1. You drive down a road whose speed limit is 60 miles per hour. An observer sees you at 12pm, and a second observer 35 miles away sees you at 12:30pm. Assuming they've watched their analysis lectures, how can they prove you were speeding?

2. Prove using l'Hôpital's rule that $\lim_{x\to\infty}\left(1+\frac{r}{x}\right)^x = e^r$. (Hint: take logs first.)

3. Let H_n denote the harmonic sum $\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}$.

   (a) Show using the mean value theorem that $\frac{1}{n+1} < \log(n+1)-\log(n) < \frac{1}{n}$ for all $n\in\mathbb{N}$.

   (b) Prove that $H_n - 1 < \log(n) < H_{n-1}$ for all $n\geq 2$, where $H_k = \frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{k}$, and deduce that $\log(n+1) < H_n < \log(n)+1$.

   (c) Prove that the sequence $\big(H_n - \log(n)\big)$ is decreasing, and that $\lim_{n\to\infty}(H_n-\log(n))$ exists. (This limit is called the *Euler–Mascheroni constant* $\gamma\approx 0.577\ldots$.)

4. (*) Let $f:\mathbb{R}\to\mathbb{R}$ be differentiable, and suppose there is a constant $C<1$ such that $|f'(x)|\leq C$ for all $x\in\mathbb{R}$. We will prove that f has exactly one fixed point, meaning there is a unique $y\in\mathbb{R}$ such that $f(y)=y$. Pick some $x_0\in\mathbb{R}$ and let

$$x_{n+1} = f(x_n) \text{ for all } n\geq 0.$$

   (a) Prove that $|x_{n+2}-x_{n+1}|\leq C|x_{n+1}-x_n|$ for all n.

   (b) Prove that the sequence $\big(x_n\big)$ converges, and that if its limit is y then $f(y)=y$.

   (c) Prove that f cannot have two different fixed points.

5. (a) Compute the Taylor series $P(x)$ of $f(x)=\log(1+x)$ centered at $x=0$, and prove that it converges absolutely on $(-1,1)$.

   (b) Prove using Taylor's theorem that $f(x)=P(x)$ on some open neighborhood of 0, by showing that the sequence of nth order Taylor polynomials $P_n(x)$ converges uniformly to $f(x)$. Show that the same is true at $x=1$, and so $\log(2)=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots$.

6. Suppose that $f:\mathbb{R}\to\mathbb{R}$ has at least six continuous derivatives, and that $f^{(i)}(0)=0$ for $i=1,2,3,4,5$ but $f^{(6)}(0)=1$. Prove that $f(x)$ has a local minimum at $x=0$.

7. (a) Prove that $f(x)=e^x$ is convex on all of $\mathbb{R}$.

   (b) Let $a,b>0$. Prove the *arithmetic mean–geometric mean inequality*

$$\frac{a+b}{2}\geq\sqrt{ab}$$

   by using the convexity of e^x. (Hint: think about $\alpha=\log(a)$ and $\beta=\log(b)$.)

   (c) Prove for any $a,b>0$ and $s\in[0,1]$ that $sa+(1-s)b\geq a^s b^{1-s}$.

   (d) Prove *Young's inequality*: for any $x,y\geq 0$ and $p,q>0$ with $\frac{1}{p}+\frac{1}{q}=1$,

$$\frac{x^p}{p}+\frac{y^q}{q}\geq xy.$$

8. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$

   (a) Prove that for all integers $n \geq 0$, there is a polynomial $p_n(x)$ such that

   $$f^{(n)}(x) = \frac{p_n(x)}{x^{3n}} e^{-1/x^2} \text{ for all } x \neq 0.$$

   (b) Prove that $f^{(n)}(0) = 0$ for all n, and hence that $f(x)$ does not equal its Taylor series (centered at $a = 0$) at any nonzero x.

   (c) Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \begin{cases} 0, & x \leq 0 \\ e^{-1/x^2}, & x > 0. \end{cases}$ Prove that $g^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $g^{(n)}(0) = 0$ for all n.

   (d) Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = g(x)g(1-x)$. Prove that $h^{(n)}(x)$ exists for all $n \geq 0$ and all $x \in \mathbb{R}$, and that $h(x) \neq 0$ if and only if $0 < x < 1$.

   The function h is called a *bump function*: it is infinitely differentiable, and it is zero outside a compact set (namely $[0, 1]$) but also takes positive values.

9. Define functions $f_n : \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$ for all $n \geq 1$.

   (a) Prove that f_n is continuously differentiable, and that $|x| \leq f_n(x) \leq |x| + \frac{1}{n}$.

   (b) Prove that $\left(f_n\right)$ converges uniformly to a continuous function f.

   (c) Prove that $\left(f_n'\right)$ doesn't converge uniformly on $[-1, 1]$, so the theorem from lecture about limits of differentiable functions doesn't apply to tell us that f should be differentiable on $[-1, 1]$. (Is f differentiable there?)

10. In an upcoming lecture, we'll need to know that $\lim_{x \to \infty} x s^{x-1} = 0$ for all $s \in (0, 1)$.

    (a) Prove that for all $c > 0$, there exists $N > 0$ such that $\log(x) < cx$ for all $x \geq N$.

    (b) Prove for $s \in (0, 1)$ that $\lim_{x \to \infty} x s^x = 0$, and that this implies the above claim.