1. Prove that if $f : [a, b] \to [0, \infty)$ is continuous and $f(c) \neq 0$ for some $c \in [a, b]$, then $\int_a^b f(x)\,dx > 0$.

2. Suppose for some $f : [a, b] \to \mathbb{R}$ and integer $n \geq 1$ that the nth power f^n of f is integrable. Prove that if n is odd, then f is integrable. Why doesn't this work for n even, and can you find additional hypotheses on f that make it true in that case?

3. Let $C[a, b]$ denote the set of continuous functions $f : [a, b] \to \mathbb{R}$, and define a function $d : C[a, b] \times C[a, b] \to \mathbb{R}$ by $d(f, g) = \displaystyle\int_a^b |f(x) - g(x)|\,dx$.

   (a) Prove that $d(f, g) = d(g, f)$ for all $f, g \in C[a, b]$.

   (b) Prove that $d(f, g) \geq 0$, with equality if and only if $f = g$.

   (c) Prove the triangle inequality $d(f, g) + d(g, h) \geq d(f, h)$.

   These properties say that d is a *metric*, which is a notion of distance on $C[a, b]$.

   (d) Prove that if $f_n \to f$ uniformly on $[a, b]$, then $\lim_{n\to\infty} d(f_n, f) = 0$.

4. Evaluate $\displaystyle\int_1^x \frac{\sqrt{t^2 - 1}}{t}\,dt$ for $x \geq 1$. (Hint: what is the inverse of the integrand?)

5. In problem sheet 4 we constructed a smooth (i.e., infinitely differentiable) function $f : \mathbb{R} \to [0, \infty)$ such that $f(x) > 0$ if and only if $x \in (0, 1)$.

   (a) Construct a smooth, monotone increasing function $g : \mathbb{R} \to [0, \infty)$ such that $g(x) = 0$ for all $x \leq 0$ and $g(x) = 1$ for all $x \geq 1$.

   (b) Given $a < b < c < d$, construct a smooth function $h : \mathbb{R} \to [0, \infty)$ satisfying

   $$h(x) = 0 \text{ for all } x \notin [a, d], \qquad h(x) = 1 \text{ for all } x \in [b, c],$$

   and with h monotone increasing on $(-\infty, b]$ and decreasing on $[c, \infty)$.

6. (a) Given $a < b < 0$, evaluate $\displaystyle\int_a^b \frac{1}{x}\,dx$. Be careful not to take the logarithm of a negative number along the way!

   (b) Check that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ is strictly monotone increasing on the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, with

   $$\lim_{x\downarrow -\frac{\pi}{2}} \tan(x) = -\infty \quad \text{and} \quad \lim_{x\uparrow \frac{\pi}{2}} \tan(x) = +\infty.$$

   (c) Let $\tan^{-1} : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$ be the inverse function to $\tan(x)$. Prove for all $x \in \mathbb{R}$ that

   $$\cos(\tan^{-1}(x)) = \frac{1}{\sqrt{1 + x^2}}.$$

   (d) Fix $\theta \in (0, \frac{\pi}{2})$. Find a convenient substitution which proves that

   $$\int_0^\theta \tan(x)\,dx = -\log(\cos(\theta)).$$

(e) Prove for $x > 0$ that $\displaystyle\int_0^x \tan^{-1}(t)\,dt = x\tan^{-1}(x) - \frac{1}{2}\log(1+x^2)$.

7. (a) Check that the derivative of $x\log(x) - x$ is $\log(x)$.
   (b) Use Darboux sums to prove for all integers $n \geq 1$ that
   $$\log((n-1)!) \leq \int_1^n \log(x)\,dx \leq \log(n!).$$
   (c) Evaluate the integral in (b) and deduce that
   $$\frac{1}{n} \leq \frac{\log(n!)}{n} - \log\left(\frac{n}{e}\right) \leq \log\left(1+\frac{1}{n}\right) + \frac{\log(n+1)}{n}$$
   for all $n \geq 1$.
   (d) Conclude that $\displaystyle\lim_{n\to\infty} \frac{n}{\sqrt[n]{n!}} = e$.

   Remark: this is a weak version of *Stirling's formula* $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$.

8. (*) Let $f : [N,\infty) \to [0,\infty)$ be a nonnegative, monotone decreasing function.
   (a) Let $\displaystyle S_n = \sum_{k=N}^{n} f(k)$ for all integers $n \geq N$. Use Darboux sums to prove that
   $$S_n - f(N) \leq \int_N^n f(x)\,dx \leq S_{n-1}.$$
   (b) Prove that the series $\displaystyle\sum_{k=N}^{\infty} f(k)$ converges if and only if the limit
   $$\int_N^\infty f(x)\,dx \stackrel{def}{=} \lim_{x\to\infty} \int_N^x f(t)\,dt$$
   (called an *improper integral*) exists. This is the *integral test* for convergence.
   (c) Prove that if the series $\displaystyle S = \sum_{k=N}^{\infty} f(k)$ converges, so $I = \int_N^\infty f(x)\,dx$ exists, then $I \leq S \leq I + f(N)$.

9. Consider for any real s the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s}$.
   (a) Prove that this series is not convergent if $s \leq 0$.
   (b) Use the integral test to prove that for $s > 0$, the series converges if and only if $s > 1$. If $s > 1$, show that $\displaystyle\frac{1}{s-1} < \sum_{n=1}^{\infty} \frac{1}{n^s} < \frac{s}{s-1}$.
   (c) Prove for any $a > 1$ that the series converges uniformly to a continuous function on $[a,\infty)$, and hence it defines a continuous function $\zeta : (1,\infty) \to \mathbb{R}$ called the *Riemann zeta function.* Can it be extended continuously to $[1,\infty)$?
   (d) (Harder!) Prove that $\zeta(s)$ is continuously differentiable, and compute its derivative. It may help to first show that $\displaystyle\lim_{x\to\infty} \frac{\log(x)}{x^\epsilon} = 0$ for any $\epsilon > 0$.