

Math40002 Analysis 1

Problem Sheet 1

1. What is the biggest element of the set $\{x \in \mathbb{R}: x < 1\}$? Give a careful proof.
2. Prove that for every positive integer $n \neq 3$, the number $\sqrt{n} - \sqrt{3}$ is irrational.
3. * Show that any positive *eventually periodic* decimal expansion is rational, and in fact can be written as the fraction

$$p/99 \dots 9900 \dots 00 \quad (m \text{ 9s and } n \text{ 0s})$$

for some integers $p, m, n \geq 0$.

Deduce that any integer divides some number of the form $99 \dots 9900 \dots 00$.

4. Irrational Kevin tries to show $\sqrt{12} - \sqrt{3}$ is rational, by the following argument.

$$\begin{aligned} \sqrt{12} - \sqrt{3} &= p/q, \quad p, q \in \mathbb{N}, \\ \Rightarrow 12 - 2\sqrt{12}\sqrt{3} + 3 &= p^2/q^2, \\ \Rightarrow 15 - 2\sqrt{36} &= p^2/q^2. \end{aligned}$$

Since $\sqrt{36} = 6$ is indeed rational, this looks good to him. Can you help him by pointing out three ways in which he's gone wrong? Be kind to him!

5. Suppose the sets S_n , $n = 1, 2, 3, \dots$ are all disjoint and countable. Show that $S = \bigcup_{n=1}^{\infty} S_n$ is also countable. (Hint: recall the diagonal argument used in lectures.)
6. Suppose that S and T are countable. Show that $S \times T$ is countable. Hence show that $\bigcup_{n=1}^{\infty} S^n$ is countable, where $S^n := S \times \dots \times S$ (n times).
7. † Show the set of polynomials $p(x)$ with integer coefficients is countable. (Hint: use Q6.)

A real number is called *algebraic* if it is a root of a polynomial with integer coefficients. Show that rational numbers n/m and n th roots $\sqrt[n]{m}$ are algebraic. Show that the set of algebraic real numbers is countable.

A real number is called *transcendental* if it is not algebraic. (Examples include π and e , but this is hard to prove.) Prove that transcendental numbers exist, and that in fact there are uncountably many of them.

8. Let $S^1 = \{s_1^1, s_2^1, s_3^1, \dots\}$, $S^2 = \{s_1^2, s_2^2, s_3^2, \dots\}$, \dots , $S^n = \{s_1^n, s_2^n, s_3^n, \dots\}$, \dots be subsets of \mathbb{N} . Here the elements are ordered so that $s_i^n < s_{i+1}^n$ for all i and n .

Define t_n recursively to be strictly larger than s_n^n and t_{n-1} (e.g. set $t_n = \max\{s_n^n, t_{n-1}\} + 1$), or $t_{n-1} + 1$ if $\nexists s_n^n$ (i.e. if S^n has $< n$ elements).

Show that $T = \{t_1, t_2, \dots\} \subseteq \mathbb{N}$ is not equal to any S^i . Conclude that the set of subsets of \mathbb{N} is *not* countable.

*Starred questions * are good to prepare to discuss at your Problem Class. Questions marked † are slightly harder (more fun).*