1. (a) Suppose that some function $f : (-R, R) \to \mathbb{R}$ is equal to the power series $\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$, which converges absolutely on $(-R, R)$. Prove that the Taylor series of f centered at $a = 0$ is precisely $\sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$.

   (b) Compute the Taylor series of $f(x) = \frac{1}{1-x^2}$ centered at $a = 0$. What is $f^{(100)}(0)$?

2. (*) Let (a_n) denote the Fibonacci sequence, with $a_0 = 0$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for all $n \geq 0$.

   (a) Prove by induction that $a_n < 2^n$ for all $n \geq 0$. What is the radius of convergence of the *exponential generating function*

$$F(x) = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!} = 0 + 1x + \frac{1x^2}{2} + \frac{2x^3}{6} + \frac{3x^4}{24} + \ldots ?$$

   (b) Prove that $F''(x) = F'(x) + F(x)$, and that $F(0) = 0$ and $F'(0) = 1$.

   (c) Solve this differential equation for $F(x)$.

   (d) Use the solution from part (c) to prove *Binet's formula*:

$$a_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right).$$

3. Recall that we defined $\pi = \inf\{y > 0 \mid \sin(y) = 0\}$.

   (a) Prove that $\sin(n\pi) = 0$ for all $n \in \mathbb{Z}$.

   (b) Prove that if $\sin(y) = 0$, then $y = n\pi$ for some $n \in \mathbb{Z}$. (Hint: write $y = q\pi + r$.)

   (c) Prove that $\cos(x) = 0$ if and only if $x = \frac{(2k+1)\pi}{2}$ for some $k \in \mathbb{Z}$.

4. In this problem we will show that the mysterious constant π lies strictly between $2\sqrt{2} \cong 2.828\ldots$ and 3.2. (Can you use these same ideas to do better?)

   (a) Use the third-order Taylor polynomial for $\cos(x)$, centered at $x = 0$, to prove that if $0 < x \leq \frac{\pi}{2}$ then

$$1 - \frac{x^2}{2} < \cos(x) \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}.$$

   (b) Evaluate one or both of these inequalities at $x = \frac{\pi}{2}$ and conclude that $\pi > 2\sqrt{2}$.

   (c) Show that $\cos(2) < 0$ and hence that $\frac{\pi}{2} < 2$. Once you've done this, use a calculator to do the same for $\cos(1.6)$ and deduce that $\pi < 3.2$.

5. Fix an integer $r \geq 0$ and define $f : [1, b] \to \mathbb{R}$ by $f(x) = x^r$, where $b > 1$.

   (a) Let $P_n = (1, b^{1/n}, b^{2/n}, \ldots, b^{(n-1)/n}, b)$ be a partition of $[1, b]$. Compute the lower Darboux sum $L(f, P_n)$, and show that $U(f, P_n) = b^{r/n} L(f, P_n)$.

   (b) Prove that $\lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n)$, and compute their common value.