1. Define a function $f : [a, b] \to \mathbb{R}$ by $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q}. \end{cases}$

   Prove that f is not integrable, but that f^2 is.

2. Prove that any monotone increasing function $f : [a, b] \to \mathbb{R}$ is integrable, by considering its Darboux sums for partitions where every subinterval $[x_i, x_{i+1}]$ has the same length.

3. Define the *mesh* of a partition $P = (x_0, \ldots, x_k)$ to be the maximal length of any subinterval:

$$\text{mesh}(P) = \max_{0 \le i \le k-1} \Delta x_i = \max_{0 \le i \le k-1} (x_{i+1} - x_i).$$

   Show that if $f : [a, b] \to \mathbb{R}$ is continuous and $\big(P_n\big)$ is any sequence of partitions of $[a, b]$ such that $\text{mesh}(P_n) \to 0$ as $n \to \infty$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P_n) = \lim_{n\to\infty} U(f, P_n).$$

   The proof should follow the argument we used in lecture to show that continuous functions are integrable.

4. (a) Prove for any $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$ that if $\sin(\frac{\theta}{2}) \neq 0$, then

$$\sin(\theta) + \sin(2\theta) + \cdots + \sin(n\theta) = \frac{\sin(n\theta/2)\sin((n+1)\theta/2)}{\sin(\theta/2)}$$

   using the formula $\sin(\alpha)\sin(\beta) = \frac{1}{2}(\cos(\alpha - \beta) - \cos(\alpha + \beta))$.

   (b) Fix $t \in (0, \frac{\pi}{2}]$ so that $\sin(x)$ is monotone increasing on the interval $[0, t]$, and consider the partition $P_n = (0, \frac{t}{n}, \frac{2t}{n}, \ldots, \frac{(n-1)t}{n}, t)$ of $[0, t]$. Compute the upper Darboux sum $U(\sin(x), P_n)$, and show that

$$\lim_{n\to\infty} U(\sin(x), P_n) = 2\sin^2(\tfrac{t}{2}).$$

   Remark: This limit is equal to $1-\cos(t)$ by the double-angle formula $\cos(2\theta) = 1 - 2\sin^2(\theta)$, so problem 3 tells us that

$$\int_0^t \sin(x)\,dx = 2\sin^2\left(\frac{t}{2}\right) = 1 - \cos(t)$$

   for all $t \in (0, \frac{\pi}{2}]$.

5. Let $f, g : [a, b] \to \mathbb{R}$ be bounded functions such that $f(x)$ and the product $f(x)g(x)$ are both integrable, and $f(x) \geq 0$ for all $x \in [a, b]$. If $c \leq g(x) \leq d$ for all $x \in [a, b]$, prove that

$$c\int_a^b f(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq d\int_a^b f(x)\,dx.$$

6. (*) Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & x \notin \mathbb{Q} \\ 1/|q|, & x = \frac{p}{q} \in \mathbb{Q}. \end{cases}$

We proved in problem sheet 1 that f is discontinuous at all rational numbers.

(a) Compute the lower Darboux integral $\underline{\int_0^1} f(x)\,dx$.

(b) Consider the partition $P_n = (0, \frac{1}{n^3}, \frac{2}{n^3}, \dots, \frac{n^3-1}{n^3}, 1)$ of $[0,1]$. Show for n large that there are at most n^2 subintervals $[\frac{i}{n^3}, \frac{i+1}{n^3}]$ on which

$$M_i = \sup_{\frac{i}{n^3} \le t \le \frac{i+1}{n^3}} f(t)$$

is at least $\frac{1}{n}$.

(c) Prove that $U(f, P_n) \le \frac{2}{n}$ for n large. (Hint: break the sum into terms where $M_i \ge \frac{1}{n}$ and terms where $M_i < \frac{1}{n}$.)

(d) Conclude that f is integrable, and compute $\int_0^1 f(x)\,dx$.