Calcusus 1 - Concise Notes

MATH40001

Term 1 Content

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Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

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January 31, 2022

Contents

1 Chapter 1, Limits of Functions, continuity

Definition 1.0.1. $\epsilon - \delta$ Definition of Limit

Let f be a function defined at all points near X_0, except possible at x_0, and let l be a real number. We say that l is a limit of $f(x)$ as x approaches x_0, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - l| < \epsilon$ whenever $|x - x_0| < \delta$ and $x \neq x_0$. We write $lim_{x \to x_0} f(x) = l$

Basic Properties of Limits

1. *Sum rule*
   $\lim_{x \to x_0}[f(x) + g(x)] = \lim_{x \to x0} f(x) + \lim_{x \to x0} g(x)$

2. *Product rule*
   $\lim_{x \to x0}[f(x)g(x)] = \lim_{x \to x0} f(x) \lim_{x \to x0} g(x)$

3. *Reciprocalrule* If $\lim x \to x_0 f(x) \neq 0$ then
   $\lim_{x \to x0}[1/f(x)] = 1/\lim_{x \to x0} f(x)$

4. *Quetient rule* If $\lim_{x \to x0} g(x) \neq 0$ then
   $\lim_{x \to x0}[f(x)/g(x)] = \lim_{x \to x0} f(x)/\lim_{x \to x0} g(x)$

5. *Composite function rule* If $h(x)$ is conitnuous at $\lim_{x \to x0} f(x)$ then
   $\lim_{x \to x0} h(f(x)) = h(\lim_{x \to x0} f(x))$

Definition 1.0.2. The $\epsilon - A$ definition of $\lim x \to \infty f(x) = l$

Let $f(x)$ bedefined on a domain containing, the interval (a, ∞).A real number $l$is the limit of $f(x)$ as x approaches ∞ if for every $\epsilon > 0$ there exists a $A > a$, such that $|f(x) - l| < \epsilon$ whenver $x > A$. We write
$limx \to \infty f(x) = l$

Definition 1.0.3. $\epsilon - B$ definition of $\lim_{x \to x0} f(x) = \infty$

Let $f(x)$ be a function defined in an interval containing x_0 except possibly at $x = x_0$. We saythat $f(x)$ approaches ∞ as x approaches x_0, if given any real number $B > 0$, there exists $\epsilon > 0$, so that whenever $|x - x_0| < \epsilon$ and $x \neq x_0$, we have $f(x) > B$. We write $\lim_{x \to x0} f(x) = \infty$

Definition 1.0.4. One-Sided limit

Let $f(x)$ be defined for all x in an interval (x_0, a). We say that $f(x)$ approaches l as x approaches x_0 from the right if for any $\epsilon > 0$ there exists $\delta > 0$, such that for all $x_0 < x < x_0 + \delta$ we have $|f(x) - l| < \epsilon$. We Write $\lim_{x \to x0-} f(x) = l$
Comparison Test for Limits

1. $\lim_{x \to x0} f(x) = 0$ and $|g(x) \leq |f(x)|$ for all x near x_0 with $x \neq x_0$, then $\lim_{x \to x0} g(x) = 0$

2. $\lim_{x \to \infty} f(x) = 0$ and $|g(x)| \leq |f(x)|$ for all large enough x then$\lim_{x \to \infty} g(x) = 0$

Two Basic Trigonometric Limits
$\lim_{h \to 0} \frac{sinh}{h} = 1$ $\lim_{h \to 0} \frac{cosh-1}{h} = 0$

Definition 1.0.5. Continuity

We say that f is continuous at x_0 if $\lim_{h \to 0} f(x_0 + h) = f(x_0)$ Equivalently $\lim_{x \to x0} f(x) = f(x_0)$ A totally equivalent definition is: $f(x)$ is continuous at a point x_0 if for every $\epsilon > 0$ there exists a number $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for all x in the domain of f for which $|x - x_0| < \delta$

2 Differentiation

2.1 Definition with limits

Definition 2.1.1. Differentiability

The function $f(x)$ is differentiable at x if "Newton's quotient"
$\lim h \to 0 \frac{f(x+h)-f(x)}{h}$ exists. We call this $f'(x)$ the derivative of f at point x

2.1.1 Polynomials

Theorem 2.1.

Let n be an integer ≥ 1 and let $f(x) = x^n$. Then $f'(x) = \frac{df}{dx} = nx^{n-1}$

Theorem 2.2.

Let $f(x) = x^a$ where a is any real number and $x > 0$. Then $f'(x) = ax^{a-1}$.

2.2 General rules, chain rule, rate of changes

2.2.1 General rules

1. If c is a constant $(cf)'(x) = cf'(x)$
2. if $f(x), g(x)$ are given functions and $f'(x), g'(x)$ exist, then
   $(f+g)'(x) = f'(x) + g'(x)$
3. $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
4. Let g(x) be a function that has a derivative $g'(x)$ and such that $g(x) \neq 0$
   Then $\frac{d}{dx}(\frac{1}{g(x)}) = -\frac{g'(x)}{(g(x))^2}$
5. $\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{g(x)f'(x)-f(x)g'(x)}{(g(x))^2}$

2.2.2 The Chain Rule

$\frac{d}{dx}(f)(x) = (f \circ g)'(x) = f'(g(x))g'(x)$

Theorem 2.3.

If $f(x)$ is differentiable at $x = x_0$ then it is also continuous there.

2.3 Implicit differentiation, related rates of change

Not much notable here. You can prove the derivative of polynomials with fractional powers using implicit differentiation.

3 Mean Value and Intermediate Value Theorems

For a function $f(x)$ which is defined at a point c, we say that c is *maximum* of f if
$f(c) \geq f(x) \forall x$ where f is defined
The minimum is obvious

Theorem 3.1.

Let f be a function which is defined and differentiable on the open interval (a, b). Let c be a number in the interval which is a maximum for the function. Then $f'(c) = 0, f'(c) = 0$ also if c is a minimum of f

Theorem 3.2.

Let $f(x)$ be continuous on the close interval [a,b]. Then $f(x)$ has a maximum and a minimum on this interval.

Theorem 3.3.

Let $f(x)$ be continuous over the closed interval$a \leq x \leq b$ and differentiable on the interval $a < x < b$. Assume also that $f(a) = f(b) = 0$. Then there exists a point $c, a < c < b$ such that $f'(c) = 0$

Theorem 3.4.

Suppose f is continuous on $[a, b]$ and differentiable on a, b Then there exists $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Definition 3.0.1.

We say that f is increasing over a given interval if given x_1, x_2 in the interval with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$ If it is strictly increasing it is the same with $<$ insteaf of $\leq$ Same for decreasing and strictly decreasing

Definition 3.0.2.

Let $f(x)$ be continuous in some interval, and differentiable there(even possible at the end points.)
If $f'(x) = 0$ in the interval(except possible at endpoints) then f is constant
If $f'(x) > 0$ in the interval(except possible at endpoints) then f is strictly increasing
If $f'(x) < 0$ in the interval(except possible at endpoints) then f is striclty decreasing

Theorem 3.5. Intermediate value theorem

Let f be continuous on the close interval $a \leq x \leq b$. Given any number $y*$ between $f(a) and f(b)$, there exists a point $x*$ between a and b such that $f(x*) = y*$

4 Inverse Functions

Definition 4.0.1.

Let $y = f(x)$ be defined on some interval. Given any y_0 in the range of f, if we can find a unique value x_0 in its domain such that $f(x_0) = y_0$, then we an define the ***inverse funtion*** $x = g(y)$(sometimes written $x = f^{-1}(y)$

Theorem 4.1.

Let $f(x)$ be strictly increasing or strictly decreasing. Then the inverse function exists.

Theorem 4.2.

If $f(x)$ is continuous $[a, b]$ and is strictly increasing(or decreasing), and $f(a) = y_a$ and $f(b) = y_b$, then $x = g(y)$ is defined on $[y_a, y_b]$

4.0.1 Derivative of inverse functions

Theorem 4.3.

let $f(x)$ be differentiable on (a, b) and $f'(x) > 0$ or $f'(x) < 0$ for all x in (a, b). Then the inverse function exists and we have $g'(y) = \frac{d}{dy} f^{-1}(y) = \frac{1}{f'(x)}$

5 Exponentials and Logarithms

5.1 Geometrical Definition, Derivative

Definition 5.1.1.

The quantity $log(x)$ is the area under the curse $\frac{1}{x}$ between 1 and x if $x \geq 1$ and the negative the area under the curve $\frac{1}{x}$ between 1 and x if x is in the interval $(0, 1)$. In particular $log(1) = 0$

Theorem 5.1.

$log(x)$ is differentiable and $\frac{d}{dx} log(x) = \frac{1}{x}$

Theorem 5.2.

If $a, b > 0$, then $log(ab) = log(a) + log(b)$

Theorem 5.3.

$log(x)$ is strictly increasing for all $x > 0$. Its range is $(-\infty, \infty)$

Theorem 5.4.

If n is an integer(positive or negative) then $log(a^n) = nlog(a)$ for all $a > 0$

5.2 Exponential as Inverse of $logx$

Theorem 5.5.

If x_1, x_2 are two numbers then $exp(x_1 + x_2) = exp(x_1)exp(x_2)$

Theorem 5.6.

$exp(x)$is differentiable and $\frac{d}{dx}exp(x) = exp(x)$

Theorem 5.7.

$\frac{d}{dx}a^x = a^x(log(a))$

Corollary:
$\lim_{h\to 0}\frac{a^h-1}{h} = log(a)$ for $a > 0$

Theorem 5.8.

Let a be any real number and let $f(x) = x^a$ for $x > 0$.Then $f'(x)$ exists and $f'(x) = ax^{a-1}$

5.3 Function estimates for Small and Large Arguments

Theorem 5.9. Let a be any real number. Then $\frac{(1+a)^n}{n} \to \infty$ as $n \to \infty$

Corollary:$\frac{e^n}{n} \to \infty$ as $n \to \infty$ since $e = 1 + a$ for some $a > 0$

Theorem 5.10.

The function $f(x) = \frac{e^x}{x}$ is strictly increasing for $x > 1$ and $\lim_{x\to\infty} f(x) = \infty$
Corollary
The function $x - log(x)$ becomes arbitrarily large as x becomes arbitrarily large. x beats log.
Corrolary
The function $\frac{x}{log(x)}$ becomes large as x becomes large. x beats log
Corolary
As x becomes large $x^{1/x}$ approches the limit 1.

Theorem 5.11. exp(x) beats any power of x

Let m be a positive integer. Then the function $f(x) = \frac{e^x}{x^m}$ is strictly increasing for x¿m and becomes arbitrarily large as x becomes arbitrarily large.

5.4 Logarithmic Differentiation

not much here

5.5 L'Hopital's Rule

Theorem 5.12.

If f, g are differnetiable on an open interval containing x_0, $g(x_0) = f(x_0) = 0$, and $g'(x_0) \neq 0$, then

$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$

Theorem 5.13.

Let $f(x)$ and $g(x)$ be a differentiable on an open interval containing x_0(except possible at x_0). Assume that $g(x) \neq 0$ and $g'(x) \neq 0$ for x in an interval about x_0 but with $x \neq x_0$. Assume also that $f.g$ are continous at x_0 with $f(x_0) = g(x_0) = 0$ and $\lim_{x\to x_0}\frac{f(x)}{g(x)} = l$. Then also:
$\lim_{x\to x_0}\frac{f(x)}{g(x)} = l$

Theorem 5.14. L'Hopital's Rule-general case

To find $limx \to x_0\frac{f(x)}{g(x)}$ when $limx \to x_0 f(x)$ and $\lim x \to x_0 g(x)$ are both zero or both infinite, differentiate numberator and denominator and take the limit of the new function. Repeat as many times as needed as long as it satisfies the conditions. Note that x_0 may be replaced by $\pm\infty$ or $x_0\pm$

Theorem 5.15. Cauchy Mean Value Theorem

Let f,g be continuous on $[a, b]$ and differentiable on (a, b) with $g(a) \neq g(b)$. Then there exists c in (a, b) such that $g'(c)\frac{f(b)-f(a)}{g(b)-g(a)} = f'(c)$

6 Integration

6.1 Anti-derivative and Geometrical Interpretation

Definition 6.1.1.

Given $f(x)$ defined over some interval then if we can find a function $F(x)$ defined over the same interval such that $F'(x) = f(x)$ then $F(x)$ is the *indefinite integral* of $f \longrightarrow F = \int f(x)dx$. Then
$\frac{d}{dx}(F - G) = 0 \Rightarrow F(x) = G(x) + constant$

6.1.1 Area under a curve

Theorem 6.1.

The function $F(x)$ is differentiable and its derivatives is equal to $f(x)$. Another way to state this is $\frac{d}{dx}\int_a^x f(t)dt = f(x)$

Definition 6.1.2. Signed Area

If $f(x) < 0$ then the area is below the $x - axis$.Define $F(x)$ to be **minus** the area. This leads to the definite integral.

7 The Riemann Sum

Given $f(x), a \leq x \leq b$, take the partition of the interval $[a.b]$ to be $x_i = a + ih \ i = 0, 1, \ldots\ldots, n \ h = \frac{b-a}{n}$ Take any sub-interval $[x_{i-1}, x_i]$ and let $x_i \in [x_{i-1}, x_i]$. Then **the Riemann sum** is $\Sigma_{i=1}^{n} f(x_i*)h$ There are three ways of picking x_i*

1. $x_i* = x_i$ the right hand Riemann Summ
2. $x_i* = x_{i-1}$ left hand RS
3. $x_i* = \frac{1}{2}(x_i + x_{i-1})$ mid point RS

The Limit as $n \to \infty, h \to 0$ $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i*)h = \int_a^b f(x)dx$ This can be probed using squeeze theorem between the Lower Riemann sum and Right Sum

8 Properties of the definite Integral; Fundamental Theorrem of Calculus

1. $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ c a constant
2. $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
3. If $c \in (a, b)$ then
   $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. If $f(x) \leq g(x)$ for $x \in [a, b]$ then
   $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Theorem 8.1. Suppose $g(x)$ is defined for all $x \in [a, b]$ and is differentiable on $[a, b]$. Then
$\int_a^b g'(x)dx = g(b) - g(a)$

Theorem 8.2. *Fundamental Theorem of Calculus*

Suppose F is differentiable on [a.b] and F' is integrable on $[a, b]$ Then
$\int_a^b F'(x)dx = F(b) - F(a)$
If f is integrable on $[a, b]$ and has *anti-derivative* F then
$\int_a^b f(x)dx = F(b) - F(a)$ **Useful Theorem** $\frac{d}{dx}\int_a^{g(x)} f(t)dt = f(g(x))g'(x)$

9 Some Application

just do practise questions for these. this aint fucking physics note

10 Improper Integrals

Definition 10.0.1. Improper Integral

$\int_a^b f(x)dx$ is an ***improper integral*** if
(i) $a = -\infty$ and/or $b = \infty$
(ii) $f(x) \to \pm\infty$ in (a, b)

11 Mean value theorem for Integrals

Given a function f that is integrable on $[a.b]$ we define its average $\langle f \rangle_{[a,b]}$ by the formula $\langle f \rangle_{[a,b]} = \frac{1}{b-a} \int_a^b f(x)dx$

Theorem 11.1.

Let f be continuous on $[a, b]$ then there exists a point $x_0 \in (a.b)$ such that

$$f(x_0) = \frac{1}{b-a} \int_a^b f(x)dx$$

12 Techniques of Integration

lmao it's just integration just get good

13 Application of Integration

Length of Curves $L = \int_a^b [1 + (f'(x))^2]^{1/2} dx$
$L = \int_{t_0}^{t_1} [(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2]^{1/2} dt$ **Volumes of Revolution**
$V = \int_a^b \pi(f(x))^2 dx$ Rotating around x-axis
$V = \int_a^b 2\pi x f(x) dx$ Revolving around the y-axis
swith the x and y for rotating around y-axis
Surface area of revolution $S = \int_a^b 2\pi f(x)\sqrt{1 + (f(x))^2} dx$

13.1 Centre of Mass

1D case-simple If centre of mass is at x=, then we must have zero total moment $\Sigma m_k(\bar{x} - x_k) = 0$ i.e $\bar{x} = \frac{\Sigma m_k x_k}{\Sigma m_k}$ 2D-case-disrete masses
If there are n-masses of mass m_k and cordinates (x_k, y_k). Assume the centre of mass is $(\bar{(x)}, \bar{y})$. So must balance the moments in $x - axis$ and $y - axis$
Therefore:
$\bar{x} = \frac{\Sigma m_i x_i}{\Sigma m_i}$,$\bar{y} = \frac{\Sigma m_i y_i}{\Sigma m_i}$
Now for continuous mass distribution.
Define the density per unit area as $\rho(x, y)$
Dividing the region into small rectangles with sides $\Delta y, \Delta x$
So the moment of one of these rectangles about the y-axis is $x_i \rho(x_i, y_i)\Delta y$
Adding all of them gives $_{ij} x_i \rho(x_i, y_i)\Delta y$
The moment of the whole plate about the y-axis $x \int\int \rho(\bar{x}, y)dxdy$
Therefore $\bar{x} \int\int \rho dxdy = \int\int x\rho dxdy$
similar result for $\bar{y}$

Theorem 13.1. Theorem of Pappus

Let R be a region that lies on one side of line l $A =$ area of R
$V =$ Volume obtained by rotating about l
$d =$ distane travelled by the centre of mass when R is rotated
then $V = Ad$

13.2 Mment of Inertia

Consider an object of mass m at a distane y from the x-axis rotating at an angular velocity of ω.
Then the velocity of the the object is
$v = y\omega$
And thus the kinetic energy of the object is
$KE = \frac{1}{2}m(y\omega)^2$.
The coefficient of $\frac{1}{2}\omega^2$ ia defined to be the **moment of inertia**. Hence for the single particle considered here, we define the moment of inertia I to be
$I = my^2$
And therefore $KE = \frac{1}{2}\omega^2 I$

So using this, we an express the moment of inertia of a string.
Moment of Inertia about x-axis - $I_x = \int_{x_0}^{x_1} \rho(x)y^2\sqrt{1+y^2}$
Moment of Inertia about y-axis - $I_y = \int_{x_0}^{x_1} \rho(x)x^2\sqrt{1+y^2}$

13.3 Length Of curves and areas ousing polar coorindates

Length of polar curve: $L = \int_{\theta=\alpha}^{\beta}[(\frac{dr}{d\theta})^2 + r^2]^{1/2}d\theta$
Area of polar curve
$A = \frac{1}{2}\int_{alpha}^{\beta} r^2 d\theta$

14 Series

Definition 14.0.1.

Given a sequence $a_{n n\geq 1}$of real numbers, define the sequence of partial sums $S_N = \Sigma_{n=1}^{N} a_n$ If $S_N \to S$ as $N \to \infty$ we say the series converges to the sum S $S = \Sigma_{n=1}^{\infty} a_n$

Theorem 14.1.

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$

Theorem 14.2.

If $\alpha > 1$ is a rational number, then $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges

14.0.1 Elemental algebraic rules for series

Theorem 14.3.

If the series $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$ as $n \to \infty$

14.1 Cauchy sequences and convergence of series

Definition 14.1.1. Cauchy Sequence

Cauchy Sequence if and only if:
$\forall \epsilon > 0 N \in \mathbb{N}$ such that for any $m, n > N$
$|S_m - S_n| < \epsilon$

Theorem 14.4. Every cauchy sequence converges

Theorem 14.5. The alternating series test

A series thats alternating and $a_n \to 0$ as $n \to \infty$ converges

14.2 Convergence tests

Theorem 14.6. Comparison test

Llet $\sum_{n=1}^{\infty} b_n$ be convergent with b_n non-negative. If $|a_n| \leq b_n$then the series for a_n converges

Theorem 14.7.

Every absolutely convergent series is convergent

Theorem 14.8. Integral test

Let $f(x)$ be a function which is defined for all $x \geq 1$, and is positive and decreasing. $\sum_{n=1}^{\infty} f(n)$ converges if and only if the indefinite integral to infinity converges

Theorem 14.9. The ratio test

Let $S = \sum_{n=1}^{\infty} a_n$ $\lim n \to \infty |\frac{a_{n+1}}{a_n}| = L$ Then:

1. If $L < 1$ the series converges absolutely
2. If $L > 1$ the series diverges
3. If $L = 1$ the test is inconclusive

Theorem 14.10. The root test

Suppose:

$$\lim n \to \infty |a_n|^{1/n} = L$$

Then:

1. If $L < 1$ the series converges absolutely
2. If $L > 1$ the series diverges
3. If $L = 1$ the test is inconclusive

15 Power Series

Definition 15.0.1.

let x be a real number and $a_{n n \geq 0}$ be a sequence of numbers. Then we can form the **power series** $\sum_{n=0}^{\infty} a_n x^n$. The partial sums S_N are polynomials of degree N

Theorem 15.1.

Assume that there is a number $R > 0$ such that $\sum_{n=0}^{\infty} a_n R^n$ converges. Then for all $|x| < R$ the series$_{n=0}^{\infty} a_n x^n$ converges absolutely

Definition 15.0.2.

The greatest such R(mentioned above) is called the ***radius of convergence***.

Theorem 15.2. Ratio test for power series

Let $_{n=0}^{\infty} a_n$ be a power series and asuume that $\lim n \to \infty |\frac{a_{n+1}}{a_n}| = L$ exists. Let $R = \frac{1}{L}$
Then

1. If $|x| < R$ the series converges absolutely
2. If $|x| > R$ the serues diverges
3. If $x = \pm R$ could converge or diverge

15.1 Differentiation and integration of power series

We can differentiate power series if $|x| < R$

Theorem 15.3.

Let $f(x) = \sum_{n=0}^{\infty} a_n$
THen $f'(x) = \sum_{n=0}^{\infty} n a_{n-1}$ The integral is the opposite

Theorem 15.4.

If two power series with radi of convergence R_1, R_2 are added or multiplied together then the radi of convergence of the new series is $min(R_1, R_2)$

16 Taylor series

Theorem 16.1. Taylor's theorem with remainder

Let f be a function defined on a closed interval between two numbers x_0 and x. Assume that the function has $n+1$ derivatigves on the

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f^2(x_0)}{2!}(x-x_0) + \ldots + \frac{f^n(x_0)}{n!}(x-x_0)^n + R_n$$

where the remaineder R_n is given by

$$R_n = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{n+1}(t)dt$$

16.1 Exponentials and logarithms. Binomial theorem

$e^x = 1 + x + \frac{x^2}{2} + \ldots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}x^{n+1}$
$log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \ldots + (-1)^{n-1}\frac{x^n}{n} + (-1)^n \int_0^x \frac{t^n}{1+t}dt$
$(1+x)^n = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}x^n$

17 Orthogonal and orthonormal function spaces

Definition 17.0.1.

If f, g are real value functions that are Riemann integrabale then the inner product of f, g are

$$(f, g) = \int_a^b f(x)g(x)dx$$

Definition 17.0.2.

Let $\mathcal{S} = \phi_0, \phi_1, \ldots,$ be a collection of functions that are Riemann integrable on [a,b] If

$$(\phi_n, \phi_m) = 0 \text{ whenver } m \neq n$$

Then $\mathcal{S}$ is an ***Orthogonal*** system on [a,b]. Additionally if $||\phi_n|| = 1$ then $\mathcal{S}$ is said to be ***Orthonormal***

18 Periodic functiuons and periodic extensions

At points of discontinuity define

$$f(\xi) = \frac{1}{2}[f(\xi_+) + f(\xi_-)]$$

19 Trigonometric polynomials

19.1 Euler's relation

$cos(\theta) + isin(\theta) = e^{i\theta}$ **Orthogonality** $\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = 0$ if $n, = 2\pi$if $n = m$

20 Fourier series

Consider the trigonetrix polynomial

$$f(x) = S_N(x) = \frac{1}{2}a_0 +_{n=1}^{N} a_n cos(nx) + b_n sin(nx)$$
$$\text{where } a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cos(nx)dx$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sin(nx)dx$$

Orthogonality properties If m, n are integers then

$$\int_{-\pi}^{\pi} sin(mx)sin(nx)dx = \int_{-\pi}^{\pi} cos(mx)cos(nx)dx = 0, m \neq n, \pi, m = n$$
$$\int_{-\pi}^{\pi} sin(mx)cos(nx)dx = 0$$

Theorem 20.1. The fourier series formed by the fourier coefficients converges to the value $f(x)$ for any piecewise continuous function $f(x)$ over period period 2π which has piecewise continuous derivatives of first and second order. At any discontinuities the function must be defined by $f(x) = \frac{1}{2}[f(x^+)f(x^-)]$

If $c_n(x) = \frac{1}{2} + cos(x) + cos(2X) + cos(3x) + \ldots = \frac{sin(n+\frac{1}{2})x}{2sin(0.5x)}$ Define the poits where $\frac{1}{2}x = n\pi$ define c_n by $n + 1/2$ **Riemann-Lebesgue Lemma**

$I_\lambda = \int_a^b g(x)sin(\lambda x)dx$ tends to 0 as $\to \infty$

Lemma 2

$\int_0^\infty \frac{sin(z)}{z}dz = \frac{\pi}{2}$

Parseval's indentity

If $f(x) = S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^N a_n cos(nx) + b_n sin(nx)$

Then $\frac{1}{\pi}\int_{-pi}^{\pi} f^2 dx = \frac{1}{2}a^2 + 0 + \sum_{n=1}^\infty (a_n^2 + b_n^2)$

Fourier Transform pair

$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)e^{i\omega t}dte^{i\omega x d\omega}$