Math40002 Analysis 1 **Problem Sheet 6**

1. Let $a_1 = 1$ and $a_{n+1} = \sqrt{2a_n}$. Prove that (a_n) converges and compute the limit.

2. Fix $r > 1$. By the ratio test prove that $n/r^n \to 0$ as $n \to \infty$.
   Conclude that $n^{1/n} < r$ for sufficiently large n. Hence prove $n^{1/n} \to 1$ as $n \to \infty$.

3. Fix $M \in \mathbb{R}$. Prove $M^n/n! \to 0$. Hence show the sequence $(n!)^{1/n}$ is unbounded.

4.* Which of the statements (a)–(d) imply $(*)$ and which are implied by $(*)$?

$$\exists\, a \in \mathbb{R} \text{ such that } \forall \epsilon > 0\ \forall N \in \mathbb{N}\ \exists\, n \geq N,\ |a_n - a| < \epsilon. \qquad (*)$$

   (a) $\exists\, a \in \mathbb{R}$ such that $\forall \epsilon > 0\ \exists\, N \in \mathbb{N}$ such that $\forall n \geq N,\ |a_n - a| < \epsilon$.
   (b) $\exists\, a \in \mathbb{R}$ and $\exists\, \epsilon > 0$ such that $\forall N \in \mathbb{N}\ \forall n \geq N,\ |a_n - a| < \epsilon$.
   (c) $\forall a \in \mathbb{R}\ \exists\, \epsilon > 0$ such that $\forall N \in \mathbb{N}\ \forall n \geq N,\ |a_n - a| < \epsilon$.
   (d) $\exists\, a \in \mathbb{R}$ such that $\exists\, N \in \mathbb{N}$ such that $\forall \epsilon > 0,\ \forall n \geq N,\ |a_n - a| < \epsilon$.

5. We saw in lectures that the series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ diverges.
   What about $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots$? Prove your answer.

6.† Let $\sum_{n \geq 1} a_n$ be the series obtained from $\sum_{n \geq 1} \frac{1}{n}$ deleting all the terms $\frac{1}{n}$ such that the base 10 expansion of n contains the digit 4. Prove this series converges.

7. Prove *from first principles* that you can multiply a series by a constant $c \in \mathbb{R}$ term by term, i.e. if $\sum_{n=1}^{\infty} a_n$ is convergent then $\sum_{n=1}^{\infty} c a_n$ is convergent to $c \sum_{n=1}^{\infty} a_n$.

8. Given a real sequence (a_n), define a new sequence $b_n := \frac{1}{n} \sum_{i=1}^{n} a_i$ by averaging.

   (a) For any $a \in \mathbb{R}$, $N > 1$ and $n \geq N$, let $A(N) := \sum_{i=1}^{N-1} |a_i - a|$. Show that
$$|b_n - a| \ \leq \ \frac{A(N)}{n} + \frac{\sum_{i=N}^{n} |a_i - a|}{n}.$$
   (b) Suppose that $a_n \to a$. Prove carefully that $b_n \to a$.
   (c) Give (without proof) an example with a_n divergent but b_n convergent.
   (d) Suppose $\sum_{n=1}^{\infty} a_n$ is convergent, does it follow that $\sum_{n=1}^{\infty} b_n$ is also convergent, and to the same value ? *Hint: consider the sequence* $a_n = \begin{cases} 1 & n = 1, \\ 0 & n > 1. \end{cases}$

9. For which values of $a, b \in \mathbb{R}$ does $\sum_{n=1}^{\infty} n^a/b^n$ converge or diverge ? (*Give a proof in the MATH40004 sense, and a proof in the proof sense when* $a \in \mathbb{Z},\ b \in \mathbb{R}$.)

10. **MATH40004 question for fun.** Write down the unique degree $d{+}1$ polynomial $p(x)$ with roots $0, \lambda_1, \lambda_2, \ldots, \lambda_d$ and $p'(0) = 1$.
   "Apply" your formula to $d = \infty$ and $p(x) = \sin x$, and compare coefficients of x^3 or x^5 on both sides to evaluate

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \qquad\qquad \text{(b)}\dagger \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \ ?$$

*Starred questions * are good to prepare to discuss at your Problem Class.*
Questions marked † are slightly harder (closer to exam standard), but good for you.