1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(\mathbb{R}) \subset \mathbb{Q}$. Prove that $f$ is constant.

2. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$. Prove that $f(x) = g(x)$ for all $x \in \mathbb{R}$. Is this still true if we only assume that $f(x) = g(x)$ for $x \in \mathbb{Z}$?

3. Consider the function $f : [1, 2] \cap \mathbb{Q} \to \mathbb{R}$ defined by $f(x) = \left|x - \sqrt{2}\right|$. Prove that $f$ does *not* have a minimum value. Why doesn't the extreme value theorem apply?

4. (*) Define a function $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ 1/n, & x = m/n. \end{cases}$$
Here all rational numbers $x = \frac{m}{n}$ are written in lowest terms, with $n > 0$.
   (a) Prove that if $x$ is rational, then $f$ is not continuous at $x$.
   (b) Prove that if $x$ is irrational, then $f$ is continuous at $x$.

5. Let $f : [a, b] \to \mathbb{R}$ be continuous, and suppose that $f(a) \le y \le f(b)$.
   (a) Let $(a_0, b_0) = (a, b)$, and for all $n \ge 0$, define $m_n = \frac{a_n+b_n}{2}$ and
$$(a_{n+1}, b_{n+1}) = \begin{cases} (a_n, m_n), & f(m_n) > y \\ (m_n, b_n), & f(m_n) \le y. \end{cases}$$
Prove that the sequences $\big(a_n\big)$ and $\big(b_n\big)$ converge to the same limit $L \in [a, b]$.
   (b) Prove that $f(L) = y$, concluding a new proof of the intermediate value theorem.

6. For any nonempty set $S \subset \mathbb{R}$, define $d_S : \mathbb{R} \to \mathbb{R}$ by $d_S(x) = \inf_{s \in S} |x - s|$.
   (a) Describe or draw graphs of $d_S$ when $S$ is each of $\{0\}$, $\{-1, 3\}$, $\mathbb{Z}$, $\mathbb{Q}$.
   (b) Prove that $|d_S(y) - d_S(x)| \le |y - x|$ for all $x, y \in \mathbb{R}$, and conclude that $d_S$ is continuous.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotonically increasing function, not necessarily continuous. Define $S(x) = \sup_{y<x} f(y)$ and $I(x) = \inf_{y>x} f(y)$.
   (a) Prove for all $x \in \mathbb{R}$ that $S(x) \le f(x) \le I(x)$.
   (b) Prove for all $x \in \mathbb{R}$ that $S(x) = I(x)$ if and only if $f$ is continuous at $x$.
   (c) Find an injective mapping
$$\{x \in \mathbb{R} \mid f \text{ is not continuous at } x\} \to \mathbb{Q}.$$
Conclude that $f$ is continuous at all but at most countably many real numbers.

8. Prove that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set $U \subset \mathbb{R}$, the preimage
$$f^{-1}(U) = \{x \in \mathbb{R} \mid f(x) \in U\}$$
is open.