sum a_n$ which is

1. Convergent
2. Divergent but bounded
3. Divergent to $+\infty$ ✓
4. Divergent to $-\infty$
5. Other

It is alternating but $|a_n|$ is **not** monotonically decreasing, so the alternating series test does **not** apply.

It *does* apply to $\sum \frac{(-1)^n}{n^2}$ of course, so this sum converges. Use this, and the fact that the other bit $\sum \frac{1}{n}$ diverges to $+\infty$, to show that $\sum a_n$ diverges to $+\infty$.

Theorem 4.23: Ratio Test

If a_n is a sequence such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$, then $\sum a_n$ is absolutely convergent.

Idea: Expect, eventually, $a_{N+k} \approx a_N r^k$ so that $\sum_{k\geq 0} |a_{N+k}| \approx |a_N| \sum_{k\geq 0} r^k = \frac{|a_N|}{1-r}$. More realistically, bound $|a_{N+k}|$ by $|a_N|(r+\epsilon)^k$, choosing ϵ so that $r+\epsilon < 1$.

Proof. Let $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left|\frac{a_{n+1}}{a_n} - r\right| < \epsilon \implies |a_{n+1}| < (r+\epsilon)|a_n| = \widetilde{r}|a_n|,$$

where we set $\widetilde{r} := r + \epsilon = \frac{1+r}{2} < 1$.

Inductively

$$|a_{N+k}| < \widetilde{r}|a_{N+k-1}| < \ldots < \widetilde{r}^k|a_N|.$$

So, setting $C := \widetilde{r}^{-N}|a_N|$,

$$|a_k| < \widetilde{r}^{k-N}|a_N| = C\widetilde{r}^k \quad \forall k \geq N.$$

Therefore, for $n \geq N$,

$$\sum_{k=N}^{n} |a_k| \leq C\sum_{k=N}^{n} \widetilde{r}^k = \frac{C(\widetilde{r}^N - \widetilde{r}^{n+1})}{1-\widetilde{r}} \leq \frac{C\widetilde{r}^N}{1-\widetilde{r}}$$

since $\widetilde{r} < 1$. So partial sums $\sum_{i=1}^{n} |a_n|$ are monotonically increasing, and bounded above once $n \geq N$ (and therefore for all n). Thus they converge. □

Remark 4.24. The ratio test only applies when a_n decays exponentially in n. But many convergent series like $\sum \frac{1}{n^2}$ do not decay so fast.

Example 4.25. Consider the complex sequence

$$a_n = \frac{100^n(\cos n\theta + i \sin n\theta)}{n!} = \frac{(100e^{i\theta})^n}{n!}.$$

Then

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{100^{n+1}/(n+1)!}{100^n/n!} = \frac{100}{n+1} \longrightarrow 0.$$

So by the ratio test, $\sum a_n$ is absolutely convergent (and so convergent).

Theorem 4.26: Root Test

If $|a_n|^{1/n} \to r < 1$, then $\sum a_n$ is absolutely convergent.

Remark 4.27. Again, writing $|a_n| \to r^n$ does not make sense.

Proof. Fix $\epsilon = \frac{1-r}{2} > 0$. Then $\exists N \in \mathbb{N}$ such that $\forall n \geq N$,

$$\left||a_n|^{1/n} - r\right| < \epsilon \implies |a_n|^{1/n} < r + \epsilon$$

Set $\widetilde{r} := r + \epsilon = \frac{1+r}{2} < 1$, so that $|a_n| < \widetilde{r}^n$, and we can conclude just as in the proof of the Ratio Test. □

4.4 Rearrangement of Series

Warning. Do not rearrange series and sum them in a different order unless you are a professional who knows what you are doing and can *prove* the result is the same.

Without a license you can rearrange partial sums only; they are finite so $a+b = b+a$ makes them behave. Infinite sums are more difficult beasts.

Example 4.28. $\sum(-1)^{n+1} = 1 - 1 + 1 - 1 + \ldots$

either this " = " $(1-1) + (1-1) + \cdots = 0$,

or this " = " $1 - (1-1) + (1-1) + \cdots = 1$.

A better (convergent) example:

Example 4.29. Let $a_n = \frac{(-1)^{n+1}}{n}$ so that $\sum a_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is convergent by the Alternating Series Test.

Exercise 4.30. $\sum_{n=1}^{\infty} a_n > \frac{1}{2}$.

(In fact $\sum a_n = \log 2$ can be seen by substituting $x = 1$ into the Taylor series $\log(1+x) = x - \frac{x^2}{2} + \ldots$ even though $x = 1$ is on its radius of convergence.)

Reorder $\sum a_n$ as follows:

$$\begin{array}{cccccccccc}
 & 1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \ldots \\
 & & -\frac{1}{2} & & -\frac{1}{4} & & -\frac{1}{6} & & \ldots \\
 & & & & & & & & \\
= & 1 & & +\frac{1}{3} & & +\frac{1}{5} & & +\frac{1}{7} & \ldots \\
-\frac{1}{2}\Big[& & 1 & & +\frac{1}{2} & & +\frac{1}{3} & & \ldots\Big]
\end{array}$$

Terms with even denominator appear only in bottom row $(\times\frac{-1}{2})$.

Terms with odd denominator appear in the top row $(\times 1)$ and bottom row $(\times\frac{-1}{2})$, so $(\times\frac{1}{2})$ in total.

So we get $\frac{1}{2}[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots] = \frac{1}{2}\sum a_n$.

Thus reordering the sum can lead to a different result.

This happened because when I reordered I went along the bottom row twice as fast as I went along the top row (check you see this!). Since the top and bottom rows are both series which diverge to ∞, I'm computing $\infty - \infty$, and this can give me anything depending on how quickly I add up the first ∞ and how quickly I take away the second.

In fact we can rearrange $\sum \frac{(-1)^{n+1}}{n}$ to converge to anything we like.

Example 4.31. Rearrange $\sum \dfrac{(-1)^{n+1}}{n}$ to make it converge to your favourite number.

Pick your favourite number; call it π say. Then reorder the sum as follows.

1. Take only odd terms $a_{2n+1} > 0$ until their sum is $> \pi$. We can do this as $1 + \frac{1}{3} + \ldots$ diverges to ∞!
2. Now take only even terms $a_{2n} < 0$ until sum gets $< \pi$.
3. Repeat 1 and 2 to fade.

We can do each step because $\sum a_{2n+1} \to +\infty$ and $\sum a_{2n} \to -\infty$. If we did not eventually use all the terms a_n then we must eventually only take terms of one type (without loss of generality the even terms < 0), but the even terms sum to $-\infty$ so our sum eventually drops below π and we start taking odd terms > 0 again.

Finally we sketch the proof that this reordered sum converges to π. Since $a_n \to 0$,

$$\forall \epsilon > 0 \ \exists N \in \mathbb{N} \text{ such that } n \geq N \implies |a_n| < \epsilon. \qquad (*)$$

Go down the reordered sequence to a point where we have used all $a_1, a_2, \ldots, a_N$, and then further to the point where the partial sum crosses π. At this point, $(*)$ holds, so the sum is within ϵ of π. From this point on the sum is always within ϵ of π by design and by $(*)$. But this is the dictionary definition of the sum converging to π.

Definition (Rearrangement of a sequence) Given a bijection $n \colon \mathbb{N} \to \mathbb{N}$, define $b_i := a_{n(i)}$. Then $(b_i)_{i\geq 1}$ is a *rearrangement* or *reordering* of $(a_n)_{n\geq 1}$.

Then the method of Example 4.31 shows that if (a_n) is any sequence such that

- $a_n \to 0$,
- $\displaystyle\sum_{n\colon a_n \geq 0} a_n \to +\infty$,
- $\displaystyle\sum_{n\colon a_n < 0} a_n \to -\infty$,

then we can rearrange the series $\sum a_n$ to make it converge to *any* real number we like by the algorithm above.

And we can make it diverge to $+\infty$ or to $-\infty$.
For instance, here's an algorithm to make it diverge to $+\infty$:

1. Pick only $a_n \geq 0$ terms until the partial sum is > 1,
2. Now pick only $a_n < 0$ terms until the partial sum is < 0,
3. Pick only $a_n \geq 0$ terms until the partial sum is > 2,
4. Now pick only $a_n < 0$ terms until the partial sum is < 1,

$$\vdots$$

$2k-1$. Pick only $a_n \geq 0$ terms until the partial sum is $> k$,

$2k$. Now pick only $a_n < 0$ terms until the partial sum is $< k - 1$,

$$\vdots$$

> **Exercise 4.32.** Show this is a reordering and the sum diverges to $+\infty$.

> **Exercise 4.33.** If (a_n) is a sequence such that
>
> - $a_n \to 0$,
> - $\displaystyle\sum_{n\colon a_n \geq 0} a_n \to +\infty$,
> - $\displaystyle\sum_{n\colon a_n < 0} a_n$ converges,
>
> then any reordering of $\sum a_n$ will diverge to $+\infty$.

The "good case" is when

- $\displaystyle\sum_{n\colon a_n \geq 0} a_n$ converges,
- $\displaystyle\sum_{n\colon a_n < 0} a_n$ converges,

which imply $a_n \to 0$ of course. Together these are equivalent to $\sum_n a_n$ being absolutely convergent, and in this case any reordering will give the same sum.

Theorem 4.34

$\sum a_n$ is absolutely convergent $\iff$ (1) + (2) $\implies$ (3) + (4), where

(1) $\sum_{a_n \geq 0} a_n$ is convergent (to A say),

(2) $\sum_{a_n < 0} a_n$ is convergent (to B say),

(3) $\sum a_n = A + B$,

(4) $\sum b_m = A + B$ where (b_m) is any rearrangement of (a_n).

Idea: $\sum |a_n|$ is convergent so has a significant finite part and then a small "insignificant" tail. Any reordering covers all the finite part after finitely many terms, and then all that remains is insignificant: just a reordering of part of the tail.

Proof. Let $p_1, p_2, p_3, \ldots$ be the nonnegative $a_n \geq 0$ (so p_i is the ith nonnegative element of the sequence (a_n)).

Similarly let $n_1, n_2, n_3, \ldots$ be the negative $a_n < 0$.

Suppose $\sum a_n$ is absolutely convergent, and set $R := \sum_n |a_n|$. For any $n \in \mathbb{N}$ the partial sum of the p_i satisfies

$$\sum_{i=1}^{n} p_i \;\leq\; \sum_{i=1}^{N} |a_i| \;\leq\; R,$$

for any N sufficiently large that $\{p_1, \ldots, p_n\} \subseteq \{a_1, \ldots, a_N\}$. Therefore the partial sums of the p_i are monotonically increasing, bounded above and so convergent (to A say), proving (1).

Similarly the partial sums of the n_i are monotonically decreasing, bounded below and so convergent (to B say), proving (2).

So if we fix any $\epsilon > 0$, then

$$\exists\, N_1 \text{ such that } n \geq N_1 \implies A - \epsilon < \sum_{i=1}^{n} p_i \leq A, \qquad \text{(A)}$$

$$\exists\, N_2 \text{ such that } n \geq N_2 \implies B < \sum_{i=1}^{n} n_i < B + \epsilon. \qquad \text{(B)}$$

In particular, by monotonicity,

$$0 \leq \sum_{i \in I} p_i < \epsilon \quad \text{for any } I \subset \{N_1 + 1,\, N_1 + 2,\, \ldots\}, \qquad \text{(C)}$$

$$-\epsilon \;<\; \sum_{j\in J} n_j \;<\; 0 \quad \text{for any } J\subset\{N_2+1,\, N_2+2,\, \dots\}. \qquad \text{(D)}$$

Using (A-D) we next show that any rearrangement (b_m) of (a_n) sums to $A+B$. This will prove (3) and (4).

Pick N is sufficiently large that both $\{p_1,\dots,p_{N_1}\}$ and $\{n_1,\dots,n_{N_2}\}$ are subsets of $\{b_1,\dots,b_N\}$. (I.e. go far enough down the sequence (b_m) that we've included all the "significant" p_i and n_j.) Then write the complement as $\{p_i\}_{i\in I}\cup\{n_j\}_{j\in J}$, where I is a set of indices $> N_1$ and J is a set of indices $> N_2$.

Hence $\forall n\geq N$,

$$\begin{aligned}
\left|\sum_{i=1}^{n} b_i \;-\; (A+B)\right| \;&=\; \left|\sum_{i=1}^{N_1} p_i \;-A+\; \sum_{j=1}^{N_2} n_j \;-B \;+\; \sum_{i\in I} p_i \;+\; \sum_{j\in J} n_j\right| \\
&\leq\; \left|\sum_{i=1}^{N_1} p_i \;-A\right| \;+\; \left|\sum_{j=1}^{N_2} n_j \;-B\right| + \sum_{i\in I} p_i \;+\; \sum_{j\in J} |n_j| \\
&<\; \epsilon+\epsilon+\epsilon+\epsilon
\end{aligned}$$

by (A), (B), (C) and (D) respectively.

Finally we prove that (1)+(2) $\Longrightarrow$ $\sum|a_n|$ is convergent. We fix $\epsilon>0$ and use the same N_1, N_2, N as above so that $\forall n\geq N$, $\{a_1,\dots,a_n\}$ contains both $\{p_1,\dots,p_{N_1}\}$ and $\{n_1,\dots,n_{N_2}\}$. Therefore

$$\sum_{i=1}^{n}|a_i| \;=\; \sum_{i=1}^{N'} p_i \;-\; \sum_{i=1}^{N''} n_i,$$

where $N'\geq N_1$ and $N''\geq N_2$. Applying (A) and (B) to the RHS then gives

$$(A-\epsilon)-(B+\epsilon) \;<\; \sum_{i=1}^{n}|a_i| \;<\; A-B,$$

so $\sum|a_i|$ converges to $A-B$. □

4.5 Power Series

Let $[0,\infty]$ denote the set $[0,\infty)\cup\{+\infty\}$.

Theorem 4.35: Radius of Convergence

Fix a real or complex series (a_n) and consider the series $\sum a_n z^n$ for $z \in \mathbb{C}$.

Then $\exists R \in [0, \infty]$ such that

- $|z| < R \implies \sum a_n z^n$ is absolutely convergent, and
- $|z| > R \implies \sum a_n z^n$ is divergent.

Proof. Let $S = \{|z| : a_n z^n \to 0\}$, nonempty since $0 \in S$. Then define

$$R = \begin{cases} \sup S & \text{if } S \text{ bounded,} \\ \infty & \text{if } S \text{ unbounded.} \end{cases}$$

Now suppose $|z| < R$. Since $|z|$ not an upper bound for S there exists w such that $|w| > |z|$ and $a_n w^n \to 0$. In particular $|a_n w^n|$ is bounded by some A for all n. Thus

$$|a_n z^n| = |a_n w^n| \left|\frac{z}{w}\right|^n \leq A \left|\frac{z}{w}\right|^n.$$

Therefore by comparison with the convergent series $\sum \left|\frac{z}{w}\right|^n$ (recall $\left|\frac{z}{w}\right| < 1$) we find $\sum |a_n z^n|$ is convergent.

On the other hand, if $|z| > R$ then $a_n z^n \not\to 0$ as $n \to \infty \implies \sum a_n z^n$ diverges. □

Notice how simple this was. If $|a_n w^n|$ is just bounded (so nowhere near convergent!) then $\sum |a_n z^n|$ is convergent for $|z| < |w|$ because $\left(\frac{|z|}{|w|}\right)^n$ decays exponentially as $n \to \infty$.

Remark 4.36. R is called the *radius of convergence* of $\sum a_n z^n$. Note we are not saying anything about its behaviour on the circle $|z| = R$.

Exercise 4.37. Consider the sequences

(a) $a_n = \frac{1}{n^2}$,

(b) $a_n = \frac{1}{n}$,

(c) $a_n = 1$.

Show their power series $\sum a_n z^n$ all have radius of convergence $R = 1$, and on $|z| = 1$ their behaviour is as follows,

(a) convergent everywhere on $|z| = 1$,
(Absolutely convergent because $\sum \frac{1}{n^2} < \infty$.)

(b) convergent somewhere,
(Convergent at $z=-1$ by alternating series test, not convergent at $z=1$.)

(c) convergent nowhere on $|z|=1$.
($a_n z^n \not\to 0$ as $n\to\infty$.)

The exponential-in-n behaviour of z^n makes the **ratio test** particularly useful for testing convergence of power series, for instance readily giving the following.

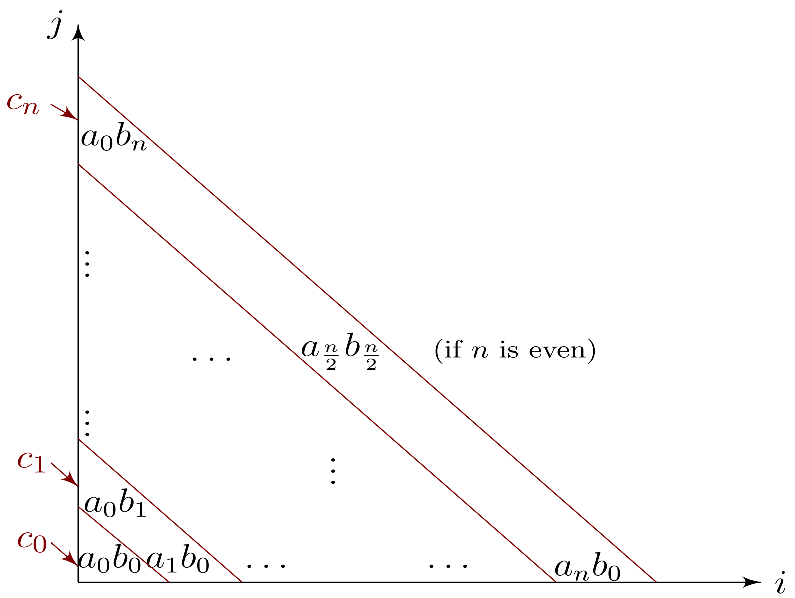
Exercise 4.38. Suppose $\left|\frac{a_{n+1}}{a_n}\right| \to a \in [0,\infty]$ as $n\to\infty$.
Then $R=\frac{1}{a}$ is the radius of convergence of $\sum a_n z^n$.

4.5.1 Products of Series

Consider

$$\begin{aligned}\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n &= (a_0 + a_1 z + a_2 z^2 + \dots)(b_0 + b_1 z + b_2 z^2 + \dots) \\ \text{“}=\text{”} &\; a_0 b_0 + (a_0 b_1 + a_1 b_0) z + (a_0 b_2 + a_1 b_1 + a_2 b_0) z^2 + \dots \\ &= \sum_{n=0}^{\infty} c_n z^n,\end{aligned}$$

where $c_0 = a_0 b_0,\ c_1 = a_0 b_1 + a_1 b + 0, \dots,\ c_n = a_0 b_n + a_1 b_{n-1} + \dots + a_n b_0$.



So we set $c_n = \sum_{i=0}^{n} a_i b_{n-i}$ and ask when is the product $\sum a_n z^n \sum b_n z^n$ equal to $\sum c_n z^n$? We can also do this without the z^ns:

Definition. Given series $\sum a_n$, $\sum b_n$ their *Cauchy Product* is the series $\sum c_n$ where $c_n := \sum_{i=0}^{n} a_i b_{n-i}$.

Notice we used power series to motivate this definition; it is not the only way we could collect all the terms $a_i b_j$ to turn $\sum a_i \sum b_j$ into a single sum. This is why we give it the specific name "Cauchy product".

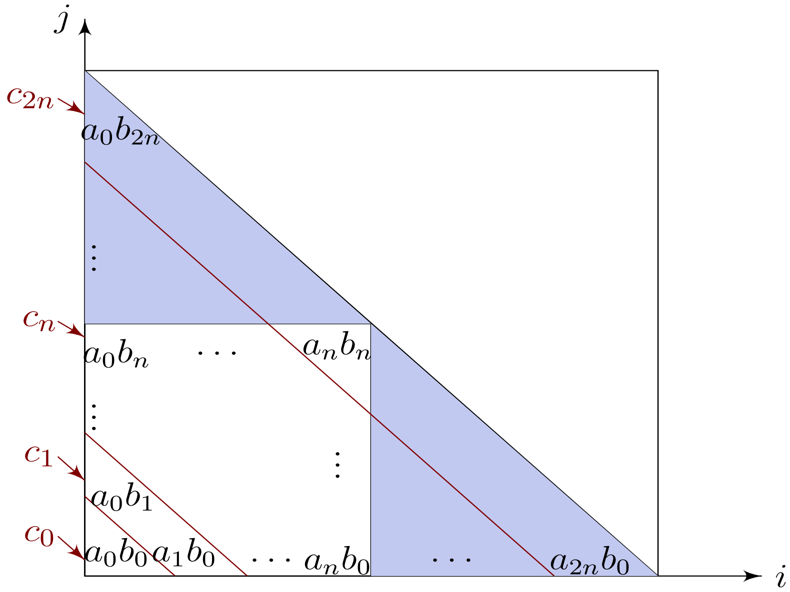
Theorem 4.39: Cauchy Product

If $\sum a_n, \sum b_n$ are absolutely convergent, then their Cauchy product $\sum c_n$ is absolutely convergent to $\left(\sum a_n\right) \cdot \left(\sum b_n\right)$.

Proof. (Non-examinable.) We try to control

$$\sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_i b_j.$$

The first term is the sum of $a_i b_j$ over all (i, j) below the diagonal in the diagram below. The second term is the sum over the small square. Therefore the difference is the sum of $a_i b_j$ over (i, j) in the two shaded triangles.



By the triangle inequality

$$\left| \sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_i b_j \right| \leq \sum |a_i b_j|,$$

where the right hand sum is over i, j in (2 shaded triangles) $\subset$ (big square minus small square). Thus it is less than the sum over (big square minus small square),

$$\left|\sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_i b_j\right| \leq \sum_{i=0}^{2n}\sum_{j=0}^{2n} |a_i b_j| - \sum_{i=0}^{n}\sum_{j=0}^{n} |a_i b_j|. \tag{1}$$

Now we're in good shape because we're summing over the complement of the small square, i.e. we're in the tail of at least one of $\sum a_n$ or $\sum b_n$, and these are (absolutely) small. Since the partial sums $\sum_{i=0}^n |a_i|$ and $\sum_{j=0}^n |b_j|$ converge, their product $\sum_{i,j=0}^n |a_i b_j|$ also converges by the Algebra of limits for sequences (Theorem 3.19). In particular it defines a Cauchy sequence; fixing $\epsilon > 0$, there exists N_1 such that

$$m \geq n \geq N_i \implies \sum_{i,j=0}^{m} |a_i b_j| - \sum_{i,j=0}^{n} |a_i b_j| < \epsilon.$$

Taking $m = 2n$ and substituting into (1) gives us

$$n \geq N_1 \implies \left|\sum_{i=0}^{2n} c_i - \sum_{i,j=0}^{n} a_j b_j\right| \leq \epsilon. \tag{2}$$

Now we know that the partial sums $\sum_{i=0}^n a_i \to A$ and $\sum_{j=0}^n b_j \to B$, so by the Algebra of limits again,

$$\sum_{i=0}^{n} a_i \sum_{j=0}^{n} b_j \longrightarrow AB.$$

This means that $\exists N_2$ such that

$$n \geq N_2 \implies \left|\sum_{i,j=0}^{n} a_i b_j - AB\right| < \epsilon.$$

Combined with (2) and the triangle inequality this gives

$$\left|\sum_{i=0}^{2n} c_i - AB\right| < 2\epsilon$$

for all $n \geq \max(N_1, N_2)$.

We can deal with $\sum_{i=0}^{2n+1} c_i$ in the same way by sandwiching it between the squares $0 \leq i, j \leq n$ and $0 \leq i, j \leq 2n+1$. The upshot is that $\exists N$ such that for all $k \geq N$,

$$\left|\sum_{i=0}^{k} c_i - AB\right| < 2\epsilon.$$

Thus $\sum_{i=0}^{k} c_i \to AB$. Finally to prove that $\sum c_n$ is absolutely convergent, just replace a_n, b_n by $|a_n|, |b_n|$ in the above proof. $\square$

Corollary 4.40. *If $\sum a_n z^n$ and $\sum b_n z^n$ have radius of convergence R_A and R_B respectively, then $\sum c_n z^n$ has radius of convergence $R_C \geq \min\{R_A, R_B\}$.*

Proof. By Theorem 4.39, for any $|z| < \min\{R_A, R_B\}$ we have $\sum a_n z^n$ and $\sum b_n z^n$ absolutely convergent $\implies \sum c_n z^n$ absolutely convergent $\implies |z| \leq R_C$. $\square$

Exercise 4.41. Fix $\alpha, \beta \in \mathbb{R}$. Prove that if $\left[x < \alpha \implies x \leq \beta\right]$ then $\alpha \leq \beta$.

Proof. If $\alpha > \beta$ then let $x := \frac{1}{2}(\alpha + \beta)$ so that $\beta < x < \alpha$ ※ $\square$

Example 4.42. $\sum z^n$ has $R_A = 1$.
$1 - z$ has $R_B = \infty$.
So their Cauchy product $\sum c_n z^n$ has $R_C \geq 1$.

Exercise: Check $c_0 = 1, c_n = 0\ \forall n \geq 1$, so the Cauchy product is 1 and in fact $R_C = \infty$.

Nonetheless, we can only say that $\sum c_n z^n = 1 = \left(\sum z^n\right)(1 - z)$ when $|z| < 1 = \min(R_A, R_B)$.

4.6 Exponential Power Series

Definition (Exponential Series) For any $z \in \mathbb{C}$ set

$$E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

For any *fixed* $z \in \mathbb{C}$ the ratio test gives

$$\frac{|z^{n+1}|/(n+1)!}{|z^n|/n!} = \left|\frac{z}{n+1}\right| \longrightarrow 0 \text{ as } n \to \infty,$$

so $E(z)$ is absolutely convergent $\forall z \in \mathbb{C}$.

Proposition 4.43. *$E(z)E(w) = E(z+w)$.*

Proof. By the Cauchy product Theorem 4.39 $E(z)E(w) = \sum_{n=0}^{\infty} c_n$, where

$$c_n = \sum_{i=0}^{n} \frac{z^i}{i!} \frac{w^{n-i}}{(n-i)!} = \frac{(z+w)^n}{n!}.$$

□

Corollary 4.44. *$E(z) \neq 0$ and $\dfrac{1}{E(z)} = E(-z)$.*

Proof. $E(z)E(-z) = E(0) = 1$. □

Definition. $e := E(1) = \sum \frac{1}{n!} \in (0, \infty)$.

You will prove on the problem sheet that e is *irrational.*

Corollary 4.45. *$E(n) = e^n$ for $n \in \mathbb{N}$.*

Proof. $E(n) = E\big(1 + (n-1)\big) = E(1)E(n-1) = \cdots = (E(1))^n$. □

Proposition 4.46. *$E(q) = e^q$ for $q \in \mathbb{Q}$ (where e^q is defined using the rational powers of $a \in \mathbb{R}$ introduced on Question sheet 2).*

Proof. For $q > 0$ we write $q = \frac{m}{n},\ m, n \in \mathbb{N}$. Then

$$E(q) \;=\; E\Big(\underbrace{\tfrac{1}{n} + \cdots + \tfrac{1}{n}}_{m \text{ times}}\Big) \;=\; E\left(\tfrac{1}{n}\right)^m \;=\; \Big(E\left(\tfrac{1}{n}\right)^n\Big)^{\frac{m}{n}} \;=\; E(1)^{\frac{m}{n}} \;=\; e^q.$$

For $q < 0$ write $q = \frac{-m}{n}$ then $E(q) = \frac{1}{E(m/n)} = \frac{1}{e^{m/n}} = e^{-m/n} = e^q$. □

So we know that $E(x) = e^x\ \forall x \in \mathbb{Q}$. Later we define real powers from rational powers by continuity. We will show both e^x and $E(x)$ are continuous, and since they're equal $\forall x \in \mathbb{Q}$ it will follow that they're equal $\forall x \in \mathbb{R}$.

Proposition 4.47. *$E(x)$ has the following properties for $x \in \mathbb{R}$.*

1. *$E(x) > 0\ \ \forall x \in \mathbb{R}$,*
2. *$x \geq 0 \Longrightarrow E(x) \geq 1$ and $x > 0 \Longrightarrow E(x) > 1$,*
3. *$E(x)$ is strictly increasing for $x \in \mathbb{R}$,*
4. *$|E(x) - 1| \leq \frac{|x|}{1-|x|}$ for $|x| < 1$,*
5. *$x \mapsto E(x)$ is a continuous bijection $\mathbb{R} \xrightarrow{\sim} (0, \infty)$ (proved later).*

Proof. 1 and 2 are obvious from the series definition of $E(x)$. For 3 notice that $E(y) = E(x)E(y - x)$, which is $> E(x).1$ when $y > x$. For 4 we bound

$$|E(x) - 1| \;=\; \left|\sum_{n=1}^{\infty} \frac{x^n}{n!}\right| \;\leq\; \sum_{n=1}^{\infty} |x|^n \;=\; \frac{|x|}{1 - |x|} \quad \text{for } |x| < 1.$$ □

Point 5 enables us to define

$$\log\colon (0, \infty) \xrightarrow{\sim} \mathbb{R}$$

as the inverse to E, i.e. $y = \log x \iff x = e^y$. The usual properties (such as $\log xy = \log x + \log y$) follow from the corresponding properties for E (like $E(x+y) = E(x)E(y)$).

This then further allows us to define a^x for $a \in (0, \infty)$ and $x \in \mathbb{R}$ by

$$a^x \;=\; E(x \log a).$$

> **Exercise 4.48.** When $x \in \mathbb{Q}$ this definition agrees with the definition of a^x on Question sheet 2.

Finally this also allows us to define the trigonometric functions by

$$\cos\theta \;:=\; \operatorname{Re} E(i\theta), \quad \sin\theta \;:=\; \operatorname{Im} E(i\theta).$$

Exercise 4.49. Using these definitions, carefully derive the power series for sin, cos.

What does $E(i\theta + i\phi) = E(i\theta)E(i\phi)$ imply for sin, cos?

5 Continuity

I have no special talent. I am only passionately curious.

- Albert Einstein, 1952

5.1 Limits

Given a discrete family of real numbers (a_n) parametrised by $\mathbb{N} \ni n$, we now know how to take a limit.

Put another way, if we have a function $a\colon \mathbb{N} \to \mathbb{R}$ we know how to define $\lim\limits_{n\to\infty} a(n)$.

Given an uncountable family of real numbers (f_x) parameterised by $\mathbb{R} \ni x$, can we find a limit as $x \to a$?

Put another way, if we have a function $f\colon \mathbb{R} \to \mathbb{R}$ how do we define $\lim\limits_{x\to a} f(x)$?

Definition. Fix a function $f\colon \mathbb{R} \to \mathbb{R}$ and points $a, b \in \mathbb{R}$.

We say that $f(x) \to b$ as $x \to a$ (or "$\lim_{x\to a} f(x) = b$") if and only if

$$\boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } 0 < |x-a| < \delta \implies |f(x)-b| < \epsilon}$$

I.e. x close to a (but not equal!) $\implies f(x)$ close to b.

More precisely, *however close* ($\forall \epsilon > 0$) *I want* $f(x)$ *to be to* b*, there's a distance* ($\exists\, \delta > 0$) *from* a *such that, inside that distance* ($0 < |x-a| < \delta$) $f(x)$ *is indeed that close to* b ($|f(x)-b| < \epsilon$).

We exclude $x = a$ so that we can compare $\lim_{x\to a} f(x)$ to $f(a)$. I.e. if we allowed $x = a$ we could only test whether $\lim_{x\to a} f(x) = f(a)$, which is slightly less general. For instance,

Exercise 5.1. $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$

Then $\lim_{x\to 0} f(x)$ is

1. 0 ✓
2. 1
3. Both 0 and 1

4. Nonexistent

(So we can talk about $\lim_{x\to a} f(x)$ for $f : \mathbb{R}\backslash\{a\} \to \mathbb{R}$, because we don't need an $f(a)$ in the definition.)

Exercise 5.2. Limits are unique when they exist. I.e. if $f(x) \to b$ and $f(x) \to c$ as $x \to a$ then $b = c$.

Definition (Unconventional) Fix a function $f \colon \mathbb{R} \to \mathbb{R}$ and a point $a \in \mathbb{R}$.

We say that f is **continuous** at a if and only if $\boxed{\lim_{x\to a} f(x) \;=\; f(a)}$

Theorem 5.3

$f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if and only if

$$\boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon}$$

Proof. We are asked to show that

$$\boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } 0 < |x - a| < \delta \implies |f(x) - f(a)| < \epsilon}$$

$$\iff \boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon}$$

Notice $\Longleftarrow$ is obvious. And $\Longrightarrow$ is nearly as obvious because the first box it only misses out the case $0 = |x - a|$, i.e. $x = a$, and when this holds $|f(x) - f(a)| = 0$ so it is certainly $< \epsilon$. ☐

5.2 Continuity

So that leads to the more conventional definition of continuity. (Use this one in exams!) By Theorem 5.3 it is entirely equivalent to the previous one.

Definition. Given a function $f : \mathbb{R} \to \mathbb{R}$, we say that f is *continuous at* $a \in \mathbb{R}$ if and only if

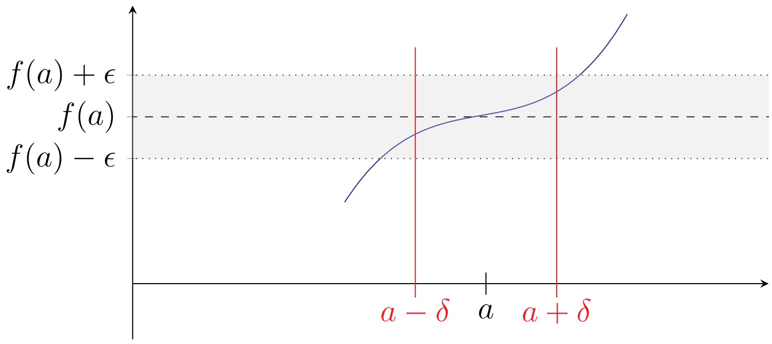
$$\boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } |x - a| < \delta \implies |f(x) - f(a)| < \epsilon}$$

We say that f is continuous on $\mathbb{R}$ (or just "continuous") if it is continuous at all $a \in \mathbb{R}$.

Notice δ depends on ϵ (and a)!

"Once x is close to a, then $f(x)$ is close to $f(a)$".

More precisely, *however close (i.e. within ϵ) I want $f(x)$ to be to $f(a)$, I can arrange it by taking x close (i.e. within δ) to a.*



Equivalently: $\boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } |f(x) - f(a)| < \epsilon\ \forall x \in (a-\delta, a+\delta)}$

Or: $\boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } f(a-\delta, a+\delta) \subseteq (f(a)-\epsilon, f(a)+\epsilon)}$

(Here we use the notation $f(S)$ for the set $\{f(x) : x \in S\}$ whenever $S \subseteq \mathbb{R}$.)

Or: $\boxed{\forall \epsilon > 0\ \exists\, \delta > 0 \text{ such that } f^{-1}(f(a)-\epsilon, f(a)+\epsilon) \supseteq (a-\delta, a+\delta)}$

(Here we use the notation $f^{-1}(T)$ for the set $\{x \in \mathbb{R} : f(x) \in T\}$ whenever $T \subseteq \mathbb{R}$. Notice we do not need an inverse function f^{-1} to exist to define $f^{-1}(T)$.)

We have already seen a simple "jump discontinuity"; let's take another and prove it is discontinuous using the conventional definition.

Example 5.4.

$$f(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$$

Then f is not continuous at $x = 0$.

Proof. Take $\epsilon = 1$ (or $0 < \epsilon < 1$).
If f is continuous at $x = 0$ then $\exists\, \delta > 0$ such that $|f(x) - f(0)| < 1\ \forall x \in (-\delta, \delta)$.

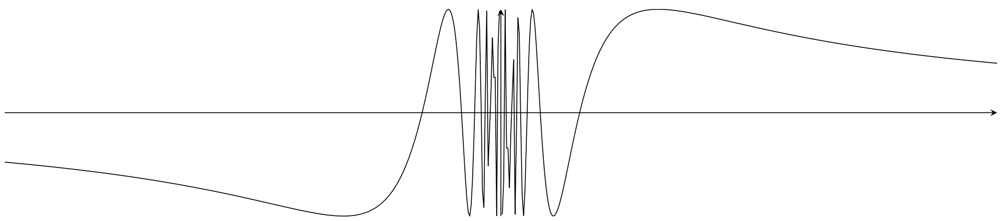
Taking $x = \delta/2$ gives $|1 - 0| < 1$ ※ □

There is another important type of discontinuity, which has no "jump".

Example 5.5.

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & x \neq 0 \\ r & x = 0 \end{cases}$$

Then f is discontinuous at $x = 0$ (for any r).



Idea of proof: If f is continuous at $x = 0$, then $f(x) \in (r - \epsilon, r + \epsilon)$ is close to $f(0) = r$ for $x \in (-\delta, \delta)$. In particular, $f(x)$ and $f(y)$ are close to each other (within 2ϵ). But $f(x)$ could be $+1$ and $f(y)$ could be -1 ※

Proof. Fix $\epsilon \in (0, 1]$. If f is continuous at 0, then

$$\exists\, \delta > 0 \quad \text{such that} \quad |f(x) - f(0)| \;<\; \epsilon \;\; \forall x \in (-\delta, \delta).$$

In particular, $\forall x, y \in (-\delta, \delta)$ we have $|f(x) - f(y)| < 2\epsilon \leq 2$, by the triangle inequality.

Now choose $n \in \mathbb{N}$, $n > \frac{1}{\delta}$ and $x = \frac{1}{(4n+1)\pi/2}$, $y = \frac{1}{(4n+3)\pi/2}$ so that both $x, y \in (0, \delta)$. But

$$\left| \sin\frac{1}{x} - \sin\frac{1}{y} \right| \;=\; |1 - (-1)| \;=\; 2 \quad ※$$

□

Example 5.6. $f : \mathbb{R} \to \mathbb{R}$, $f = mx + c$ is continuous.

Rough working: We want, at *any* $a \in \mathbb{R}$,

$$\begin{aligned} |f(x) - f(a)| < \epsilon \iff & |(mx + c) - (ma + c)| \ < \ \epsilon \\ \iff & |m(x - a)| \ < \ \epsilon \\ \iff & |x - a| \ < \ \frac{\epsilon}{|m|} \quad \text{if } m \neq 0 \\ \Longleftarrow & |x - a| \ < \ \frac{\epsilon}{|m| + 1}. \end{aligned}$$

So set $\delta := \epsilon/(1 + |m|)$. Then $|x - a| < \delta \implies |f(x) - f(a)| < \epsilon$.

Proof. Fix any $a \in \mathbb{R}$ and $\epsilon > 0$. Set $\delta := \frac{\epsilon}{1+|m|} > 0$. Then for $|x - a| < \delta$,

$$\begin{aligned} |(mx + c) - (ma + c)| \ &= \ |f(x) - f(a)| \\ &= \ |m||x - a| \\ &< \ |m|\delta \ = \ \epsilon\frac{|m|}{|m| + 1} \ < \ \epsilon. \end{aligned}$$

□

Example 5.7. $f : \mathbb{R} \to \mathbb{R}, \ f(x) = x^2$ is continuous.

Rough working:

$$|f(x) - f(a)| \ = \ |x^2 - a^2| \ = \ |x + a||x - a|$$

we want this to be $< \epsilon$, i.e. $|x - a| < \frac{\epsilon}{|x+a|}$ $(*)$. But we can't let δ depend on x!!

Problem: If $|x - a| < \frac{\epsilon}{R} \ \forall R > 0$ then $|x - a| = 0$.

Solution: We only care about x close to a; within 1 say.

So, so long as I choose $\delta \leq 1$, then I know that

$$|x - a| \ < \ \delta \ \implies \ |x + a| \ \leq \ |x - a| + 2|a| \ \leq \ 1 + 2|a|.$$

So now $|x - a| < \dfrac{\epsilon}{1 + 2|a|} \implies (*)$.

So to ensure both conditions we set $\delta = \min\{1, \epsilon/(1 + 2|a|)\}$.

Proof. Fix $a \in \mathbb{R}$ and $\epsilon > 0$. Set $\delta = \min\left\{1, \frac{\epsilon}{1+2|a|}\right\} > 0$.

Then $|x - a| < \delta$ implies:

1. $|x-a|<1 \implies |x+a|<1+2|a|$, and
2. $|x-a|<\dfrac{\epsilon}{1+2|a|}$.

Therefore

$$|x^2-a^2| \;=\; |x-a||x+a| \;<\; \frac{\epsilon}{1+2|a|}\cdot(1+2|a|) \;=\; \epsilon. \qquad \square$$

Exercise 5.8. Q: A student attempts to prove that

$$f(x)=\begin{cases}\frac{1}{x} & x\neq 0\\ 0 & x=0\end{cases}$$

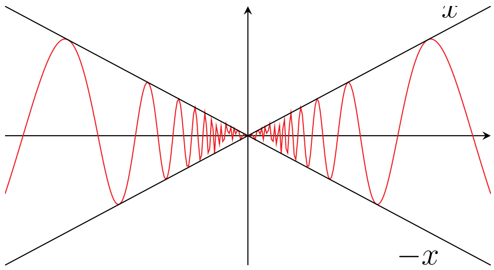
is discontinuous at $x=0$. *Where does the proof first go wrong?*

1. Suppose for a contradiction that f is continuous at 0.
2. Then $\forall\epsilon>0,\ \exists\,\delta>0$ such that
3. $|x|<\delta \implies |f(x)-f(0)|=\left|\frac{1}{x}\right|<\epsilon$.
4. Thus $\left|\frac{1}{x/2}\right|=\left|\frac{2}{x}\right|<2\epsilon$. $\quad(*)$
5. But $|x|<\delta \implies \left|\frac{x}{2}\right|<\delta$.
6. So we should get that $|f(x/2)-f(0)|=\left|\frac{1}{x/2}\right|=\left|\frac{2}{x}\right|<\epsilon$.
7. This contradicts $(*)$. ✓
8. So f is not continuous at 0.
9. Nothing wrong, a correct proof.

Answer: (7) is the problem. (6) implies (4), it doesn't contradict it.

A better proof would have been to use (1),(2),(3) and then finish off by taking $x=\min\left(\frac{\delta}{2},\frac{1}{\epsilon}\right)$. Then $|x|\le\frac{\delta}{2}<\delta$ but also $|x|\le\frac{1}{\epsilon} \implies \left|\frac{1}{x}\right|\ge\epsilon$, contradicting (3).

Example 5.9. $f(x)=\begin{cases}x\sin\frac{1}{x}, & x\neq 0,\\ 0, & x=0.\end{cases}$



Claim: f is continuous at 0.

Proof. Fix $\epsilon > 0$. Then

$$|f(x) - f(0) \;=\; \left|x \sin \frac{1}{x}\right| \;\leq\; |x|.$$

Take $\delta = \epsilon$. Then $|x| < \delta \implies |x| < \epsilon \implies |f(x) - f(0)| < \epsilon$. □

Proposition 5.10. *$E : \mathbb{C} \to \mathbb{C}$ defined by $E(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is continuous on $\mathbb{C}$ (i.e. continuous at a, $\forall a \in \mathbb{C}$).*

Exercise: From this show that $x \mapsto \sin x$ is continuous on $\mathbb{R}$.

Rough working:

$$\begin{aligned} |E(z) - E(a)| &= |E(a)(E(z-a) - E(0))| \\ &= |E(a)||E(z-a) - 1| \\ &\leq |E(a)| \cdot \frac{|z-a|}{1 - |z-a|} \end{aligned}$$

for $|z - a| < 1$ (using results from Propositions 4.43 and 4.47).

We want this to be

$$\begin{aligned} < \epsilon \iff |z-a| &< \frac{\epsilon}{|E(a)|}(1 - |z-a|) \\ \iff \left(1 + \frac{\epsilon}{|E(a)|}\right)|z-a| &< \frac{\epsilon}{|E(a)|} \\ \iff |z-a| &< \frac{\epsilon}{|E(a)|(1 + \epsilon/|E(a)|)} = \frac{\epsilon}{|E(a)| + \epsilon}. \end{aligned}$$

Proof. Fix $\epsilon > 0$. Set $\delta = \dfrac{\epsilon}{|E(a)| + \epsilon} > 0$. $(*)$

Then we calculate that

$$\begin{aligned}|E(z)-E(a)| &\leq |E(a)|\frac{|z-a|}{1-|z-a|} \\ &< |E(a)|\cdot\frac{\delta}{1-\delta}\end{aligned}$$

for all z with $|z-a|<\delta$. But by $(*)$, $\dfrac{\delta}{1-\delta}=\dfrac{\epsilon}{|E(a)|}$. So

$$|z-a| < \delta \implies |E(z)-E(a)| < \epsilon.$$

□

Theorem 5.11

$f:\mathbb{R}\to\mathbb{R}$ is continuous at $a\in\mathbb{R} \iff f(x_n)\to f(a)\ \forall$ sequences $x_n\to a$.

In one direction this is somewhat easy: if $x_n\to a$ and f is continuous at a, then $f(x_n)$ gets close to $f(a)$ as x_n gets close to $a \implies f(x_n)\to f(a)$.

The converse is *much harder.* If I want to see if f is continuous, I can test with a sequence $x_n\to a$ to see if $f(x_n)$ if close to $f(a)$ when n is large. But the x_ns don't cover all xs in $(-\delta,\delta)$! Because the x_ns are countable and $(-\delta,\delta)$ is uncountable. But if I use *all* sequences $x_n\to a$ then I do cover all x and get a theorem.

Proof. Suppose f is continuous at a and fix $\epsilon>0$. Then $\exists\,\delta>0$ such that

$$|x-a| < \delta \implies |f(x)-f(a)| < \epsilon.$$

Now $x_n\to a$ so $\exists N\in\mathbb{N}$ such that

$$\begin{aligned}n\geq N &\implies |x_n-a| < \delta \\ &\implies |f(x_n)-f(a)| < \epsilon,\end{aligned}$$

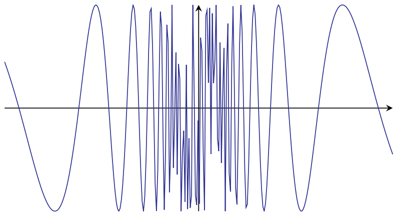
i.e. $f(x_n)\to f(a)$ as required.

Conversely, we suppose $f(x_n)\to f(a)$ for all sequences $x_n\to a$ and – for a contradiction – that f is not continuous at $a\in\mathbb{R}$.

Then $\exists\,\epsilon>0$ such that $\forall\delta>0,\ \exists x\in(a-\delta,a+\delta)$ such that $|f(x)-f(a)|\geq\epsilon$.

Choose $\delta=\frac{1}{n}$. Then $\exists\,x_n\in(a-\frac{1}{n},a+\frac{1}{n})$ such that $|f(x_n)-f(a)|\geq\epsilon$. So $|x_n-a|<\frac{1}{n}\ \forall n$ and therefore $x_n\to a$. But $f(x_n)\not\to f(a)$ ⚔ □

Example 5.12. $f(x) = \begin{cases} \sin 1/x, & x \neq 0, \\ 0, & x = 0. \end{cases}$



This is *not* continuous at 0. But if we take $x_n = \frac{1}{n\pi} \to 0$ then $f(x_n) = \sin(n\pi) = 0\ \forall n$, so $f(x_n) \to f(0)$. So this sequence does not detect the discontinuity.

We have to choose a different sequence such as $x_n = \frac{2}{\pi}, \frac{2}{3\pi}, \frac{2}{5\pi}, \dots$ giving

$$\sin(1/x_n) = (-1)^{n+1} \not\to 0 = f(0)$$

Therefore f is discontinuous at 0.

Avoiding this problem of sequences not covering the whole of an interval $(a-\delta, a+\delta)$ (so having to consider all sequences at once – nasty) was why we introduced the notion of $\lim_{x\to a} f(x)$, allowing x to run through all of $\mathbb{R}$ (instead of only countably many values).