Math40002 Analysis 1 **Problem Sheet 2**

1. Fix $S \subset \mathbb{R}$ with an upper bound, and suppose that $S \neq \emptyset$ and $S \neq \mathbb{R}$. Give proofs or counterexamples to each of the following statements.

   (a) If $S \subset \mathbb{Q}$ then $\sup S \in \mathbb{Q}$.

   (b) If $S \subset \mathbb{R}\backslash\mathbb{Q}$ then $\sup S \in \mathbb{R}\backslash\mathbb{Q}$.

   (c) If $S \subset \mathbb{Z}$ then $\sup S \in \mathbb{Z}$.

   (d) $S \cap \left\{\frac{n}{m} \in \mathbb{Q} : \ n, m \in \mathbb{N}, \ m \leq 10^{100}\right\}$ has a minimum if it is nonempty.

   (e) There exists a $\max S$ if and only if $\sup S \in S$.

   (f) $\sup S = \inf(\mathbb{R}\backslash S)$.

   (g) $\sup S = \inf(\mathbb{R}\backslash S) \iff S$ is an interval of the form $(-\infty, a)$ or $(-\infty, a]$.

2. Fix nonempty sets $S_n \subset \mathbb{R}, \ n = 1, 2, 3, \ldots$. Prove that

$$\sup\left\{\sup S_1, \sup S_2, \sup S_3, \ldots\right\} \ = \ \sup\left(\bigcup_{n=1}^{\infty} S_n\right),$$

   in the sense that if either exists then so does the other, and they are equal.

3. Take bounded, nonempty $S, T \subset \mathbb{R}$. Define $S+T := \{s+t \colon s \in S, \, t \in T\}$. Prove

$$\sup(S+T) \ = \ \sup S + \sup T.$$

4. $*$ Fix $a \in (0, \infty)$ and $n \in \mathbb{N}$. We will prove $\exists\, x \in \mathbb{R}$ such that $x^n = a$. Set

$$S_a \ := \ \{s \in [0, \infty) : \ s^n < a\}$$

   and show S is nonempty and bounded above, so we may define $x := \sup S_a$.

   For $\epsilon \in (0, 1)$ show $(x + \epsilon)^n \leq x^n + \epsilon[(x + 1)^n - x^n]$. (Hint: multiply out.)
   Hence show that if $x^n < a$ then $\exists\, \epsilon \in (0, 1)$ such that $(x + \epsilon)^n < a$. $(*)$

   If $x^n > a$ deduce from $(*)$ that $\exists\, \epsilon \in (0, 1)$ such that $(\frac{1}{x} + \epsilon)^n < \frac{1}{a}$. $(**)$

   Deduce contradictions from $(*)$ and $(**)$ to show that $x^n = a$.

5. Suppose $0 < q \in \mathbb{Q}$ and $a \in (0, \infty)$. Write $q = \frac{m}{n}$ with $m, n \in \mathbb{N}$ and define

$$a^q \ := \ x^m,$$

   where $x =: a^{1/n}$ is defined in the last question. Show this is well defined, and make a definition of a^{-q}.

   Show that $(ab)^q = a^q b^q$ and $(a^{q_1})^{q_2} = a^{q_1 q_2}$ for any $a, b \in (0, \infty)$ and $q, q_1, q_2 \in \mathbb{Q}$.

6. For real numbers x, y, z, consider the following inequalities.

(a) $|x+y| \ \leq \ |x|+|y|$

(b) $|x+y| \ \geq \ |x|-|y|$

(c) $|x+y| \ \geq \ |y|-|x|$

(d) $|x-y| \ \geq \ \Big||x|-|y|\Big|$

(e) $|x| \ \leq \ |y|+|x-y|$

(f) $|x| \ \geq \ |y|-|x-y|$

(g) $|x-y| \ \leq \ |x-z|+|y-z|$

Prove (a) from first principles. Why is it called the "triangle inequality"?

Deduce (b,c,d,e,f,g) from (a).

Starred questions * *are good to prepare to discuss at your Problem Class.*