Calculus and Applications I

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Contents

Part I

Preliminaries

Chapter 1

Limits of Functions, continuity

Given a function $f(x)$ we are concerned with the behaviour near a point $x = x_0$, and in particular the statement $\lim_{x\to x_0} f(x) = \ell$, *i.e. the limit of $f(x)$ as x tends to x_0, exists and is equal to ℓ.* The precise $\varepsilon - \delta$ definition is the following.

Definition 1. $\varepsilon - \delta$ **Definition of Limit**

Let f be a function defined at all points near x_0, except possibly at x_0, and let ℓ be a real number. We say that ℓ is the limit of $f(x)$ as x approaches x_0, if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ whenever $|x - x_0| < \delta$ and $x \neq x_0$. We write $\lim_{x\to x_0} f(x) = \ell$.

Example Prove that $\lim_{x\to 2} \sqrt{x} = \sqrt{2}$.

Solution: Here $f(x) = \sqrt{x}$, $x_0 = 2$ and $\ell = \sqrt{2}$. Given $\varepsilon > 0$ we need to find $\delta > 0$ so that $|\sqrt{x} - \sqrt{2}| < \varepsilon$ whenever $|x-2| < \delta$. For all $x > 0$ we have $\sqrt{x} - \sqrt{2} = (x-2)/(\sqrt{x} + \sqrt{2})$, hence

$$|\sqrt{x} - \sqrt{2}| = \frac{|x-2|}{\sqrt{x} + \sqrt{2}} \leq \frac{|x-2|}{\sqrt{2}}.$$

Hence picking $\delta = \sqrt{2}\,\varepsilon$ will do.

In practice we do not want to be doing $\varepsilon - \delta$ proofs for every limit we encounter. Instead we use the following laws of limits which can be proven easily using the $\varepsilon - \delta$ definition. (Try some! I do analogous proofs later on also.)

Basic Properties of Limits

Assume that $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist. Then

(i) *Sum rule:*
$$\lim_{x\to x_0}[f(x)+g(x)] = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x).$$

(ii) *Product rule:*
$$\lim_{x\to x_0}[f(x)\,g(x)] = \lim_{x\to x_0} f(x)\;\lim_{x\to x_0} g(x).$$

(iii) *Reciprocal rule:* If $\lim_{x\to x_0} f(x) \neq 0$ then
$$\lim_{x\to x_0}[1/f(x)] = 1/\lim_{x\to x_0} f(x).$$

(iii)′ *Quotient rule:* If $\lim_{x\to x_0} g(x) \neq 0$ then
$$\lim_{x\to x_0}[f(x)/g(x)] = \lim_{x\to x_0} f(x)/\lim_{x\to x_0} g(x).$$

This follows immediately from (ii) and (iii).

(iv) *Composite function rule:* If $h(x)$ is continuous at $\lim_{x\to x_0} f(x)$, then
$$\lim_{x\to x_0} h(f(x)) = h\left(\lim_{x\to x_0} f(x)\right).$$

Example 1

Calculate $\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right)$.

Solution. Of the form "0/0". Rationalise, i.e.

$$\lim_{x\to 1}\left(\frac{x-1}{\sqrt{x}-1}\right) = \lim_{x\to 1}\left(\frac{(x-1)(\sqrt{x}+1)}{(\sqrt{x}-1)(\sqrt{x}+1)}\right) = \lim_{x\to 1}\left(\frac{\cancel{(x-1)}(\sqrt{x}+1)}{\cancel{(x-1)}}\right) = 2.$$

Example 2

Sketch the function $f(x) = x/|x|$. Do this by considering $x > 0$ and $x < 0$ separately. What happens when $x = 0$?

The properties given above also hold as x becomes large and positive or negative. For example if $f(x) = 1/x$ then we know that $\lim_{x\to\pm\infty} f(x) = 0$. Lets make this precise.

Definition 2. The $\varepsilon - A$ definition of $\lim_{x\to\infty} f(x) = \ell$.

Let $f(x)$ be defined on a domain containing the interval (a, ∞). A real number ℓ is the limit of $f(x)$ as x approaches ∞ if, for every $\varepsilon > 0$ there exists a $A > a$, such that $|f(x) - \ell| < \varepsilon$ whenever $x > A$. We write $\lim_{x\to\infty} f(x) = \ell$. [Similarly for $\lim_{x\to-\infty} f(x) = \ell$.]

NOTE: The limit properties (1)-(1) hold for limits of $f(x)$ as $x \to \pm\infty$, when the limits are defined.

Consider next the limits $\lim_{x\to 0} \sin(1/x)$ and $\lim_{x\to 0}(1/x^2)$. The limits do not exist (I cannot plug $x = 0$ into the functions). Sketch them and determine that they behave differently: the former is bounded, the latter is unbounded. In fact $\lim_{x\to 0}(1/x^2) = \infty$. More precisely we have:

Definition 3. $\varepsilon - B$ definition of $\lim_{x\to x_0} f(x) = \infty$.

Let $f(x)$ be a function defined in an interval containing x_0, except possibly at $x = x_0$. We say that $f(x)$ approaches ∞ as x approaches x_0 if given any real number $B > 0$, there exists a $\varepsilon > 0$, so that whenever $|x - x_0| < \varepsilon$ and $x \neq x_0$, we have $f(x) > B$. We write $\lim_{x\to x_0} f(x) = \infty$. [Definition of $\lim_{x\to x_0} f(x) = -\infty$ totally analogous.]

In the example $f(x) = 1/x^2$ we found that as $x \to 0$, $f(x) \to \infty$ - the function is even, so it does not matter if I approach the limit from the right (i.e. through positive values of x) or the left (through negative x values). What about $f(x) = 1/x$? It is not hard to see that as x tends to 0 through *positive* values then $f \to +\infty$, whereas as x tends to 0 through *negative* values we have $f \to -\infty$.

Hence, we need to define

Definition 4. One-Sided Limits:

Let $f(x)$ be defined for all x in an interval (x_0, a). We say that $f(x)$ approaches ℓ as x approaches x_0 from the right if, for any $\varepsilon > 0$, there exists a $\delta > 0$, such that for all $x_0 < x < x_0 + \delta$ we have $|f(x) - \ell| < \varepsilon$. We write $\lim_{x\to x_0+} f(x) = \ell$. [Analogous definition for the left-sided limit, i.e. $\lim_{x\to x_0-} f(x) = \ell$.]

Note: If $\lim_{x\to x_0} f(x) = +\infty$ or $-\infty$, then the line $x = x_0$ is a *vertical asymptote.* Analogously, if $\lim_{x\to\pm\infty} = \ell_\pm$ then the lines $y = \ell_\pm$ are *horizontal asymptotes.*

As an **E**xample consider $f(x) = \frac{1}{(x-1)(x-2)^2}$. There are vertical asymptotes at $x = 1$ and $x = 2$ and a horizontal asymptote $y = 0$. Sketch the graph *without using the differentiation methods of finding critical points etc., that you are familiar with. Use intuition and estimation..*

In addition to the basic properties (1)-(1), there is another powerful test which is very useful in calculations:

Comparison Test for Limits (a.k.a. Squeezing Property)

1. If $\lim_{x\to x_0} f(x) = 0$ and $|g(x)| \le |f(x)|$ for all x near x_0 with $x \ne x_0$, then $\lim_{x\to x_0} g(x) = 0$.
2. If $\lim_{x\to\infty} f(x) = 0$ and $|g(x)| \le |f(x)|$ for all large enough x, then $\lim_{x\to\infty} g(x) = 0$.

Example

(i) Establish Comparison Test 1 using the $\varepsilon - \delta$ definition of a limit.

(ii) Show that $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$.

Solution

(i) Since $\lim_{x\to 0} f(x) = 0$, then given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x)| < \varepsilon$ when $|x - x_0| < \delta$. For the same ε and δ, we also have $|g(x)| < \varepsilon$ when $|x - x_0| < \delta$, since $|g(x)| \le |f(x)|$. Hence $\lim_{x\to 0} g(x) = 0$ also.

(ii) Take $g(x) = x\sin(1/x)$ and $f(x) = x$. Then $|g(x)| \le |x|$ for all $x \ne 0$, so the comparison test applies. Clearly $\lim_{x\to 0} x = 0$, hence the result follows.

Two Basic Trigonometric Limits

We will need the following results in finding derivatives of sin and cos from first principles.

$$\lim_{h\to 0} \frac{\sin h}{h} = 1 \qquad\qquad \lim_{h\to 0} \frac{\cos h - 1}{h} = 0.$$

The proof of the former is geometrical and the construction is given in Figure 1.1. OBC is the sector of a circle of radius 1 with subtended angle h. The two triangles OAB and OCD are constructed as shown with BD the extension of OB. Considering triangles OAB and OCD we have

$$\sin h = \frac{AB}{OB} = s, \qquad\qquad \tan h = \frac{DC}{OC} = t.$$

From geometry we have the following inequality

$$\text{area of triangle } OAB < \text{area of sector } OCB < \text{area of triangle } OCD,$$

which in turn provides

$$\frac{1}{2}\sin h \cos h < \frac{h}{2} < \frac{1}{2}\tan h.$$

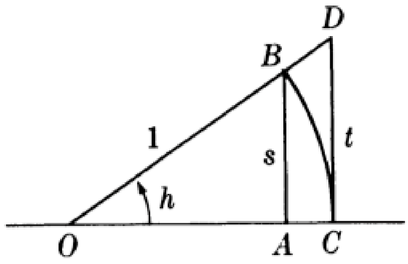


Figure 1.1: Geometrical construction.

The middle quantity follows by noting that the area of the sector OCB is equal to $h/2\pi$ times the area of a circle of unit radius which is π. Considering the first inequality (after canceling the $1/2$ factor throughout) we have

$$\sin h \cos h < h \quad \Rightarrow \quad \frac{\sin h}{h} < \frac{1}{\cos h}.$$

The above is fine since h and $\cos h$ are positive and non-zero so I can divide by them.

The second inequality gives

$$h < \frac{\sin h}{\cos h} \quad \Rightarrow \quad \cos h < \frac{\sin h}{h}.$$

Putting these together gives

$$\cos h < \frac{\sin h}{h} < \frac{1}{\cos h}.$$

As h tends to zero $\cos h$ tends to 1, hence $\sin h/h$ is squeezed between two numbers that tend to 1. By the Squeezing Property we get the desired result.

To prove the second result we write

$$\begin{aligned} \frac{\cos h - 1}{h} &= \frac{\cos h - 1}{h}\,\frac{\cos h + 1}{\cos h + 1} = \frac{\cos^2 h - 1}{h(\cos h + 1)} \\ &= \frac{-\sin^2 h}{h(\cos h + 1)} = \left(-\frac{\sin h}{h}\right)\frac{\sin h}{\cos h + 1}. \end{aligned}$$

Using the product rule for limits, it follows immediately that

$$\lim_{h\to 0}\left(-\frac{\sin h}{h}\right)\frac{\sin h}{\cos h + 1} = \underbrace{\left(\lim_{h\to 0}\frac{-\sin h}{h}\right)}_{\to -1}\underbrace{\left(\lim_{h\to 0}\sin h\right)}_{\to 0}\underbrace{\left(\lim_{h\to 0}\frac{1}{\cos h + 1}\right)}_{\to 1/2} = 0.$$

Continuity

Looking back at Definition 1, the $\varepsilon - \delta$ definition of a limit, we can see that it is equivalent to the statement

$$\lim_{h\to 0} f(x_0 + h) = \ell.$$

We can then define what we mean by continuity of a function $f(x)$ at a point x_0.

Definition 5. Continuity

We say that f is continuous at x_0 if $\lim_{h\to 0} f(x_0+h) = f(x_0)$. Equivalently $\lim_{x\to x_0} f(x) = f(x_0)$.

We have seen examples of functions that are not continuous, e.g.

$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x \le 0 \end{cases} \qquad\qquad g(x) = \begin{cases} x^2 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

are both *not continuous* at $x = 0$. If I exclude $x = 0$, then the limits as $x \to 0$ exist, $\lim_{h\to 0+ f(h)=1}$, $\lim_{h\to 0-} f(h) = 0$, and $\lim_{h\to 0} g(h) = 0$. *This may clarify some confusion I may have generated in video 1.* (Note that $\lim_{h\to 0+} = \lim_{h\to 0, h>0}$ and $\lim_{h\to 0-} = \lim_{h\to 0, h<0}$.)

Miscellaneous Examples

1. Find $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right)$, and interpret the result geometrically by considering a right angled triangle with base of length x and unit height.

   We calculate

   $$\left(\sqrt{x^2+1}-x\right) = \left(\sqrt{x^2+1}-x\right)\frac{\left(\sqrt{x^2+1}+x\right)}{\left(\sqrt{x^2+1}+x\right)} = \frac{1}{\sqrt{x^2+1}+x}.$$

   As x becomes arbitrarily large then $1/(\sqrt{x^2+1}+x)$ becomes arbitrarily small, and hence $\lim_{x\to\infty}\left(\sqrt{x^2+1}-x\right) = 0$. [Can you prove this using the $\varepsilon - A$ definition of limit?]

   Geometrical picture for you to do. Hint: The right angled triangle suggested has hypotenuse $\sqrt{x^2+1}$. Consider a circle of radius x whose arc cuts the hypotenuse at a point, and figure out what the quantity $\sqrt{x^2+1}-x$ represents geometrically.

2. Now consider $\lim_{x\to\infty}\left(x-\sqrt{x+1}\right)$. Find the limit in this case.

   Don't need to do much here. Main thing is to notice that x is much much bigger than $\sqrt{x+1}$ when x is large. Hence, $\lim_{x\to\infty}\left(x-\sqrt{x+1}\right) = \infty$.

   A precise definition in this case (for a general function $f(x)$) would be: We say $\lim_{x\to\infty} f(x) = \infty$, if given an arbitrarily large $A > 0$, there exists a number $M > 0$, so that $f(x) > A$ for all $x > M$.

   In our particular example where $f(M) = M - \sqrt{M+1}$ it is easy to see that taking $M = A^2$ will do the trick. Of course it can be proven for smaller M but we are not looking for anything sharper than a proof.

3. Find (a) $\lim_{x\to 1}\frac{1}{(x-1)^2}$, and (b) $\lim_{x\to\infty}\frac{1-x^2}{x^{3/2}}$.

For (a) as $x \to 1$ *(from above or below), then* $(x-1)^2$ *becomes arbitrarily small. Its inverse becomes arbitrarily large, so* $\lim_{x\to 1} \frac{1}{(x-1)^2} = \infty$

Intuitive answer is: For (b) as x *becomes very large then* $x^2 \gg x^{3/2}$*, hence the limit is* $-\infty$*. Can formalize as follows*

$$\lim_{x\to\infty} \frac{1-x^2}{x^{3/2}} = \lim_{x\to\infty} (x^{-3/2} - x^{1/2}) = -\infty.$$

Part II

Differentiation

Chapter 2

Derivative of a Function

2.1 Definition with limits, examples

Consider graphs of functions $y = f(x)$. We need to define the *derivative* or slope of the curve at a given point P.

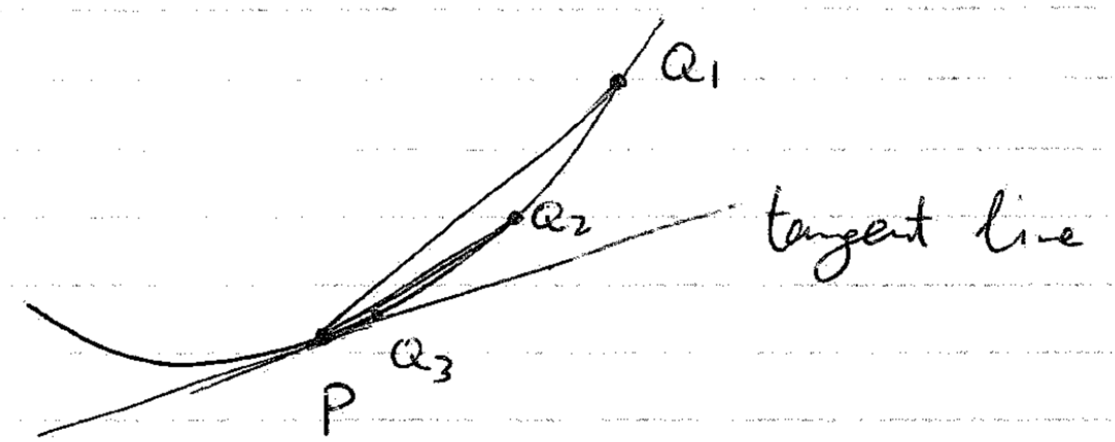


Figure 2.1: Slope of f at P is the slope of the line QP as Q tends to P. Note: The Qs are to the right of P, the definition is the same is Q_1, Q_2 etc are to the left of P.

Definition of Differentiability

The function $f(x)$ is differentiable at x if 'Newton's quotient';

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

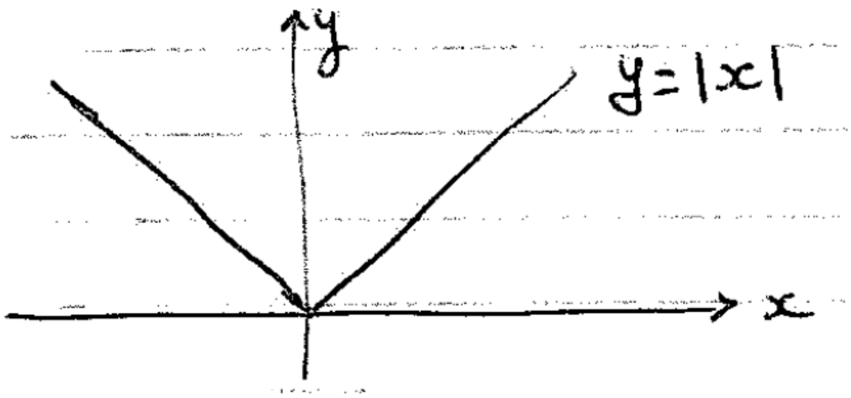
exists. We call this $f'(x)$, the derivative of f at point x.

Examples

(i) Is $f(x) = x^2$ differentiable everywhere?

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h\to 0}(2x+h) = 2x \quad \Rightarrow \textbf{YES}$$

(ii) Is $f(x) = |x|$ differentiable at $x = 0$? Draw a picture.



Need to check if the limit exists *and* the values are equal as we approach 0 from above or below.

(a)

$$\lim_{h\to 0,\, h>0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0,\, h>0} \frac{h-0}{h} = 1$$

(b)

$$\lim_{h\to 0,\, h<0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0,\, h<0} \frac{-h-0}{h} = -1$$

Right and left derivatives *exist* but are not *equal.*

A function is differentiable at x if right and left derivatives exist *and* are if the derivatives are equal.

Exercise: Sketch the derivative of $f(x) = |x|$.

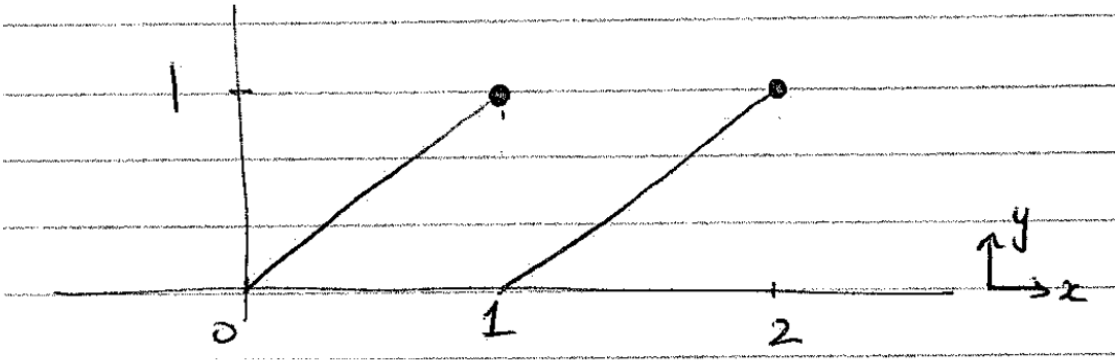
$f(x) = |x|$ is continuous at $x = 0$, but not differentiable there. Geometrically we can see this - there is a 'corner' in the graph. Now consider a function that is discontinuous at one or more points. What is the derivative there?

Note: $f(x)$ is *not continuous* at $x = x_0$ if the limit $\lim_{x\to x_0}$ *does not exist.*

Example: Consider

$$f(x) = \begin{cases} x & \text{if} \quad 0 < x \leq 1 \\ x - 1 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

Here is the graph:



What is the derivative at $x = 1$?

(a) Left derivative at $x = 1$

$$\lim_{h \to -0,\, h<0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to -0,\, h<0} \frac{1 + h - 1}{h} = 1$$

(b) Right derivative at $x = 1$

$$\lim_{h \to -0,\, h>0} \frac{f(1+h) - f(1)}{h} = \lim_{h \to -0,\, h>0} \frac{(1 + h - 1) - 1}{h} \qquad \text{as } f(1) = 1$$

$$= \lim_{h \to -0,\, h>0} (1 - \frac{1}{h})$$

which does not exist. *In fact,* $\to -\infty$. Function has no right derivative.

2.1.1 Polynomials

Theorem 1
Let n be an integer ≥ 1 and let $f(x) = x^n$. Then

$$f'(x) = \frac{\mathrm{d}f}{\mathrm{d}x} = nx^{n-1}.$$

Proof.

$$f(x+h) = (x+h)^n = x^n + nx^{n-1}h + h^2 g(x,h)$$

where $g(x,h)$ involves powers of x and h with numerical coefficient. We don't care what it is exactly but $\lim_{h\to 0} g(x,h) =$ *some number*. Then

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{x^n + nx^{n-1}h + h^2 g - x^n}{h} = nx^{n-1}.$$

□

Theorem 2
Let $f(x) = x^a$, where a is any real number and $x > 0$. Then $f'(x) = ax^{a-1}$.
If a is a negative integer then this is easy. General case is different from proof above.

2.2 General rules, chain rule, rates of change

2.2.1 General rules

(i) If c is a constant, $(cf)'(x) = cf'(x)$.

(ii) If $f(x)$, $g(x)$ are given functions and $f'(x)$, $g'(x)$ exist, then

$$(f+g)'(x) = f'(x) + g'(x).$$

(iii) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$ *(product rule)*

(iv) Let $g(x)$ be a function that has a derivative $g'(x)$ and such that $g(x) \neq 0$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{g(x)}\right) = -\frac{g'(x)}{(g(x))^2}$$

(iv)*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}.$$

Proof. (iii) - *(do the rest yourselves as an exercise)*

$$\begin{aligned}
(fg)' &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x) \overbrace{-f(x)g(x+h) + f(x)g(x+h)}^{=0}}{h} \\
&= \lim_{h\to 0} \frac{\left(f(x+h) - f(x)\right)g(x+h) + \left(g(x+h) - g(x)\right)f(x)}{h} \\
&= g(x)f'(x) + f(x)g'(x)
\end{aligned}$$

□

2.2.2 The Chain Rule

Composition $(fog)(x) = f(g(x))$ is a function constructed as follows: Take a number x, find $g(x)$ and then take the value of f at $g(x)$.

$$\begin{aligned} &e.g. \quad f(x) = x^2 \quad g(x) = \sqrt{x} \quad \text{defined for } x > 0. \\ &\text{Then} \quad (fog)(x) = (\sqrt{x})^2 = x \end{aligned}$$

Let f, g be two functions having derivatives and such that f is defined for all numbers that are values of g. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(fog)(x) = (fog)'(x) = f'(g(x))g'(x)$$

Why is this useful? **Example:** $\frac{\mathrm{d}}{\mathrm{d}x}\left[(x^3 + 9x^2 + \pi)^{51}\right]$

Last thing you want to do is multiply out the 51 factors and then differentiate! With the chain rule we identify

$$f(x) = x^{51} \quad g(x) = (x^3 + 9x^2 + \pi).$$

So that

$$\begin{aligned} (x^3 + 9x^2 + \pi)^{51} &= (fog)(x) \\ \text{Then} \quad \frac{\mathrm{d}}{\mathrm{d}x}(x^3 + 9x^2 + \pi)^{51} &= 51(x^3 + 9x^2 + \pi)^{50}(3x^2 + 18x) \end{aligned}$$

Proof.

$$\begin{aligned} (fog)'(x) &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\ &= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h} \end{aligned}$$

Let $k = g(x+h) - g(x)$, *(if $h \neq 0, k \neq 0$)*, and write $u = g(x)$. Then

$$\begin{aligned} (fog)'(x) &= \lim_{h \to 0} \frac{f(u+k) - f(u)}{k} \cdot \frac{g(x+h) - g(x)}{h} \\ &= \left(\lim_{h \to 0} \frac{f(u+k) - f(u)}{k}\right)\left(\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}\right) \\ &= f'(u)g'(x) = f'(g(x))g'(x) \end{aligned}$$

□

Analogous definition of the derivative $f'(x)$ is the following:

$$f'(x) = \lim_{y \to x} \frac{f(y)f(x)}{y - x}.$$

To see equivalence write $y = x + h$.

Application: Particle motion (rectilinear for the moment).

Position of a particle at time t is $s = f(t)$, say. Particle moved from P_1 at $t = t_1$ to P_2 at $t = t_2$. Average speed =

$$\frac{f(t_2) - f(t_1)}{t_2 - t_1}.$$

So instantaneous speed at any time t is

$$f'(t) = \lim_{t \to t_0} \frac{f(t) - f(t_0)}{t - t_0} \quad \text{rate of change.}$$

$f'(t)$ is also a function, call it $v(t)$. If it is differentiable then

$$v'(t) = \frac{\mathrm{d}^2 f}{\mathrm{d}t^2} \quad \text{is the acceleration.}$$

Can define higher derivatives (if they exist) by continuing this process.

Theorem 3
If $f(x)$ is differentiable at $x = x_0$, then it is also continuous there. *Question: Is the converse true?*

Proof.

$$\begin{aligned}
\lim_{x \to x_0} (f(x) - f(x_0)) &= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \cdot x - x_0 \right) \\
&= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} \right) \cdot \lim_{x \to x_0} (x - x_0) \\
&= f'(x_0) \cdot 0 = 0 \quad \textbf{DONE!}
\end{aligned}$$

□

2.3 Implicit differentiation, related rates of change

Recall: We saw that if n is an integer then

$$\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1} \quad \frac{\mathrm{d}}{\mathrm{d}x} (x^{-n} = -n x^{-(n+1)}.$$

This also holds if, (i) $y = x^{1/n}$ where n is an integer, and, (ii) $y = x^r$ where r is a rational number; *i.e* $r = \frac{p}{q}$, with p, q integers.

Can prove these using *implicit differentiation*. Start with (i) $y = x^{1/n}$, n integer. Assume $x^{1/n}$ is defined.

Then

$$\begin{aligned} y^n = x \quad &\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}x}(y^n) = \frac{\mathrm{d}}{\mathrm{d}x}(x), \\ ny^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad &\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{n}y^{1-n} = \frac{1}{n}x^{\frac{1-n}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}. \end{aligned}$$

e.g.

$$\frac{\mathrm{d}}{\mathrm{d}x}x^{\frac{1}{5}} = \frac{1}{5}x^{-\frac{4}{5}}$$

(ii) $y = x^{\frac{p}{q}}$. Let $g(x) = x^{\frac{1}{q}}$ q an integer. Then $y = (g(x))^p$ with p an integer. Use chain rule.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = pg^{p-1}\frac{1}{q}x^{\frac{1}{q}-1} = \frac{p}{q}x^{\frac{p}{q}-1}$$

These of course generalize to powers of the function.
e.g.

$$\frac{\mathrm{d}}{\mathrm{d}x}(f(x))^r = rf^{r-1}f' \quad r \text{ rational.}$$

Example of implicit differentiation:
Find the equation of the tangent line to the curve $2x^6 + y^4 = 9xy$ at the point $(1, 2)$.

Solution: Note that we cannot solve for y as a function of x. Hence implicit differentiation is very powerful here. Calculate the derivative

$$12x^5 + 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = 9y + 9x\frac{\mathrm{d}y}{\mathrm{d}x}.$$

Substitute point $(1, 2)$

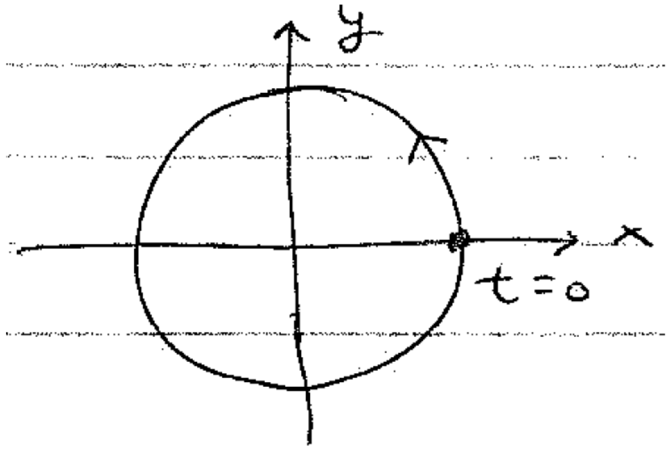
$$\begin{aligned} 12 + 32\frac{\mathrm{d}y}{\mathrm{d}x} &= 18 + 9\frac{\mathrm{d}y}{\mathrm{d}x} \\ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{6}{23} \quad \text{is the slope of the tangent line} \end{aligned}$$

Its equation is $y - 2 = \frac{6}{23}(x - 1)$.

Can also use implicit differentiation to obtain *related rates of change.*

If x and y are both functions of a parameter t, then we can differentiate implicitly with respect to t

$$\textit{e.g.} \quad x = \cos(t),\ y = \sin(t),\ t \geq 0$$



Equation is $x^2 + y^2 = 1$, where $x = x(t)$, $y = y(t)$. Differentiate implicitly with respect to t.

$$2x\frac{\mathrm{d}x}{\mathrm{d}t} + 2y\frac{\mathrm{d}y}{\mathrm{d}t} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{x}{y}\frac{\mathrm{d}x}{\mathrm{d}t}$$
$$i.e \quad \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = -\frac{x}{y}.$$

This is the derivative of $\frac{\mathrm{d}y}{\mathrm{d}x}$ if we think of $y = \pm\sqrt{1-x^2}$ as a function of x. (Another way is $x^2 + y^2 = 1 \quad \to \quad 2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \to \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{x}{y} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t}$.)

In general, if $x = f(t)$ and $y = g(t)$, describe a curve in the plane called a *parametric curve*. The slope of it's tangent line is

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} \quad \text{if} \quad \frac{\mathrm{d}x}{\mathrm{d}t} \neq 0.$$

To prove this, note that the curve can be defined (piecewise) as the graph of a function $y = h(x)$ *or* $x = H(y)$. Chain rule gives $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t}$ as required.

Example
The surface area of a cube is growing at a constant rate of $4\text{cm}^2/\text{s}$. How fast is the length of a side growing when the cube sides are 2cm long? Find the side length when the rate of change of the volume exceeds that of the area.

Solution

$$A = 6x^2 \quad \Rightarrow \quad \frac{\mathrm{d}A}{\mathrm{d}t} = 12x\frac{\mathrm{d}x}{\mathrm{d}t} \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{12x}\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{4}{12x}$$
$$\text{If } x = 2 \quad \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{6}\text{cm s}^{-1}$$
$$V = x^3 \quad \Rightarrow \quad \frac{\mathrm{d}V}{\mathrm{d}t} = 3x^2\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{3x^2}{12x}\frac{\mathrm{d}A}{\mathrm{d}t} = x \text{ cm}^3\text{s}^{-1}$$

So if $x > 4$, $\frac{\mathrm{d}V}{\mathrm{d}t} > \frac{\mathrm{d}A}{\mathrm{d}t}$ in numerical value.

Chapter 3

Mean Value and Intermediate Value Theorems

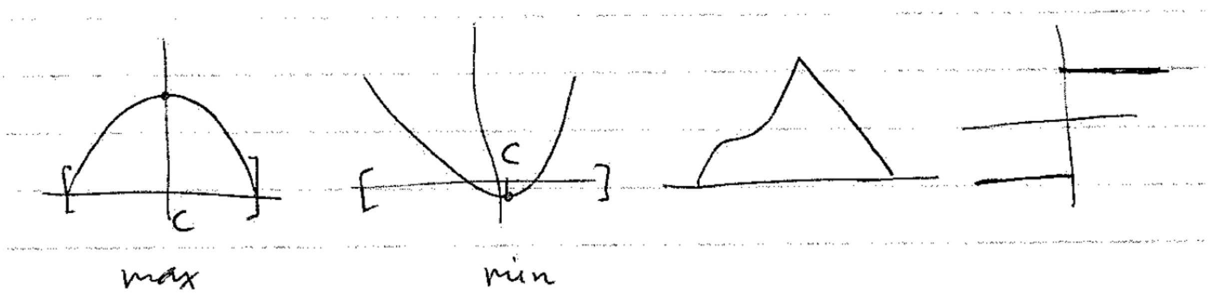
For a function $f(x)$ which is defined at a point c, we say that c is a *maximum* of f if

$$f(c) \geq f(x) \quad \forall \text{ where } f \text{ is defined.}$$

For a *minimum* we have

$$f(c) \leq f(x).$$

e.g.

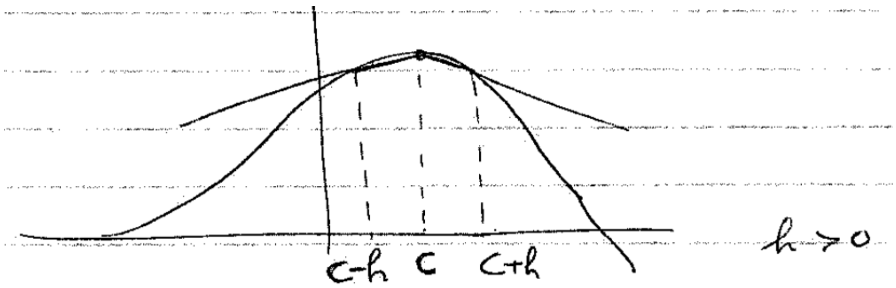


Here is a result when f is differentiable:

Theorem 1
Let f be a function which is defined and differentiable on the open interval (a, b). Let c be a number in the interval which is a maximum for the function.

Then $f'(c) = 0$. $f'(c) = 0$ also, if c is a minimum of f.

Proof. Obvious, here is a geometrical interpretation.



As $h \to 0$, the slope $\to 0$, $\Rightarrow f'(c) = 0$.

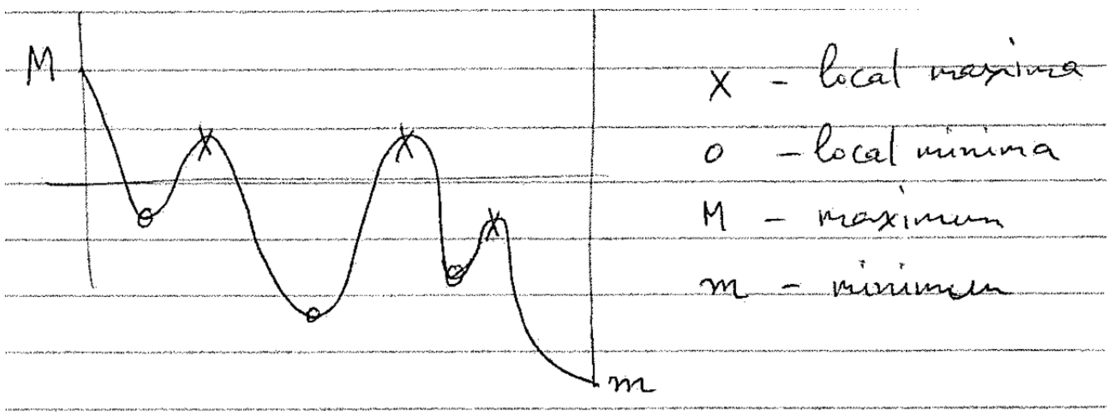
Detailed proof:
$f(c) \geq f(c+h) \Rightarrow f(c+h) - f(c) \leq 0$.

$$i.e. \quad \lim_{h\to 0,\, h>0} \frac{f(c+h) - f(c)}{h} \leq 0$$

Similarly for left limit

$$\lim_{h\to 0} \frac{f(c) - f(c-h)}{h} \geq 0.$$

As $h \to 0$ these can only be equal if $f'(c) = 0$ since the function is differentiable. □

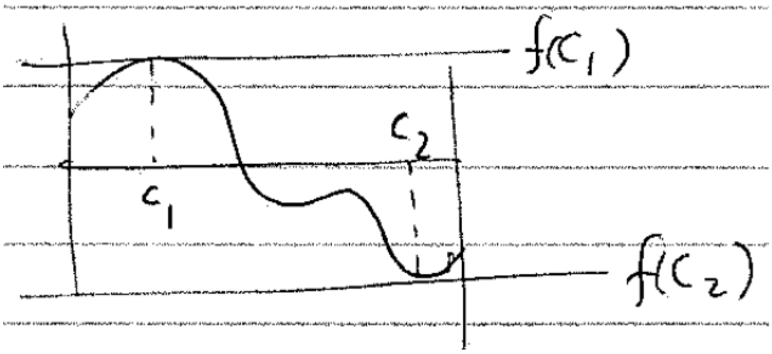


All points c such that $f'(c) = 0$ are called *critical points.*

Definition
$f(x)$ is said to be continuous on an interval $[a, b]$ if $\lim_{x\to x_0} f(x) = f(x_0)$ for all x_0 in $[a, b]$. Analogously, $\lim_{h\to 0} f(x_0 + h) = f(x_0) \quad a \leq x_0 \leq b$.

Theorem 2
Let $f(x)$ be continuous on the closed interval $[a, b]$. Then $f(x)$ has a maximum and a minimum on this interval. *i.e* There exists c_1 and c_2 so that $f(c_1) \geq f(x)$ and $f(c_2) \leq f(x)$ for all x in $[a, b]$.

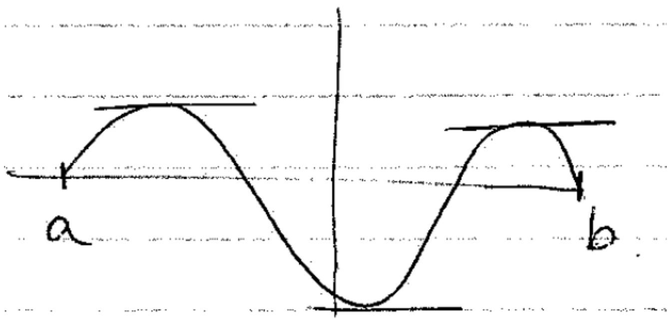
e.g



Theorem 3 (Combines Theorems 1 and 2)
Let $f(x)$ be continuous over the closed interval $a \leq x \leq b$ and differentiable on the open interval $a < x < b$. Assume also that $f(a) = f(b) = 0$.
Then there exists a point c, $a < c < b$, such that $f'(c) = 0$.

Proof. If f is a constant then nothing to prove. If f is not a constant then there exists a point in (a, b) where f is not zero. If at some point in (a, b), f is positive, then there exists a maximum c with $f(c) > 0$, and $c \neq a, b$. By Theorem 1, $f'(c) = 0$.
Similarly for a point where $f < 0$ and hence this is a minimum. □

Example:

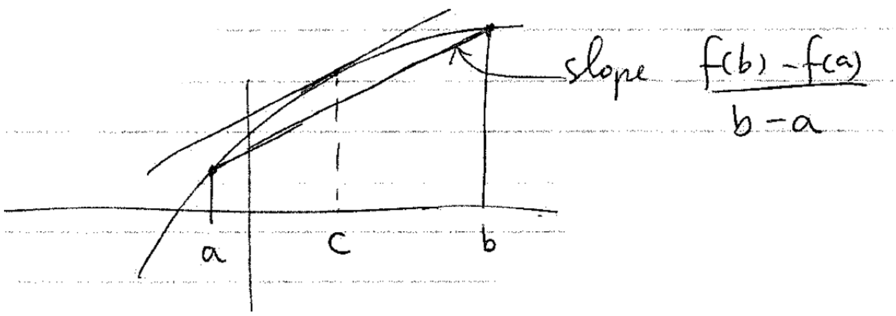


Here there are 3 such points.

Theorem 4
f is continuous on $[a, b]$ and differentiable on (a, b).
Then there exists $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation:



c is the point where the tangent has the slope $\frac{f(b)-f(a)}{b-a}$.

Proof. Straight line joining $(a, f(a))$ and $(b, f(b))$ has equation $y - \frac{f(b)-f(a)}{b-a}(x-a) + f(a)$.

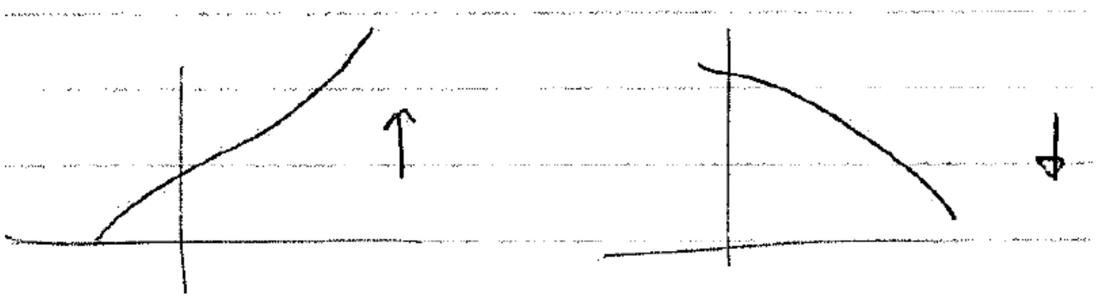
Consider $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$. Then $g(a) = 0$, $g(b) = 0$, and by Theorem 3 there exists a c with $a < c < b$ such that $g'(c) = 0$.

But $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$, and result follows. □

Definition

We say that f is increasing over a given interval if given x_1, x_2 in the interval with $x_1 \leq x_2$, we have $f(x_1) \leq f(x_2)$.

*Strictly **increasing** if $f(x_1) < f(x_2)$ when $x_1 < x_2$.*
*Strictly **decreasing** if $f(x_1) > f(x_2)$ when $x_1 < x_2$.*



Theorem 5

Let $f(x)$ be continuous in some interval, and differentiable there (even possibly at the end points).

If $f'(x) = 0$ in the interval (except end points), then f is constant.
If $f'(x) > 0$ in the interval (except end points), then f is strictly increasing.
If $f'(x) < 0$ in the interval (except end points), then f is strictly decreasing.

Proof. Use the mean value theorem.

Let x_1, x_2 be points in the interval with $x_1 < x_2$. Then there exists $x_1 < c < x_2$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_2} \quad \Longrightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

If $f'(x) = 0$ in the interval, $f'(c) = 0$ and $f(x_2) = f(x_1)$ i.e f is constant. If $f'(x) > 0$ then $f'(c) > 0$ and $f(x_2) > f(x_1)$, i.e strictly increasing. If $f'(x) < 0$ then $f'(c) < 0$ and $f(x_2) < f(x_1)$, *i.e* strictly decreasing. □

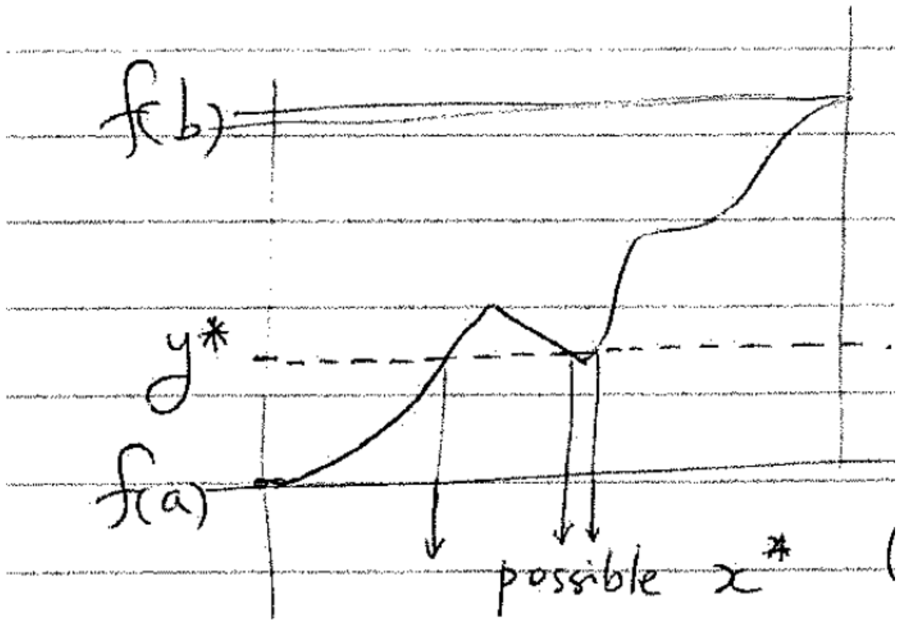
Example 1 *(Do yourself).* Determine the region of increase and decrease of the function $f(x) = x^3 - 2x + 1$.

Example 2 Prove that $\sin(x) \leq x$ for $x \geq 0$.
Solution Let $f(x) = x - \sin(x)$. Then $f(0) = 0$.

$f'(x) = 1 - \cos(x) \geq 0$ for all x. Hence $f(x)$ is an increasing function $Rightarrow$ $f(x) \geq 0$ for all x.

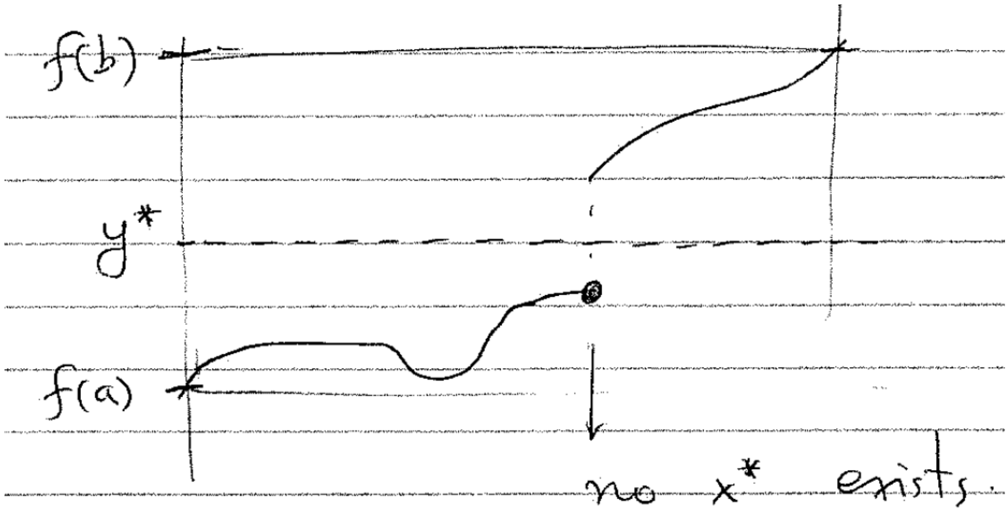
Theorem 6 - Intermediate value theorem
Let f be continuous on the closed interval $a \leq x \leq b$. Given any number y^* between $f(a)$ and $f(b)$, there exists a point x^* between a and b such that $f(x^*) = y^*$.

Picture where it works:



(there can be more than one x^*.)

Picture where is doesn't work:



Chapter 4

Inverse Functions

Given y as a function of x, when can I express x as a function of y? Here is an easy case:

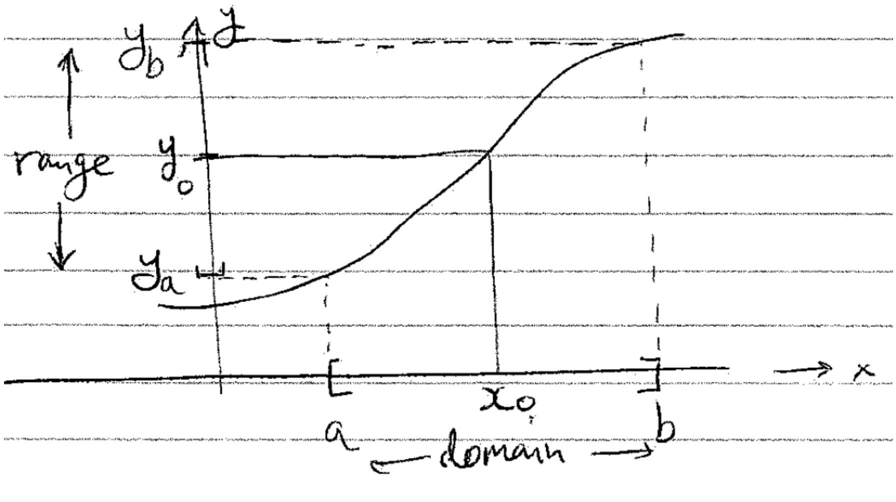
$$y = 3x + 1 \quad \Rightarrow \quad x = \frac{1}{3}(y - 1)$$

Usually we do not have a formula like this, but we can say a lot about the function $x = g(y)$.

Definition

Let $y = f(x)$ be defined on some interval. Given any y_0 in the range of f, if we can find a unique value x_0 in its domain such that $f(x_0) = y_0$, then we can define the ***inverse function***

$$x = g(y) \quad \text{(sometimes written } x = f^{-1}(y) \text{)}$$



Clearly we have

$$f(g(y)) = y \qquad \text{and} \qquad g(f(x)) = x$$
$$\text{or} \quad f(f^{-1}(y)) = y \qquad \text{and} \qquad f^{-1}(f(x)) = x$$

Question: When can we be *certain* an inverse function exists?

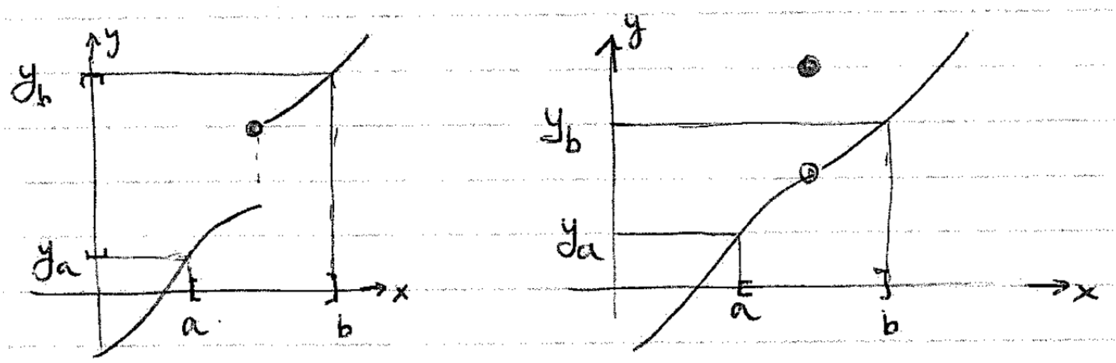
Theorem 1
Let $f(x)$ be strictly increasing or strictly decreasing. Then the inverse function exists.

Proof. Obvious from definition of strictly increasing/decreasing. □

Theorem 2
If $f(x)$ is continuous on $[a, b]$ and is strictly increasing (or decreasing), and $f(a) = y_a$ and $f(b) = y_b$, then $x = g(y)$ is defined on $[y_a, y_b]$.

Proof. Easy by the intermediate value theorem. □

Here is what goes wrong if we drop continuity



4.0.1 Derivative of inverse functions

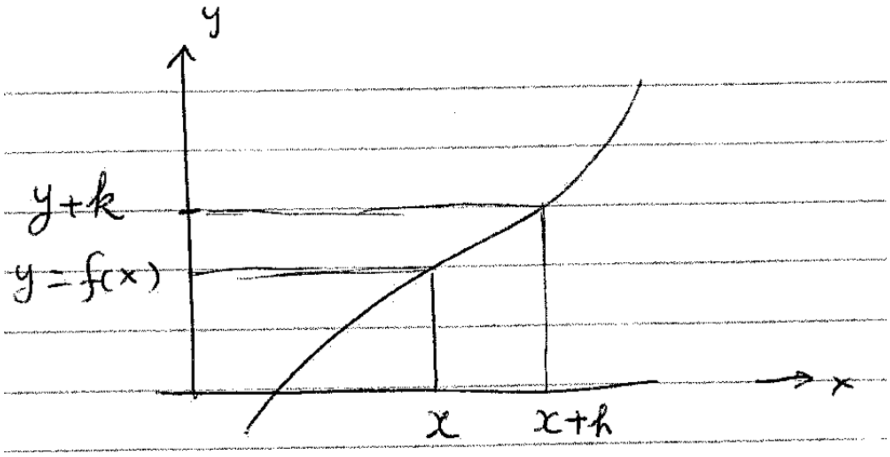
Theorem 3
Let $f(x)$ be differentiable on (a, b) and $f'(x) > 0$ or $f'(x) < 0$ for all x in (a, b). Then the inverse function exists and we have

$$g'(y)(= f^{-1}(y)') = \frac{1}{f'(x)}.$$

Proof. Need to find

$$\lim_{k\to 0} \frac{g(y+k) - g(y)}{k}$$

where we have $y = f(x)$. Here is a useful picture:



Going from y to $y+k$, increases x to $x+h$. The intermediate value theorem ensures that this is true for all k in fact.

Hence $f(x+h) = y+k \quad \Rightarrow \quad g(y+k) = x+h$. Since $f(x) = y \quad \Rightarrow \quad g(y) = x$.

Back to the limit

$$\lim_{k \to 0} \frac{g(y+k) - g(y)}{k}$$
$$\lim_{h \to 0} = \frac{h}{f(x+h) - f(x)} = \frac{1}{f'(x)}$$

□

Chain rule way: $g(y) = x$ where $y = f(x)$. *(In fact $x = g(y)$, iff $y = f(x)$). i.e*

$$\frac{\mathrm{d}g}{\mathrm{d}y} f'(x) = 1 \quad g' = \frac{1}{f'(x)}$$

Example: Consider

$$y = x^4 + 3x^3 + x - 5, \quad x > 0.$$

Find $g'(0)$ - *i.e* $\frac{\mathrm{d}}{\mathrm{d}y} f^{-1}(y)\big|_{y=0}$. **Note:** We will not even attempt to find $x = g(y)$!

Theorem says $\frac{\mathrm{d}g}{\mathrm{d}y} = g'(y) = \frac{1}{f'(x)}$ where $y = f(x)$. If $y = 0$ then need to solve $0 = f(x)$ by inspection $f(1) = 0 \Rightarrow f'(1) = 14 \Rightarrow g'(0) = \frac{1}{14}$.

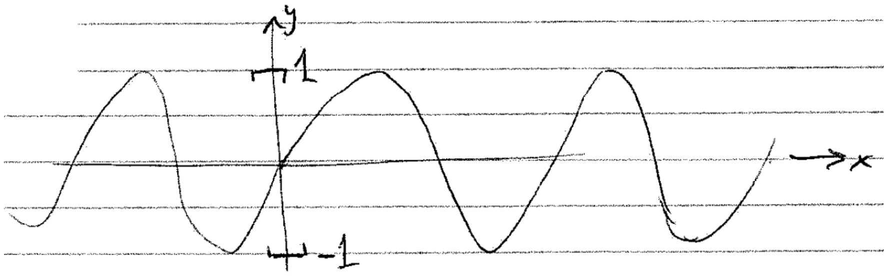
Note: Very useful in solving problems.

$x = g(y)$ *i.e.* some function of y. Our theorem really says $\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\mathrm{d}y/\mathrm{d}x}$.

4.0.2 Some special inverse functions

(i) The arcsin, or $\sin^{-1}$.

Consider $y = \sin(x)$ *(shown below)*

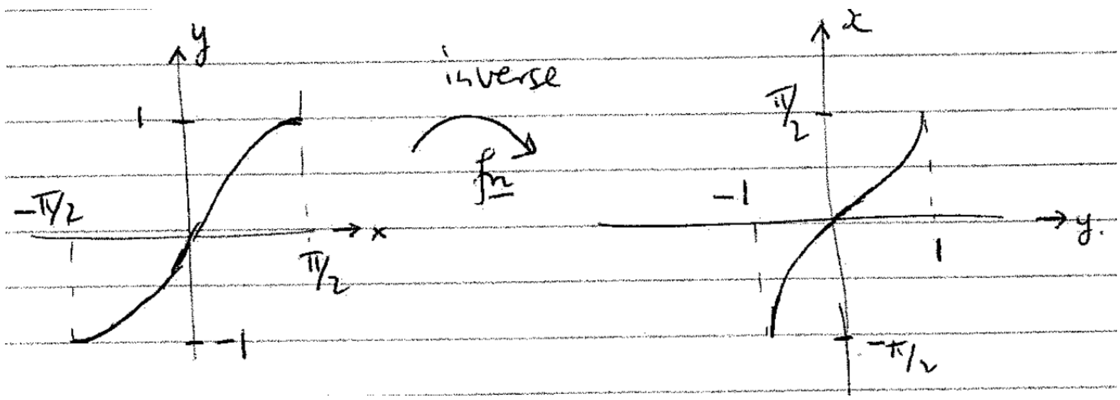


So given any $-1 \leq y \leq 1$ there are an infinite number of x such that $y = \sin(x)$. If we restrict out domain to regions where $\sin(x)$ is strictly increasing or decreasing, then we can find inverse functions - we have a theorem. By convention we take $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$ for the domain (range is of course $[-1, 1]$).

Now $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) > 0$ if $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and $\frac{\mathrm{d}}{\mathrm{d}x}\sin(x) = 0$ at $x = \pm\frac{\pi}{2}$.

Let the inverse function be $g(y) = x$. Then

$$g'(y) = \frac{1}{f'(x)} = \frac{1}{(\sin(x))'} > 0 \quad x \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$



$$g'(y) = \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\cos(x)} = \frac{1}{\sqrt{1-\sin^2(x)}} = \frac{1}{\sqrt{1-y^2}}.$$

Think of y as a dummy variable now. Then $\arcsin(y) = \sin^{-1}(y)$ is a function with domain $[-1, 1]$ and range $[-\frac{\pi}{2}, \frac{\pi}{2}]$. Instead of y use x, *i.e.* $y = f(x) = \sin^{-1}(x)$. Then

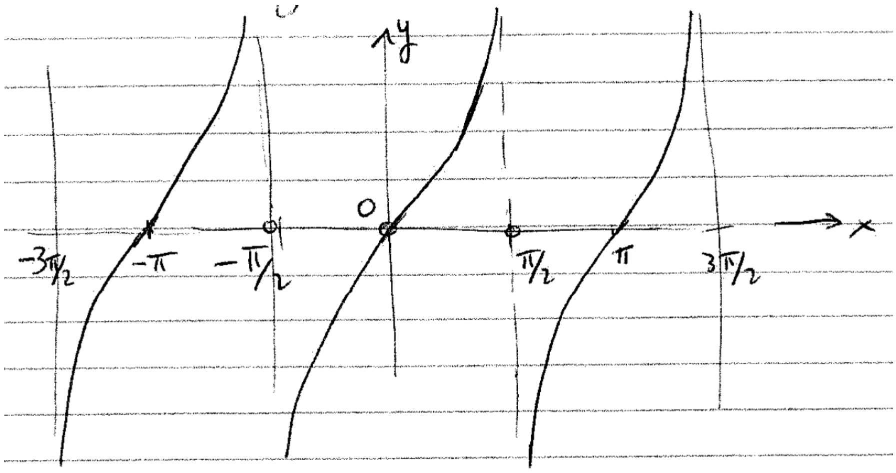
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}.$$

Once you have identified the domain and range where the inverse function exists, there is an easier way (equivalent) to find derivatives.

Start with $y = \sin^{-1}(x)$ i.e y is the angle whose sin is x. Then $\sin(y) = x$. By chain rule, $(\cos(y))\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$ as shown above.

(ii) arctan or $\tan^{-1}$

Consider $y = \tan(x)$

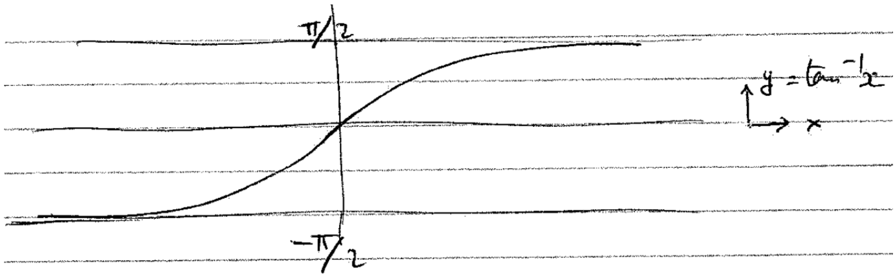


Can define $\tan(x)$ on $(-\frac{\pi}{2}, \frac{\pi}{2})$, $\frac{\mathrm{d}}{\mathrm{d}x}\tan(x) = 1 + \tan^2(x) > 0$. So $x = g(y)$ the inverse function has domain $(-\infty, \infty)$ and range $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Derivative of tan^{-1}:

$$y = \tan^{-1}(x)$$
$$\tan(y) = x$$
$$(1 + \tan^2(y))\frac{\mathrm{d}y}{\mathrm{d}x} = 1 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{1+x^2}$$

Here is the graph of $y = \tan^{-1}(x)$



Note for later material: we showed that $\frac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}(x) = \frac{1}{1+x^2}$. In other words, the *anti-derivative* of $\frac{1}{1+x^2}$ is $\tan^{-1}(x) + c$ where c is a constant. Similarly, the *anti-derivative* of $\frac{1}{\sqrt{1-x^2}}$ is $\sin^{-1}(x) + c$.

Examples: *(Do yourself.)* Find $\arctan\big(\tan(\frac{3\pi}{4})\big)$, $\arctan(\tan(2\pi))$.

Chapter 5

Exponentials and Logarithms

Summary of what we know:

If $a > 0$ is any real number, and r any rational number we know how to define a^r. What about a^x where x is *any* real number (including irrationals)? It is also a *continuous* function of x.

Intuitively, we can imagine that any irrational x can be approximated by rationals and accurate values obtained. Here is an example: $2^{\sqrt{3}}$. A decimal approximation of $\sqrt{3} \approx 1.732050808 \Rightarrow \frac{1732}{1000} < \sqrt{3} < \frac{17321}{10000}$. 2^x is an increasing function, therefore

$$2^{\frac{1732}{1000}} < 2^{\sqrt{3}} < 2^{\frac{17321}{/}10000} \quad i.e.\ 2^{\sqrt{3}} \approx 3.322 \text{ correct to 3 decimal places.}$$

There is a way of defining a^x and deriving all its properties by using properties of real numbers - though this is technical and we do not have time to do it.

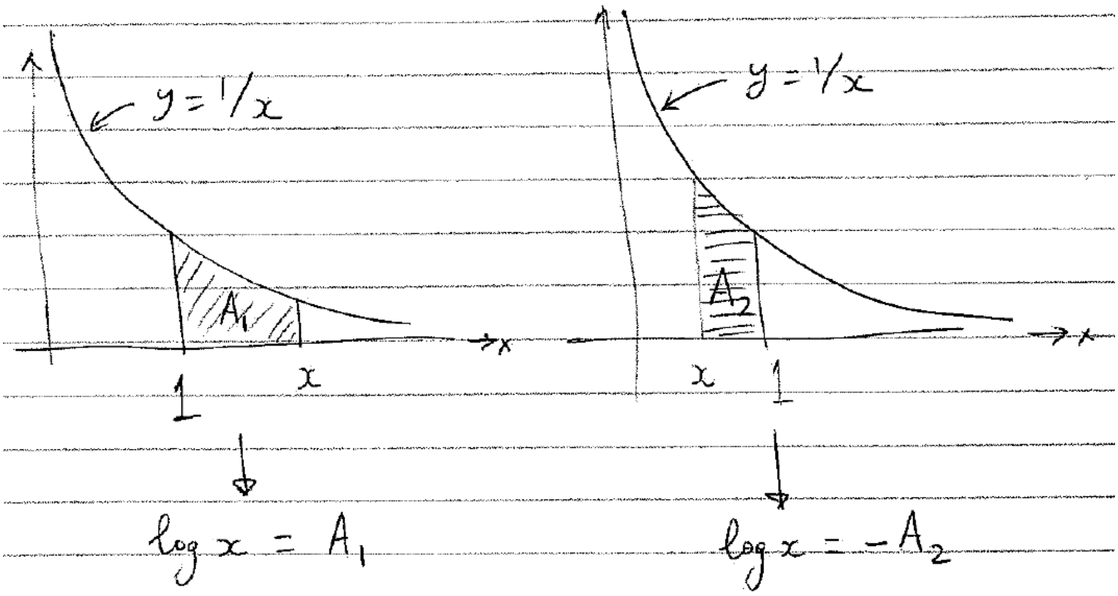
We will do it in a more intuitive way that may seem a bit unnatural at first, namely by defining a new function (the logarithm) and then defining the exponential as the inverse function of the logarithm. Advantage of this - intuitive, simple and clear arguments.

5.1 Geometrical Definition, Derivative

The following leads to the natural logarithm.

Definition

$\log(x)$ *is the area under the curve* $\frac{1}{x}$ *between 1 and* x *if* $x \geq 1$*; and* ***negative*** *the area under the curve* $\frac{1}{x}$ *between 1 and* x *if* $0 < x < 1$*. In particular,* $\log(0) = 1$.



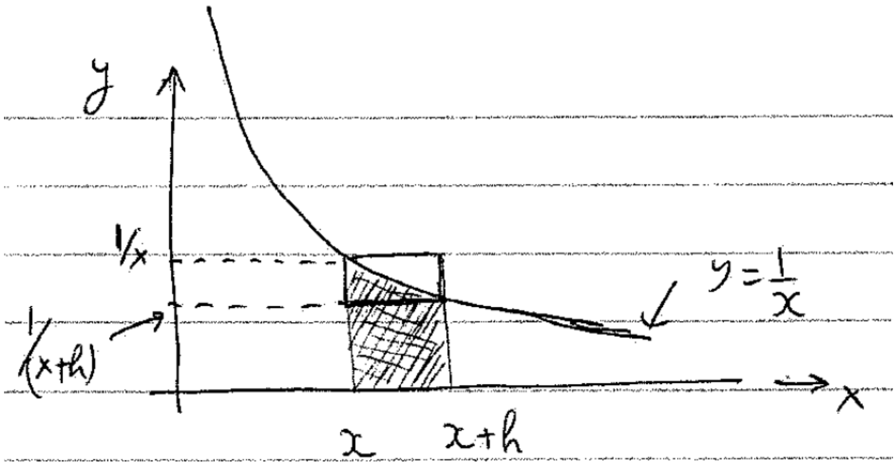
Hence $\log(x) \geq 0$ if $x \geq 1$; and $\log(x) < 0$, $0 < x < 1$.

Theorem 1
$\log(x)$ is differentiable and $\frac{\mathrm{d}}{\mathrm{d}x}\log(x) = \frac{1}{x}$.

Proof. Need to consider the Newton quotient and prove

$$\lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h} = \frac{1}{x}.$$

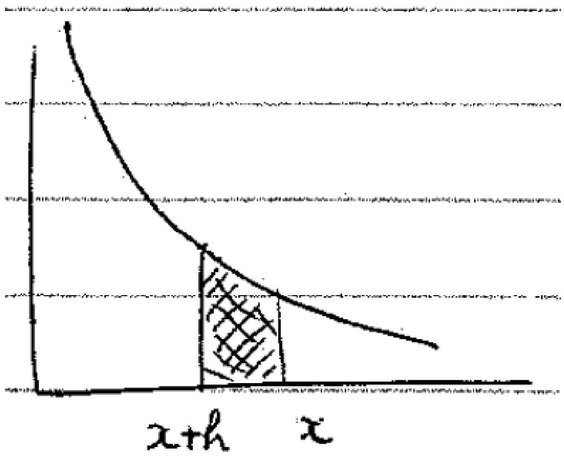
Start with $h > 0$ and consider the area under the curve between x and $x + h$.



$\log(x + h) - \log(x)$ is the shaded area above. From geometry we have *(or from the fact that $\frac{1}{x}$ is a decreasing function)*:

$$\frac{1}{x+h} \cdot h < \log(x+h) - \log(x) < \frac{1}{x} \cdot h$$
$$\Rightarrow \quad \frac{1}{x+h} < \frac{\log(x+h) - \log(x)}{h} < \frac{1}{x}$$

(here $h > 0$ so we can divide by h as done above.) As $h \to 0$ we use the squeezing theorem to get the required limit. If $h > 0$ the picture is:



So

$$-h \cdot \frac{1}{x} < \log(x) - \log(x+h) < -h \cdot \frac{1}{x+h}$$
$$\Rightarrow \frac{1}{x+h} < \frac{\log(x+h) - \log(x)}{h} < \frac{1}{x}$$

Hence limit is $\frac{1}{x}$ as $h \to 0$. □

$\log(x)$ is a function defined for $x > 0$, has $\log(1) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}x}\log(x) = \frac{1}{x}$. **(*)**

We will use (*) alone in what follows. If $g(x)$ is another such function then $g(x) = \log(x)$ uniquely. The condition $\log(1) = 0$ fixes this.

Theorem 2
If $a, b > 0$, then $\log(ab) = \log(a) + \log(b)$.

Proof. Let $f(x) = \log(ax)$, $x > 0$ and a as above.

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{1}{ax} \cdot a = \frac{1}{x} \quad \text{by the chain rule.}$$

i.e. same derivative as $\log(x) \Rightarrow$ they differ by a constant.

$$\Rightarrow \log(ax) = \log(x) + K \quad \forall x > 0$$

Put $x = 1 \Rightarrow K = \log(a)$. Put $x = b \Rightarrow \log(ab) = \log(b) + \log(a)$. □

Theorem 3
$\log(x)$ is strictly increasing for all $x > 0$. Its range is $(-\infty, \infty)$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}x}\log(x) = \frac{1}{x} > 0 \quad \text{for all } x > 0 \Rightarrow \text{strictly increasing.}$$

To prove that it takes on arbitrarily large values, note that since it is strictly increasing and $\log(1) = 0$, we must have, for example, $\log(2) > 0$.

From Theorem 2,

$$\log(2^n) = \log(2 \cdot 2 \cdot \cdots \cdot 2) = \overbrace{\log(2) + \log(2) + \cdots + \log(2)}^{n \text{ terms}} = n\log(2).$$

This holds for *any* positive integer and $\log(2) > 0 \Rightarrow$ as n becomes large, so does $\log(2^n)$. To prove it takes on arbitrarily large *negative* values, note that

$$0 = \log(1) = \log\left(2 \cdot \frac{1}{2}\right) = \log(2) + \log\left(\frac{1}{2}\right)$$

$$\Rightarrow \log\left(\frac{1}{2}\right) = -\log(2)$$

Hence

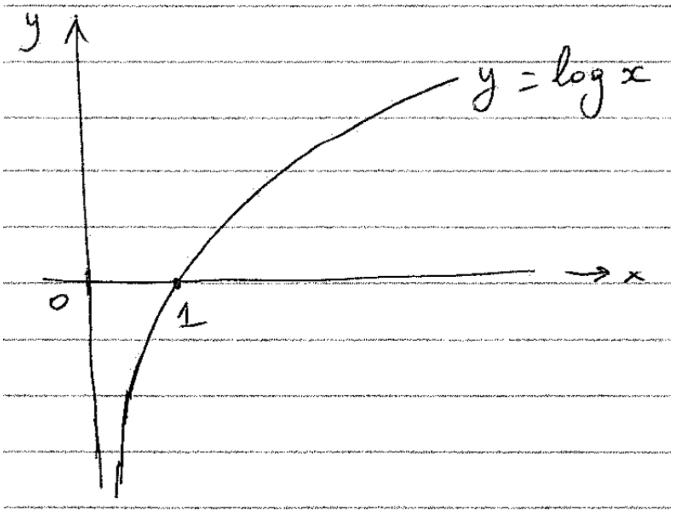
$$\log\left(\frac{1}{2^n}\right) = -n\log(2) \quad \text{by Theorem 2}$$

$$\to -\infty \text{ as } n \to \infty$$

□

Theorem 4
If n is an integer (positive or negative) then $\log(a^n) = n\log(a)$ for all $a > 0$.

Proof. As above - simple use of Theorem 2.



Domain $(0, \infty)$, range $(-\infty, \infty)$, strictly increasing. □

5.2 Exponential as Inverse of $\log x$

Define the exponential as the inverse function of $\log(x)$. We know it exists - write it as $\exp(x)$. Since $0 = \log(1)$ by inverse we have $\exp(0) = 1$.

Theorem 5
If x_1, x_2 are two numbers, then $\exp(x_1 + x_2) = \exp(x_1) \cdot \exp(x_2)$.

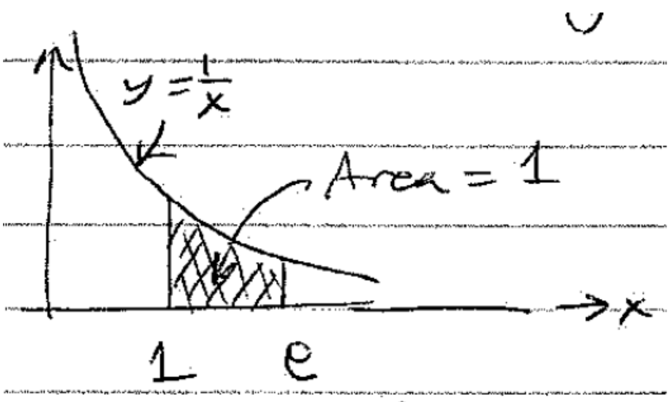
Proof. Let $a = \exp(x_1),\ b = \exp(x_2)$. By inverses, $x_1 = \log(a),\ , x_2 = \log(b)$.
By Theorem 2

$$x_1 + x_2 = \log(a) + \log(b) = \log(ab)$$

$$\Rightarrow\ ab = \exp(x_1 + x_2) \text{ as required.}$$

Define the number e to be $\exp(1)$, *i.e.* $\log(e) = 1$, or $\exp(1) = e$. □

Geometric interpretation of e shown below:



Can show easily (by induction for example) that

$$\exp(n) = e^n \quad \text{for every positive integer } n.$$

Since $1 = \exp(0) = \exp(m - m) = \exp(m)\exp(-m)$ by Theorem 5.

$$\exp(-m) = \frac{1}{e^m} = e^{-m} \quad m \text{ a positive integer.}$$

Theorem 6
$\exp(x)$ is differentiable and $\frac{\mathrm{d}}{\mathrm{d}x}\exp(x) = \exp(x)$.

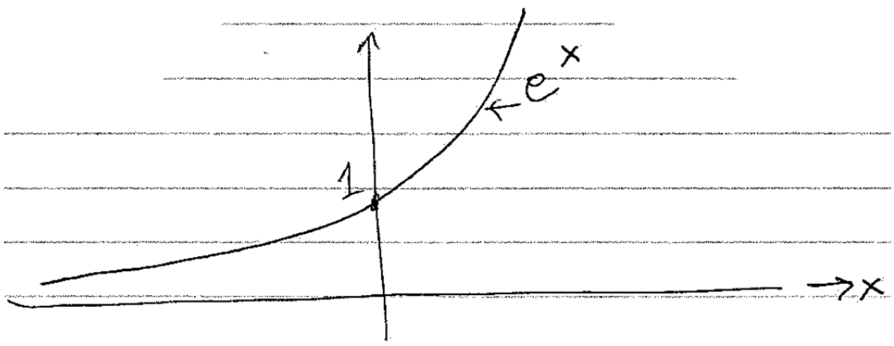
Proof. Think of $\frac{\mathrm{d}}{\mathrm{d}x}\exp(x)$ as the derivative of the inverse function to $\log(x)$. Hence it is differentiable.

We have proved that if $g(y)$ is the inverse function of $y = f(x)$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$ *i.e.* $f'(x) = \frac{1}{g'(y)}$ or $g'(y) = \frac{1}{f(x)}$. Let $y = \exp(x)$, then the inverse function is $x = \log(y)$. Chain rule $1 = \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = y = \exp(x)$. *(Equivalently* $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{d}x/\mathrm{d}y}$*).* □

Use e^x instead of $\exp(x)$ - completely analogous. Derivative for definition:

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = e^x \lim_{h\to 0}\left(\frac{e^h - 1}{h}\right).$$

By Theorem 6, we have $\lim_{h\to 0}\frac{e^h-1}{h} = 1$.[1]



Generally we may have the exponential function

$$y = a^x \quad a > 0.$$

Can write this as $y = \exp(x\log(a)) = e^{x\log(a)}$. All the usual properties $a^{x+y} = a^x a^y$ *etc.* all hold.

Theorem 7

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = a^x(\log(a))$$

[1]This suggests that $\lim_{h\to 0}(1+h)^{1/h} = e$. Prove by $(1+h)^{1/h} = \exp\left(\frac{1}{h}\log(1+h)\right) = \exp\left(\frac{\log(1+h)-\log(1)}{h}\right) \to e$.

Proof.

$$\frac{\mathrm{d}}{\mathrm{d}x}a^x = \frac{\mathrm{d}}{\mathrm{d}x}\exp(x\log(a)) = \exp(x\log(a))\cdot\log(a) \text{ by the chain rule}$$

□

Corollary:

$$\lim_{h\to 0}\frac{a^h-1}{h} = \log(a) \quad \text{for } a>0$$

Finally, the general power function.

Theorem 8
Let a be any number and let $f(x) = x^a$ for $x > 0$. Then $f'(x)$ exists and $f'(x) = ax^{a-1}$.

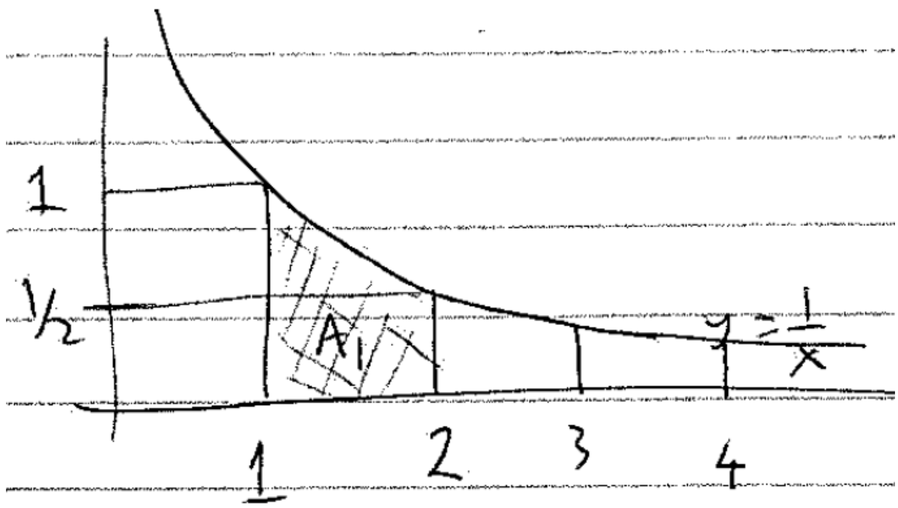
Proof.

$$\begin{aligned} f(x) &= x^a = e^{\log(x^a)} = e^{a\log(x)} \\ \Rightarrow\ f'(x) &= e^{a\log(x)}\cdot\frac{a}{x} \text{ by chain rule} \\ &= ax^{a-1} \end{aligned}$$

□

5.3 Function Estimates for Small and Large Arguments

Start with showing $2 < e < 4$. We know that $\log(e) = 1$ *i.e.* the area between $x = 1$ and $x = e$ of $y = 1/x$, is 1.



Since $\log(2) < 1$, *i.e.* area $A_1 < 1$, so we must have $e > 2$. Now

$$\log(4) = 2\log(2) > 2 \times \frac{1}{2} = 1.$$

This implies $e < 4 \Rightarrow 2 < e < 4$. Accurate calculation later using Taylor's theorem.

Theorem 9
Let a be any positive number. Then $\frac{(1+a)^n}{n} \to \infty$ as $n \to \infty$. [Analogously, $\lim_{n\to\infty} \frac{n}{(1+a)^n} = 0$.]

Proof. Write $(1+a)^n = 1 + na + \frac{n(n-1)}{2}a^2 + b$ where $b \geq 0$ is some number.

$$\Rightarrow \frac{(1+a)^n}{n} = \frac{1}{n} + a + \frac{n-1}{2}a^2 + \frac{b}{n}$$

$\frac{b}{n} \geq 0$, and so for large n $\frac{(1+a)^n}{n}$ becomes arbitrarily large. □

Corollary: $\frac{e^n}{n} \to \infty$ as $n \to \infty$, since $e = 1 + a$ for some $a > 0$.

Theorem 10
The function $f(x) = \frac{e^x}{x}$ is strictly increasing for $x > 1$ and $\lim_{x\to\infty} f(x) = \infty$. exp beats x.

Proof.

$$f'(x) = \frac{xe^x - e^x}{x^2} = \frac{e^x}{x^2}(x-1) > 0 \text{ for } x > 1$$

$\lim_{n\to\infty} f(n) = \infty$, hence result follows.

□

Corollary 1.
The function $x - \log(x)$ becomes arbitrarily large as x becomes arbitrarily large. x beats log.

Proof.

$$\log\left(\frac{e^x}{x}\right) = x - \log(x) > 0 \text{ for } x \text{ large enough}$$

Since $\log(t)$ becomes large for t large. □

Corollary 2
The function $\frac{x}{\log(x)}$ becomes large as x becomes large. x beats log.

Proof. Let $y = \log(x)$, then $x = e^y$. So $\frac{x}{\log(x)} = \frac{e^y}{y}$. $\log(x)$ becomes large as x becomes large, hence y also becomes large. By Theorem 10 the result follows. □

Corollary 3
As x becomes large, $x^{1/x}$ approaches the limit 1.

Proof.

$$x^{1/x} = e^{\log\left(x^{1/x}\right)} = e^{\frac{\log(x)}{x}}$$
$$\text{Now } \frac{\log(x)}{x} = \frac{1}{x/\log(x)} \to 0 \text{ as } x \text{ becomes large by Corollary 2}$$
$$\Rightarrow \lim_{x\to\infty} x^{1/x} = 1$$

□

Note Corollary 3 is used many times for integers. *i.e.* $n^{1/n} \to 1$ for $n \to \infty$ being integers.

Theorem 11 - $\exp(x)$ beats any power of x.
Let m be a positive integer. Then the function $f(x) = \frac{e^x}{x^m}$ is strictly increasing for $x > m$ and becomes arbitrarily large as x becomes arbitrarily large.

Proof.

$$f(x) = \frac{e^x}{e^{m\log(x)}} = e^{x-m\log(x)}$$
$$f'(x) = e^{x-m\log(x)}(1 - \frac{m}{x}) > 0 \text{if} x > m$$
$$\log(f(x)) = x - m\log(x) = (\log(x))(\frac{x}{\log(x)} - m)$$

By Corollary 2, $\log(f(x)) \to \infty \Rightarrow f(x) \to \infty$. □

5.4 Logarithmic Differentiation

Examples *(Do yourselves)*

(i) Differentiate $y = x^x\sqrt{x}$

(ii) Differentiate $y = \frac{(2x+1)^{1/2}}{(x^2+1)^{1/4}}$

5.5 L'Hôpital's Rule

Theorem 11
If f, g, are differentiable on an open interval containing x_0, $g(x_0) = f(x_0) = 0$, and $g'(x_0) \neq 0$, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$$

Proof. Write

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}}$$

$$\Rightarrow \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{\frac{f(x)-f(x_0)}{x-x_0}}{\frac{g(x)-g(x_0)}{x-x_0}} = \frac{f'(x_0)}{g'(x_0)}$$

□

Example 1

$$\lim_{x \to 0} \frac{1 - \cos(x)}{\sin(x)} \quad \text{of form} \quad \frac{0}{0}$$

$$\Rightarrow \quad = \lim_{x \to 0} \frac{\sin(x)}{\cos(x)} = 0$$

Example 2

$$\overbrace{\lim_{x \to 0} \frac{\sin(x) - x}{x^3}}^{0/0} = \underbrace{\lim_{x \to 0} \frac{\cos(x) - 1}{3x^2}}_{\text{also } 0/0} = \ \ldots \text{ carry on differentiating } \ldots$$

Problem is that we need to know that if $\lim_{x \to x_0} \frac{f'(x_0)}{g'(x_0)}$ which is of the form $\frac{0}{0}$ exists, then it is equal to the $\lim_{x \to x_0} \frac{f(x)}{g(x)}$. This is the useful form of L'Hôpital's rule.

Theorem 12
Let $f(x)$ and $g(x)$ be differentiable on an open interval containing x_0 (except possibly at x_0.) Assume that $g(x) \neq 0$ and $g'(x) \neq 0$ for x in an interval about x_0 but with $x \neq x_0$. Assume also that f, g are continuous at x_0 with $f(x_0) = g(x_0) = 0$, and $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = l$. Then also:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = l$$

Example 1:

$$\lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\lim_{x\to 0}\frac{\sin(x)}{2x}\quad \text{if latter limit exists}$$

Can use L'Hôpital's rule again to show (since limit of form $\frac{0}{0}$.)

$$\lim_{x\to 0}\frac{\sin(x)}{2x}=\lim_{x\to 0}\frac{\cos(x)}{2}=\frac{1}{2}$$

$$\Rightarrow \quad \lim_{x\to 0}\frac{1-\cos(x)}{x^2}=\frac{1}{2}\quad \text{by Theorem 2}$$

Note: Need to check that every time we apply L'Hôpital's rule the limit is of form 0/0.

Example 2:

$$\lim_{x\to 0}\left(\frac{x^2+1}{x}\right)\neq \lim_{x\to 0}\frac{2x}{1}=0.$$

Not of form 0/0, *i.e.* not indeterminate.

Example 3:

Show that $\lim_{x\to 0}\frac{\sin(x)-x}{\tan(x)-x}=-\frac{1}{2}$.

Practical note: L'Hôpital's rule holds for:

(i) One-sided limits

(ii) Limits as $x\to\infty$.

(iii) Indeterminate forms $\frac{\infty}{\infty}$.

Let us prove the rule for the form 0/0 as $x\to\infty$.

Proof. By assumption we have $\lim_{x\to\infty}\frac{f'(x)}{g'(x)}=l$ exists. If $y=\frac{1}{x}$, then $y=0_+$ as $x\to\infty$, *i.e.*

$$\begin{aligned}\lim_{x\to\infty}\frac{f'(x)}{g'(x)}&=\lim_{y\to 0_+}\frac{f'(1/y)}{g'(1/y)}=\lim_{y\to 0_+}\frac{-y^2f'(1/y)}{-y^2g'(1/y)}\\ &=\lim_{y\to 0_+}\frac{\frac{\mathrm{d}}{\mathrm{d}y}\left[f(1/y)\right]}{\frac{\mathrm{d}}{\mathrm{d}y}\left[g(1/y)\right]}=\lim_{y\to 0}\frac{f(1/y)}{g'(1/y)}\text{by L'Hôpital}\\ &=\lim_{x\to+\infty}\frac{f(x)}{g(x)}\end{aligned}$$

□

What about $\lim_{x\to x_0}\frac{f(x)}{g(x)}$ when of the form $\frac{\infty}{\infty}$? Would think that casting into $\frac{0}{0}$ form would help, but it doesn't. *i.e.* $\frac{f(x)}{g(x)}=\frac{1/g(x)}{1/f(x)}$ *i.e* this is now of form $\frac{0}{0}$. But

$$\lim_{x\to x_0}\frac{1/g}{1/f}\underbrace{=}_{\text{L'Hôp}}\lim_{x\to x+0}\frac{-g'/g^2}{-f'/f^2}$$

does not help! The proof is more technical, see later.

Theorem - L'Hôpital's Rule - general case.
To find $\lim_{x\to x_0} \frac{f(x)}{g(x)}$ when $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ are both zero or both infinite, differentiate numerator and denominator and take he limit of the new function. Repeat as many times as needed as long as L'Hôpital's rule applies at each stage. We then have

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}$$

Note that x_0 may be replaced by $\pm\infty$ or $x_0\pm$

Example 1

$$\begin{aligned} &\lim_{x\to\infty} \frac{\log(x)}{x^p},\ p > 0,\ \text{form} \frac{\infty}{\infty} \\ &= \lim_{x\to\infty} \frac{1/x}{px^{p-1}} = 0 \text{ since } p > 0 \end{aligned}$$

Example 2

$$\begin{aligned} &\lim_{x\to 0_+} x\log(x) \text{ of form } 0\times\infty \text{ so need to rewrite as} \\ &\lim_{x\to 0_+} \frac{log(x)}{1/x} \text{ which is of form } \frac{\infty}{\infty} \\ \Rightarrow &\lim_{x\to 0_+} x\log(x) = \lim_{x\to 0+} \frac{1/x}{-1/x^2} = 0. \end{aligned}$$

Example 3

(a) Find $\lim_{x\to 0_+} x^x$

Of form 0^0 (indeterminate).

(b) $\lim_{x\to 1} x^{-1/(1-x)}$

Of form 1^∞, again indeterminate.

Solutions

(a) $x^x = e^{x\log(x)}$. Have shown $\lim_{x\to 0_+} x\log(x) = 0$ and since exp is continuous, $\lim_{x\to 0_+} \exp(x\log(x)) = \exp\left[\lim_{x\to 0_+}(x\log(x)\right] = e^0 = 1$.

(b) Again use logs, namely

$$
\begin{aligned}
x^{-1/(1-x)} &= \exp\left(\frac{1}{x-1}\log(x)\right) \\
\lim_{x\to 1}\frac{\log(x)}{x-1} \text{ is of form } \frac{0}{0} &\Rightarrow \text{ by L'Hôp} \\
&= \lim_{x\to 1}\frac{1/x}{1} = 1 \\
\Rightarrow \lim_{x\to 1} x^{1/(x-1)} &= \exp\left[\lim_{x\to 1}\left(\frac{\log(x)}{x-1}\right)\right] \\
&= e \text{ by continuity of } \exp(x).
\end{aligned}
$$

Note: If we set $x = 1+\frac{1}{n}$ with n an integer, then we have shown $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.

Example 4

Find $\lim_{x\to 0}\left(\frac{1}{x\sin(x)} - \frac{1}{x^2}\right)$. *(The answer is* $1/6$ *and you need to apply L'Hôpital's rule 3 times!)*

Proof of L'Hôpital's rule when $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ i.e. $f(x_0) = g(x_0) = 0$. The case $x_0 \to \infty$ is done in the exercises. We need the following Theorem:

Theorem (Cauchy Mean Value Theorem)
Let f, g be continuous on $[a, b]$ and differentiable on (a, b) with $g(a) \neq g(b)$. Then there exists c in (a, b) such that

$$g'(c)\frac{f(b)-f(a)}{g(b)-g(a)} = f'(c).$$

Proof.

$$Leth(x) = f(a) + (g(x)-g(a))\frac{f(b)-f(a)}{g(b)-g(a)}.$$

Then $h(a) = f(a)$ and $h(b) = f(b)$. For the function $\phi(x) = h(x) - f(x)$ we have $\phi(a) = \phi(b) - 0$ and ϕ is differentiable on (a, b). Hence by the MVT, $\exists\, c \in (a, b)$ such that $\phi'(c) = 0$. *i.e.* $h'(c) = f'(c)$, and the theorem is proved. ☐

Proof. of L'Hôpital's Theorem

Here we prove the 0/0 version, *i.e.* $f(x_0) = g(x_0) = 0$.

$$\frac{f(x)}{g(x)} = \frac{f(x)-f(x_0)}{g(x)-g(x_0)} = \frac{f'(c')}{g'(c')}$$

where c' is a number between x and x_0. As $x \to x_0$, $c' \to x_0$ also. By hypothesis

$$
\begin{aligned}
\lim_{x\to x_0}\left[\frac{f'(x)}{g'(x)}\right] = l \;\Rightarrow\; &\lim_{x\to x_0}\frac{f'(c')}{g'(c')} = l \\
\Rightarrow\; &\lim_{x\to x_0}\frac{f(x)}{g(x)} = l \quad \text{as required}
\end{aligned}
$$

□

Part III

Integration

Chapter 6

Anti-derivatives and Geometrical Interpretation

The *anti-derivative* or *integral* of a function $f(x)$.

Given $f(x)$ defined over some interval, then if I can find a function $F(x)$ defined over the same interval such that

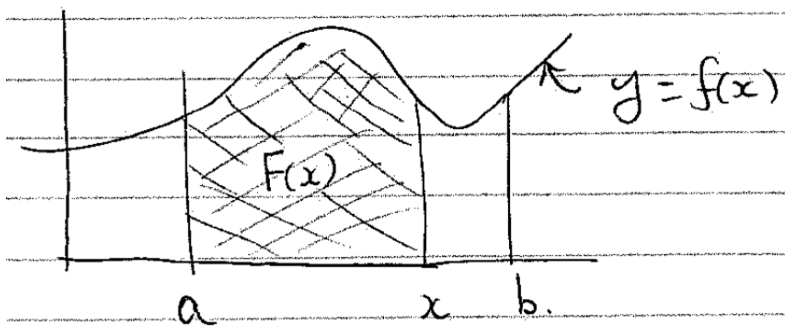
$$F'(x) = f(x),$$

then $F(x)$ is the *indefinite integral* of $f \;\Rightarrow\; F = \int f(x)\mathrm{d}x$. This is not unique. Let G be another indefinite integral, *i.e.* $G'(x) = f(x)$. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}(F - G) = 0 \;\Rightarrow\; F(x) = G(x) + \text{constant}.$$

6.0.1 Area under a curve

Suppose $f(x) \geq 0$ in some given interval $[a, b]$ and it is also continuous on $[a, b], (a < b)$. Define by $F(x)$ the area under the curve between $x = a$ and some x.



By definition, $F(a) = 0$.

Theorem 1
The function $F(x)$ is differentiable and its derivative is equal to $f(x)$. Another way to state this is $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\mathrm{d}t = f(x)$.

Proof. Newton quotient $\frac{F(x+h)-F(x)}{h}$.

Suppose $x \neq b$ and also $h > 0$. $F(x+h) - F(x)$ is the area under the graph between x and $x+h$.

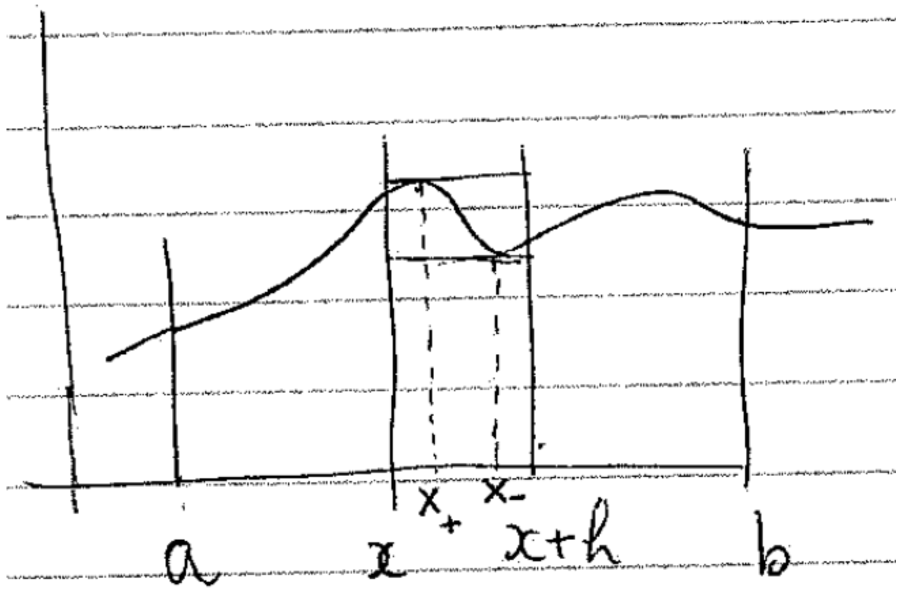


Figure 6.1:

Since $f(x)$ is continuous on $[x, x+h]$ and is defined there, it must have a maximum at some point x_+, and minimum at some point x_-. Hence, for all $t \in [x, x+h]$

$$f(x_-) \leq f(t) \leq f(x_+).$$

Can also bound the area using the rectangles shown in Figure 6.1.

$$\begin{aligned} h \cdot f(x_-) \ &\leq F(x+h) - F(x) \ \leq \ h \cdot f(x_+) \\ \textit{i.e.} \quad f(x_-) \ &\leq \frac{F(x+h)-F(x)}{h} \ \leq \ f(x_+) \end{aligned}$$

Since x_+ and x_- are contained in $[x, x+h]$, as $h \to 0$, $x_-, x_+ \to x$ and by the squeezing theorem we have $\lim_{h\to 0} \frac{F(x+h)-F(x)}{h} = f(x)$, *i.e.* $F'(x) = f(x)$. Hence the anti-derivative is connected to area under the curve. □

The constant is fixed by $F(a) = 0$. In other words, if I can guess a function $G(x)$ whose derivative is $f(x)$ *(e.g. guess* $\log(x)$ *for the anti-derivative of* $\frac{1}{x}$*)*, then since F and G differ by a constant I have

$$F(x) = G(x) + K.$$

But $F(a) = 0 \Rightarrow -G(a) = K \Rightarrow F(x) = G(x) - G(a)$. Hence

$$\int_a^b f(x)\mathrm{d}x = F(b) = G(b) - G(a).$$

This is the familiar *definite integral.*

Example 1

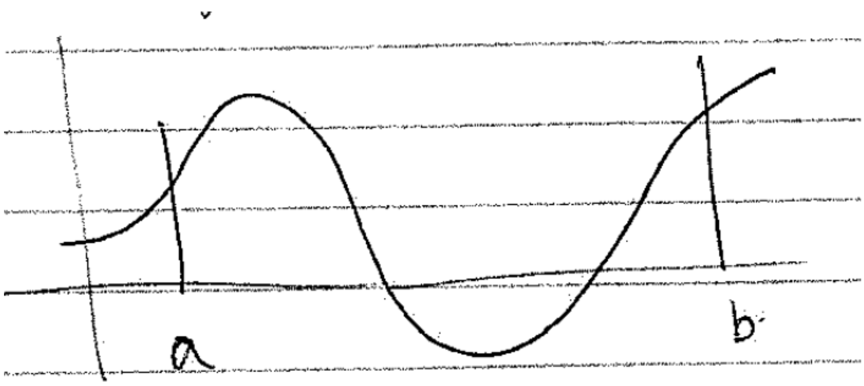
$$\int_1^2 x^2\mathrm{d}x = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3}.$$

Here $f(x) = x^2$, $G(x) = \frac{x^3}{3}$ is the guessed anti-derivative.

Definition: signed area

If $f(x) < 0$ then the area is below the x-axis. Define $F(x)$ to be ***minus*** *the area. (All very familiar). This leads to the definite integral.*

Example Draw example function $f(x)$



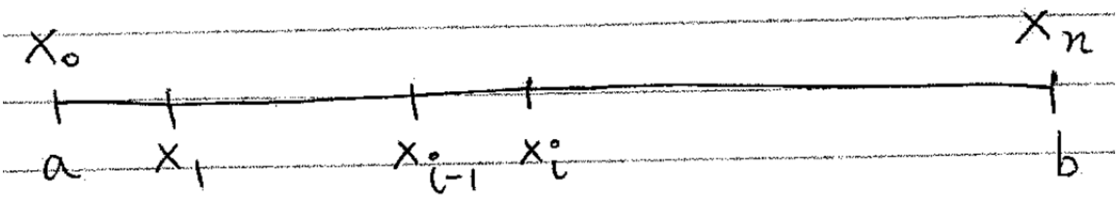
$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a) \quad \text{All negative areas are accounted for.}$$

Chapter 7

The Riemann Sum

Given $f(x)$, $a \le x \le b$, take a *partition* of the interval $[a, b]$ to be

$b - a$



Note: My partition has regular spacing. Can generalise this to have a partition defined by a sequence $\{x_k\}_{k=0,\dots,n}$ and in the limit $\max\limits_k |x_k - x_{k-1}| \to 0$. I am avoiding this technical issue which is quite irrelevant to what we want to do!

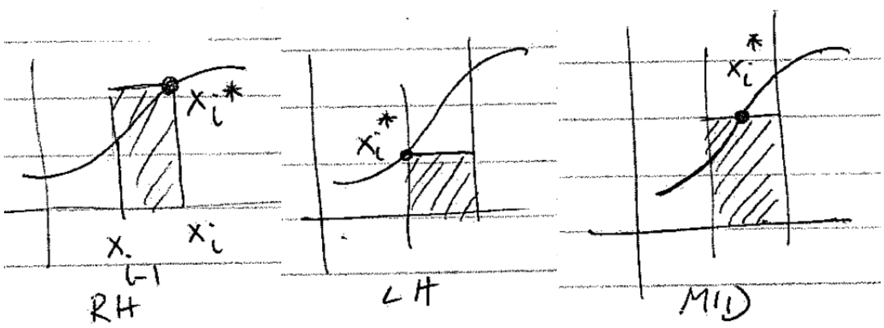
Take any sub-interval $[x_{i-1}, x_i]$ and let $x_i^* \in [x_{i-1}, x_i]$. Then the Riemann sum is $\sum_{i=1}^n f(x_i^*)h$.

Three particularly useful ways:

(i) $x_i^* = x_i$ "right-hand" RS (Riemann Sum)

(ii) $x_i^* = x_{i-1}$ "left-hand" RS

(iii) $x_i^* = \frac{1}{2}(x_i + x_{i-1})$ midpoint RS



Now in the limit $n \to \infty$, $h \to 0$, we can prove

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)h = \int_a^b f(x)\mathrm{d}x.$$

Sketch of the proof:

Lower Riemann sum $= \sum_{i=1}^{n} f(x_{i-1})h := L_n$

Upper Riemann sum $= \sum_{i=1}^{n} f(x_i)h := U_n$

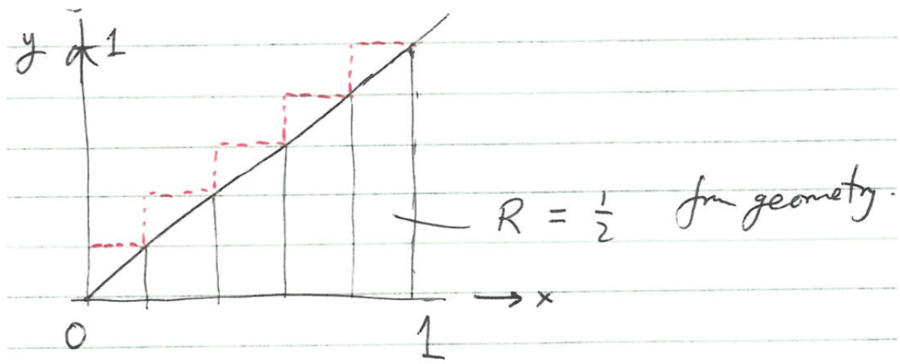
By geometry:

$$L_n \ \le \ \int_a^b f(x)\mathrm{d}x \ \le \ U_n.$$

In the limit it gets squeezed, if the limit exists then it is the integral.

Example 1:

$$f(x) = x \quad 0 \le x \le 1$$



$$a = 0,\ b = 1 \ \Rightarrow\ h = \frac{1}{n},\ x_i = ih = \frac{i}{n}$$

$$\begin{aligned}\text{Upper RS } &= \sum_{i=1}^{n} f(x_i)\frac{1}{n} = \sum_{i=1}^{n} \frac{i}{n}\frac{1}{n} = \frac{1}{n^2}\sum i = 1^n i \\ &= \frac{1}{n^2}\frac{n(n+1)}{2} = \frac{1}{2} + \frac{1}{2n} \ \to\ \frac{1}{2} \text{ as } n \to \infty\end{aligned}$$

Aside $\sum_{i=1}^{n} i = \frac{1}{2}n(n+1) := S$.

(i) Consider

$$\begin{array}{cccccc} & 1 & 2 & 3 & \ldots & n-1 & n \\ + & n & n-1 & n-2 & \ldots & 2 & 1 \\ \hline \end{array}$$

$$\text{Add} \quad 2S = \underbrace{(1+n) + (1+n) + \cdots + (1+n)}_{n \text{ times}} = n(1+n)$$

$$S = \frac{1}{2}n(n+1).$$

(ii) Consider $(i+1)^2 - i^2 = 2i + 1$.

$$\Rightarrow \sum_{i=1}^{n}(i+1)^2 - i^2 = 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1$$

$$i.e. \quad \underbrace{\cancel{2^2} - 1^2 + \cancel{3^2} - \cancel{2^2} + \cancel{\cdots} + (n+1)^2 - \cancel{n^2}}_{\text{“telescoping series”}} = 2S + n$$

$$S = \frac{1}{2}n(n+1)$$

Exercise: Re-do Example 1 but with (a) lower RS, (b) midpoint RS.

Example 2

$$\int_0^1 e^x \mathrm{d}x$$

$$\text{Upper Riemann Sum} \quad U_n = \sum_{i=1}^{n} e^{i/n}\frac{1}{n}$$

$$\int_0^1 e^x \mathrm{d}x = \lim_{n\to\infty} \frac{1}{n} \underbrace{\sum_{i=1}^{n} \left(e^{1/n}\right)^i}_{\text{geometric series}} = e - 1$$

7.0.1 Comparison between upper Riemann sum and midpoint RS

$$U_n = \sum_{i=1}^{n} f(x_i)h \quad M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)h$$

We know that $\lim_{n\to\infty} U_n = \lim_{n\to\infty} M_n = \int_a^b f(x)\mathrm{d}x$. Example for $\int_0^1 e^x \mathrm{d}x = e - 1 \approx 1.71828183 := I$ correct to 8 decimal places.

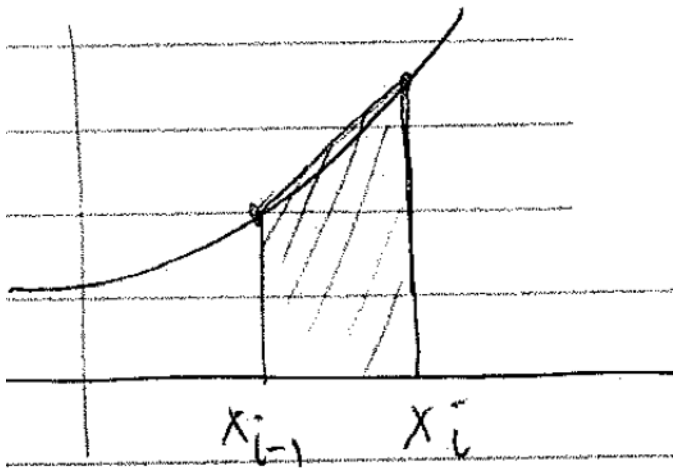
Here are some calculations

n	h	U_n	$\lvert I - U_n\rvert$	M_n	$I - M_n$
1	1	2.7183	1.0000	1.6487	0.0696
2	0.5	2.1835	0.4652	1.7005	0.0178
4	0.25	1.9420	0.2237	1.7138	0.0045

Conclusions: If h decreases by a factor of two, then the error $|I - U_n|$ decreases by 1/2, **but** $|I - M_n|$ decreased by 1/4.

Midpoint is far superior. Why? *(Geometrical explanation.)*

Question: Can you think of a better way still? We have to go beyond the Riemann sum definition - *Numerical Analysis*
Answer: Approximate the function by a linear function. *Geometry* - (trapezium rule).



Approximate $f(x),\ x \in [x_{i-1}, x_i]$

by $l_i(x) = \dfrac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}(x - x_{i-1}) + f(x_{i-1})$. *Can we get better still??*

Chapter 8

Properties of the Definite Integral; Fundamental Theorem of Calculus

1) $\int_a^b cf(x)\mathrm{d}x = c\int_a^b f(x)\mathrm{d}x \quad c$ constant.

2) $\int_a^b (f(x)+g(x))\mathrm{d}x = \int_a^b f(x)\mathrm{d}x + \int_a^b g(x)\mathrm{d}x$

3) If $c \in (a,b)$ (and here $a < b$), then

$$\int_a^b f(x)\mathrm{d}x = \int_a^c f(x)\mathrm{d}x + \int_c^b f(x)\mathrm{d}x$$

4) If $f(x) \le g(x)$ for $x \in [a,b]$ then

$$\int_a^b f(x)\mathrm{d}x \le \int_a^b g(x)\mathrm{d}x$$

Hence $\int_a^b f(x)\mathrm{d}x \le \int_a^b |f(x)|\mathrm{d}x$ and $\left|\int_a^b f(x)\mathrm{d}x\right| \le \int_a^b |f(x)|\mathrm{d}x$

5) $\int_a^b f(x)\mathrm{d}x = -\int_b^a f(x)\mathrm{d}x$.

Proofs follow easily from RS definitions and the use of signed areas.

Theorem 1
Suppose $g(x)$ is defined for all $x \in [a,b]$ and is differentiable on $[a,b]$. Then

$$\int_a^b g'(x)\mathrm{d}x = g(b) - g(a)$$

Proof. (Sketch)

Let $x_i = a + ih$, $h = \frac{b-a}{n}$.

$$\text{Upper RS} \quad \sum_{i=1}^{n} g'(x_i)h \approx \underbrace{\sum_{i=1}^{n} \frac{g(x_i+h)-g(x_i)}{\not h} \cdot \not h}_{\text{telescoping series}} = g(b+h) - g(a+h)$$

As $h \to 0$ result follows. □

Theorem: Fundamental Theorem of Calculus
Suppose F is differentiable on $[a,b]$ and F' is integrable on $[a,b]$. Then

$$\int_a^b F'(x)\mathrm{d}x = F(b) - F(a).$$

If f is integrable on $[a,b]$ and has *anti-derivative* F, then

$$\int_a^b f(x)\mathrm{d}x = F(b) - F(a).$$

Useful Theorem:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\mathrm{d}t = f(g(x)) \cdot g'(x).$$

Proof. Let $F(x) = \int_a^x f(t)\mathrm{d}t$. Then $F'(x) = f(x)$ - already proved.

Now $\int_a^{g(x)} f(t)\mathrm{d}t = F(g(x))$ by the definition of F.

$$\begin{aligned} \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{g(x)} f(t)\mathrm{d}t\right) = \frac{\mathrm{d}}{\mathrm{d}x}F(g(x) &= F'(g(x)) \cdot g'(x) \\ &= f(g(x)) \cdot g'(x). \end{aligned}$$

□

Example

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_a^{x^2} e^t \mathrm{d}t\right) = e^{x^2} \cdot 2x$$

$$\text{or} \quad \int_a^{x^2} e^t\mathrm{d}t = e^t\Big|_a^{x^2} = e^{x^2} - e^a \quad \text{same as before}$$

Chapter 9

Some Applications

Mechanics - very elementary knowledge needed! Newton's 2nd law says:
Force = mass × acceleration, Work = Force × distance.
MKS (meter-kilogram-second) system

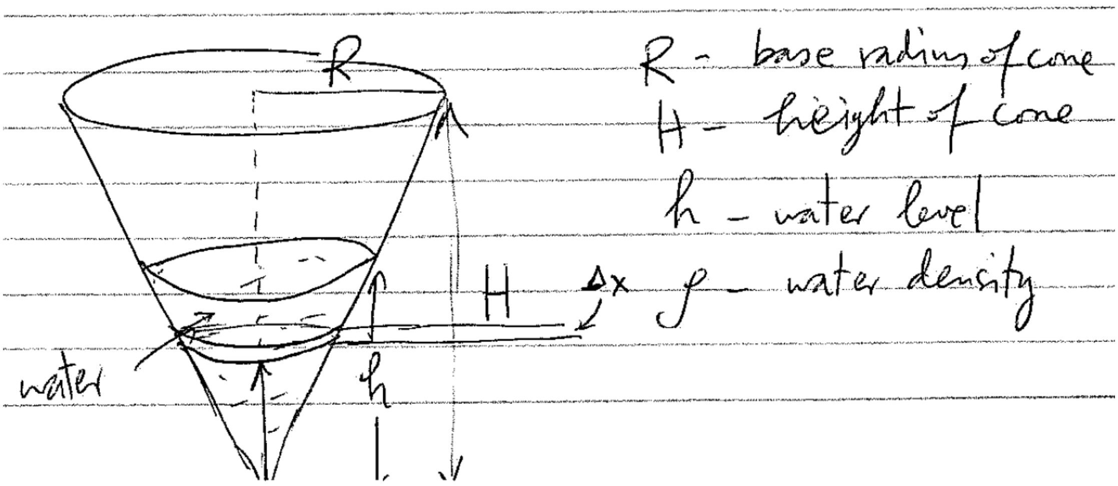
mass	kg
distance	m
time	s
force	Newton, N = kg·m/s^2
work	Joule, J = N·m

Example 1 - Find the work done in lifting a 1kg book to a height of 1m above its resting position.

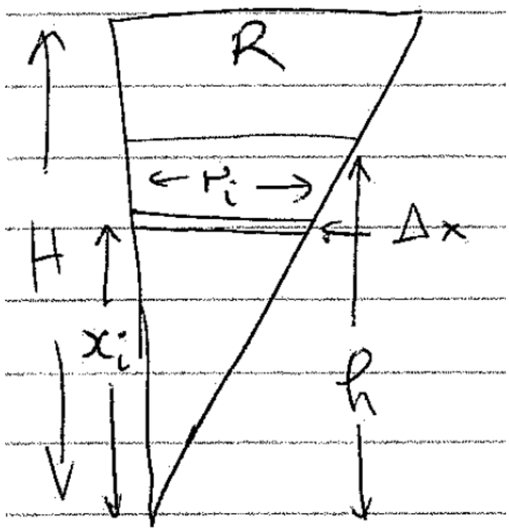
$$\mathrm{W = force \times distance = mg \times d = 1kg \times 9.8ms^{-2} \times 1m = 9.8J}$$

Example 2 - Work done in moving fluids.

Consider a water tank in the shape of an inverted cone. The tank is partially filled to a height h. If it gets emptied by taking the water to the top first, find the work done in pumping the water out.



Think of a "slice" of water of thickness Δx. Here is a cross section.



$\dfrac{r_i}{x_i} = \dfrac{R}{H}$. Mass of water slice $= \rho\pi \dfrac{R^2}{H^2} x_i^2 \Delta x$.

Work done in moving this to the top is $\Delta \mathrm{W} = \underbrace{\left(\rho\pi \dfrac{R^2}{H^2} x_i^2 \Delta x\right)}_{\text{mass}} \underbrace{g}_{\text{gravity}} \cdot \underbrace{(H - x_i)}_{\text{distance moved}}.$

$= \text{Force} \times \text{distance}$

Here, adding all the work done and sending $\Delta x \to 0$ we get

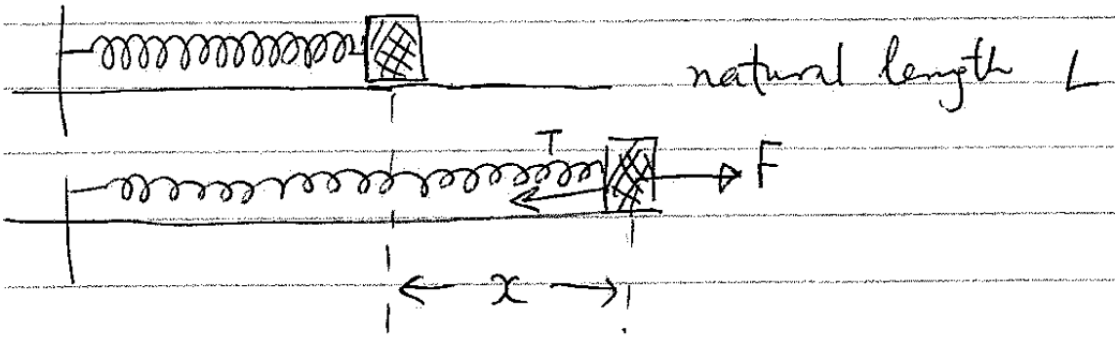
$$
\begin{aligned}
\mathrm{W} &= \int_0^h \rho g \pi \frac{R^2}{H^2} x^2 (H - x) \mathrm{d}x \\
&= \rho g \pi \frac{R^2}{H^2} \left[\frac{x^3}{3} H - \frac{x^4}{4} \right]_0^h \quad = \rho g \pi \frac{R^2}{H^2} \left(\frac{h^3}{3} H - \frac{h^4}{4} \right) \\
&\qquad\qquad\qquad\qquad = \rho g \pi \frac{R^2}{H^2} \frac{h^3}{12} (4H - 3h)
\end{aligned}
$$

Put some numbers in

$$
\begin{gathered}
R = 2\mathrm{m} \quad H = 5\mathrm{m} \quad h = 3\mathrm{m} \quad \rho = 10^3 \mathrm{kg/m^3} \quad g = 9.8 \mathrm{m/s^2} \\
\mathrm{W} = 10^3 \frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 9.8 \frac{\mathrm{m}}{\mathrm{s}^2} \pi \left(\frac{4}{25} \right) \cdot \frac{9}{12} \mathrm{m}^3 (20 - 9) \mathrm{m} \\
\approx 4.06 \times 10^4 \approx 40 \mathrm{kJ} \quad \text{(kilo Joules)}
\end{gathered}
$$

This is approximately 10,000 calories.

Example 3 Hooke's law (linear springs)



When spring is extended to x units beyond its natural length, a force is needed to keep it there, *i.e.* F must balance the tension in the string.

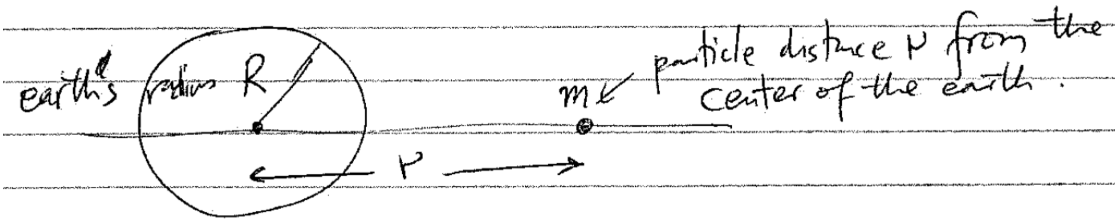
Hooke's law tells us that for *small* extensions $F \propto x$, *i.e.*

$$\text{Force} = kx, \quad k \text{ a positive constant}$$

$$\text{Work done in stretching the string by } x_0 \text{ units} \quad = \int_0^{x_0} kx\mathrm{d}x = \frac{kx_0}{2} \cdot \mathrm{J}$$

Example 4 Gravitational forces

Find the work done in moving a particle of mass m from the Earth's surface to ∞.



$$\text{Force on a particle is } f(r) = \frac{GMm}{r^2} \qquad M - \text{mass of Earth}$$

$$\mathrm{W} = \int_R^\infty GMm\frac{\mathrm{d}r}{r^2} = GMm\left[-\frac{1}{r}\right]_R^\infty = \frac{GMm}{R}$$

What velocity is needed for the particle to escape to ∞? By energy conservation: Work done = kinetic energy.

$$\Rightarrow \frac{GMm}{R} = \frac{1}{2}mV_{\text{esc}}^2 \Rightarrow V_{\text{esc}} = \left(\frac{2GM}{R}\right)^{1/2}.$$

Here are some numbers.

$$V_{\text{esc}} = \left(\frac{2 \times 6.67 \times 10^{-11}\mathrm{N} \cdot \mathrm{m}^2/\mathrm{kg}^2 \times 5.97 \times 10^{24}\mathrm{kg}}{6.37 \times 10^6\mathrm{m}}\right)^{1/2}$$

$$\approx 11,000\mathrm{m/s} = 11\mathrm{km/s} = 33 \times \text{ speed of sound}$$

Note: $R \to 0 \quad V_{\text{esc}} \to 0$. This is impossible for a black hole (Einstein).

Chapter 10

Improper Integrals

Definition

$\int_a^b f(x)\mathrm{d}x$ is an ***improper integral*** *if*

(i) $a = -\infty$ and/or $b = \infty$ (ii) $f(x) \to \pm\infty$ in (a, b)

To find improper integrals we take the limit of proper integrals. If the limit is finite, the integral *converges*, otherwise it *diverges*.

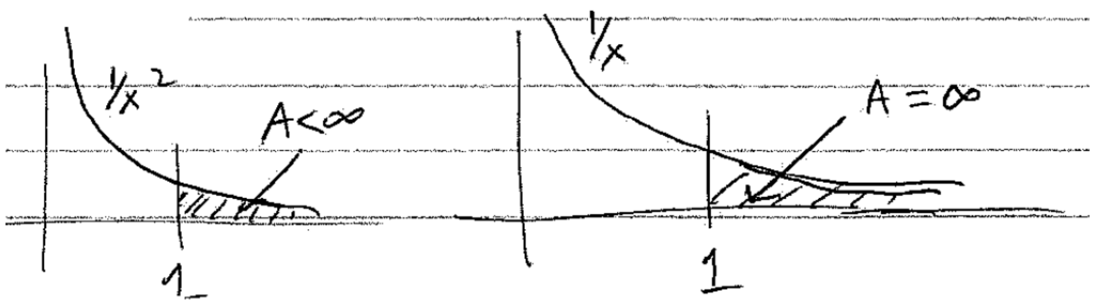
Example 1

$$\int_1^\infty \frac{1}{x^2}\mathrm{d}x = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\mathrm{d}x = \lim_{b\to\infty}\left(-\frac{1}{b}+1\right) = 1$$

Example 2

$$\int_1^\infty \frac{\mathrm{d}x}{x} = \lim_{b\to\infty}\log(b) = \infty \quad \textit{i.e. } \text{diverges.}$$

Geometrically:



In general

$$\lim_{b\to\infty}\int_1^b x^r\mathrm{d}x = \lim_{b\to\infty}\left(\frac{b^{r+1}-1}{r+1}\right) \qquad r \neq -1$$

So need $r+1 < 0$ for convergence, *i.e.* $r < -1$, ($r = -1$ is divergent - see log example earlier).

10.0.1 Comparison Theorem/Test

Suppose f and g satisfy:

(i) $|f(x)| \leq g(x)$ for all $x \geq a$

(ii) $\int_a^b f(x)\mathrm{d}x$ and $\int_a^b g(x)\mathrm{d}x$ exist for every $b > a$.

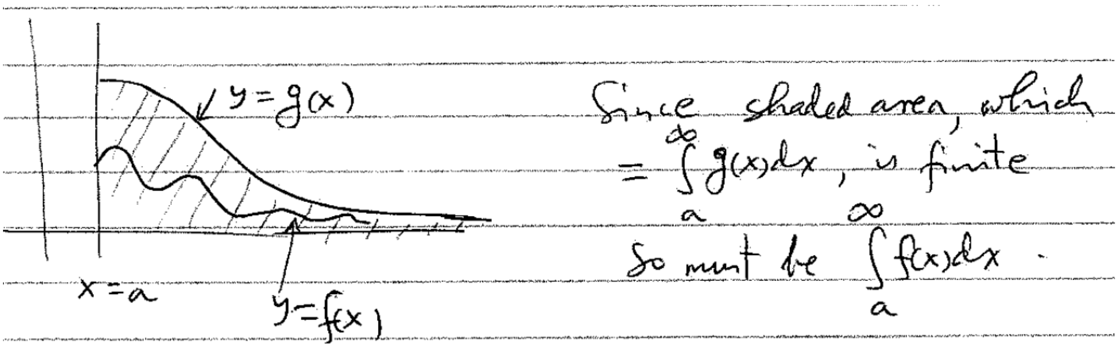
Then

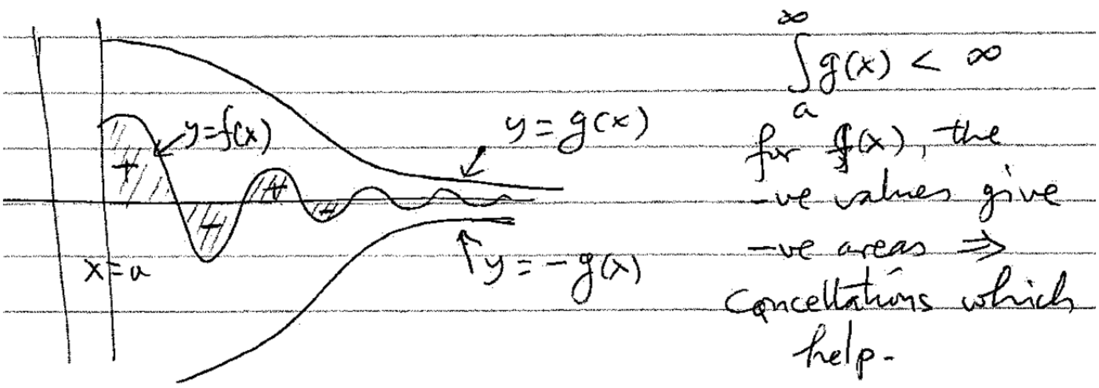
(a) If $\int_a^\infty g(x)\mathrm{d}x$ is convergent, so is $\int_a^\infty f(x)\mathrm{d}x$

(b) If $\int_a^\infty f(x)\mathrm{d}x$ is divergent, so is $\int_a^\infty g(x)\mathrm{d}x$

Similarly for $\int_{-\infty}^b f(x)\mathrm{d}x$ and $\int_{-\infty}^\infty f(x)\mathrm{d}x$.

Intuitive "proof": If f, g both positive then the picture is





Comparison test is useful if we cannot carry out the integral exactly. It will tell us if it exists, then we can find it numerically etc. *e.g*

$$\int_0^\infty \frac{\sin(x)}{(1+x)^2} \quad \text{converges}$$

First thing to note is that $\int_0^\infty \frac{\mathrm{d}x}{(1+x)^2}$ converges by comparison to $\int_1^\infty \frac{\mathrm{d}x}{x^2}$. Why? If $x \geq 1$, $\frac{1}{(1+x)^2} < \frac{1}{x^2}$. By the comparison test

$$\int_0^\infty \frac{sin(x)}{(1+x)^2}\mathrm{d}x \quad \text{converges since} \quad \frac{|\sin(x)|}{(1+x)^2} \leq \frac{1}{(1+x)^2} < \frac{1}{x^2} \quad \text{for } x \geq 1 \tag{10.1}$$

e.g.

$$\int_1^\infty \frac{\mathrm{d}x}{\sqrt{1+x^2}} \text{ is divergent}$$
$$\int_1^b \frac{\mathrm{d}x}{\sqrt{1+x^2}} \geq \int_1^b \frac{\mathrm{d}x}{\sqrt{x^2+x^2}} = \int_1^b \frac{\mathrm{d}x}{\sqrt{2}x}$$
$$\text{Now } \int_1^\infty \frac{\mathrm{d}x}{x} \text{ diverges } \Rightarrow \text{ so does } \int_1^\infty \frac{\mathrm{d}x}{\sqrt{1+x^2}}$$

e.g. $\int_1^\infty \frac{\mathrm{d}x}{\sqrt{x}}$ diverges. (Already saw this, and we can do it directly).

Here is a proof using the comparison theorem.

$$\text{If} \quad x \geq 1 \; \frac{1}{x} \leq \frac{1}{\sqrt{x}} \; \Rightarrow \; \int_1^b \frac{1}{x}\mathrm{d}x < \int_1^b \frac{\mathrm{d}x}{\sqrt{x}} \; b > 1$$
$$\text{and } \int_1^\infty \frac{1}{x}\mathrm{d}x \quad \text{diverges}$$

10.0.2 Improper integrals of unbounded functions

Without loss of generality, consider the situation where $|f(x)| \to \infty$ as $x \to 0$. Again, take limits of bounded integrals. *e.g.*

$$\int_0^1 \frac{1}{x^p}\mathrm{d}x \quad \begin{cases} \text{converges} & \text{if} \quad p < 1 \\ \text{diverges} & \text{if} \quad p \geq 1 \end{cases}$$

Proof. Left as an exercise. □

Example 1

$$\int_0^1 \log(x)\mathrm{d}x \quad \text{exists}$$
$$= \lim_{\epsilon\to 0}\int_\epsilon^1 \log(x)\mathrm{d}x = \lim_{\epsilon\to 0}\left\{[x\log(x)]_\epsilon^1 - \int_\epsilon^1 x\frac{1}{x}\mathrm{d}x\right\}$$
$$= \lim_{\epsilon\to 0}[-\epsilon\log(\epsilon) - 1 + \epsilon] = -1$$

Example 2

Show that the improper integral $I = \inf_0^\infty \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x$ converges.

$$\text{Write } I = I_1 + I_2 \text{ where } I_1 = \int_0^1 \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x, \quad I_2 = \int_1^\infty \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x$$

$$I_1 = \int_0^1 \frac{e^{-x}}{\sqrt{x}} < \int_0^1 \frac{1}{\sqrt{x}}\mathrm{d}x \quad \text{which is convergent}$$

$$\int_1^\infty \frac{e^{-x}}{\sqrt{x}}\mathrm{d}x < \int_1^\infty e^{-x}\mathrm{d}x \quad \text{which is also convergent}$$

Example 3

Find the length of the curve $y = \sqrt{1-x^2}$ for $x \in [-1, 1]$.

$$\text{Length } L = \int_{-1}^1 (1+y^2)^{1/2}\mathrm{d}x = \int_{-1}^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} \quad \text{improper at both ends}$$

$$\int_{-1}^0 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{p\to -1}\int_p^0 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{p\to -1}\left[\sin^{-1}(0) - \sin^{-1}(p)\right] = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \lim_{p\to 1}\int_0^1 \lim_{p\to 0}\left[\sin^{-1}(1) - \sin -1(p)\right] = \frac{\pi}{2} \;\Rightarrow\; \pi \text{ is the length}$$

Chapter 11

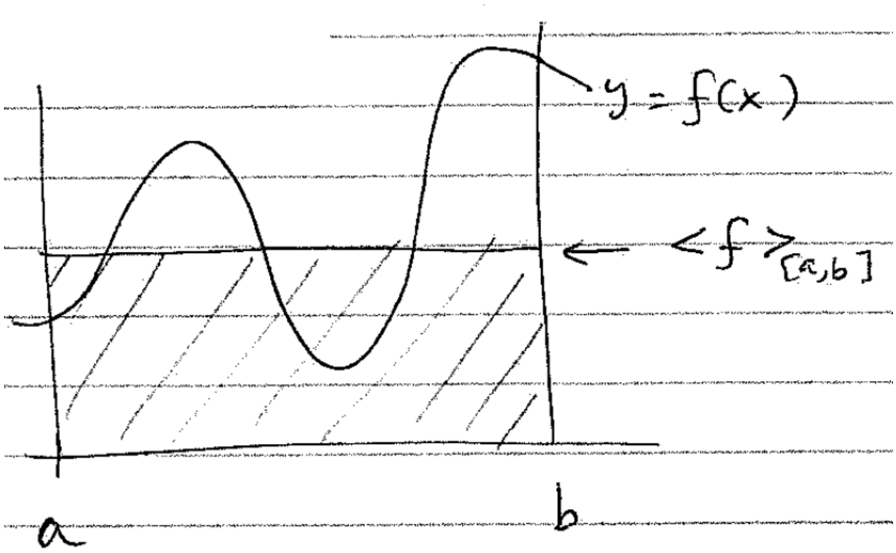
Mean Value Theorem for Integrals

Given a function f that is integrable on $[a, b]$, we define its average $\langle f\rangle_{[a,b]}$ by the formula.

$$\langle f(x)\rangle_{[a,b]} = \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x.$$

Since $\langle f\rangle_{[a,b]}$ is a number (constant), then we have

$$\int_a^b f(x)\mathrm{d}x = \int_a^b \langle f\rangle_{[a,b]}\mathrm{d}x$$



Geometrically, the area of the shaded rectangle is equal to the area under $y = f(x)$.

Theorem 1 Let f be continuous on $[a, b]$. Then there exists a point $x_0 \in (a, b)$ such that

$$f(x_0) = \frac{1}{b-a} \int_a^b f(x) \mathrm{d}x.$$

Proof. Define $F(x) = \int_a^x f(t) \mathrm{d}t$. By the Fundamental Theorem of Calculus we have $F'(x) = f(x)$ for all $x \in (a, b)$. F is continuous at a and b (proof in exercises). By MVT we have

$$F'(x_0) = \frac{F(b) - F(a)}{b-a}$$

i.e.

$$f(x_0) = \frac{\int_a^b f(t) \mathrm{d}t - \int_a^a f(t) \mathrm{d}t}{b-a} = \frac{1}{b-a} \int_a^b f(t) \mathrm{d}t$$

□

Chapter 12

Techniques of Integration

Will assume familiarity with substitution and integration by parts.

12.0.1 Trigonometric Integrals

$$\int \sin^m(x)\cos^n(x)\mathrm{d}x \quad m,\ n \text{ integers}$$

For:

$n = 1$	substitute $u = \sin(x) \;\Rightarrow\; \int \sin^m(x)\cos(x)\mathrm{d}x = \int u^m \mathrm{d}u$
$n = 1,\ m = -1$	$\int \frac{\cos(x)}{\sin(x)}\mathrm{d}x = \log\lvert\sin(x)\rvert + c$
$m,\ n \neq 1$	Write in terms of $\sin^p(x)\cos(x)$ or similar, or use double angle formulae

Important - use trig formulas! - a reminder of double angle formulas:

$$\begin{aligned}
\cos(2x) &= 2\cos^2(x) - 1 = 1 - 2\sin^2(x)\\
\sin(2x) &= 2\sin(x)\cos(x)\\
\sin(x \pm y) &= \sin(x)\cos(y) \pm \cos(x)\sin(y)\\
\cos(x \pm y) &= \cos(x)\cos(y) \mp \sin(x)\sin(y)
\end{aligned}$$

From these we have

$$\begin{aligned}
\sin(x)\cos(y) &= \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]\\
\sin(x)\sin(y) &= \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]\\
\cos(x)\cos(y) &= \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]
\end{aligned}$$

Example 1

$$\int \sin^2(x)\cos^3(x)\mathrm{d}x = \int \sin^2(x)(1-\sin^2(x))\cos(x)\mathrm{d} \underbrace{\Rightarrow}_{u=\sin(x)} \int u^2(1-u^2)\mathrm{d}u$$

Example 2

$$\int \sin^2(x)\cos^2(x) = \int \frac{\sin^2(2x)}{4} = \int \frac{1}{4}\frac{1-\cos(4x)}{2}\mathrm{d}x$$

Example 3

$$I = \int \tan^3(\theta)\sec^3(\theta)\mathrm{d}\theta$$

i) Write $\displaystyle\int \frac{\sin^3(\theta)}{\cos^6(\theta)}\mathrm{d}\theta = \int \frac{1-\cos^2(\theta)}{\cos^6(\theta)}\sin(\theta)\mathrm{d}\theta \underbrace{\Rightarrow}_{u=\cos(\theta)} = -\int \frac{1-u^2}{u^6}\mathrm{d}u$

ii) Notice $\displaystyle\frac{\mathrm{d}}{\mathrm{d}\theta}\sec(\theta) = \tan(\theta)\sec(\theta)$

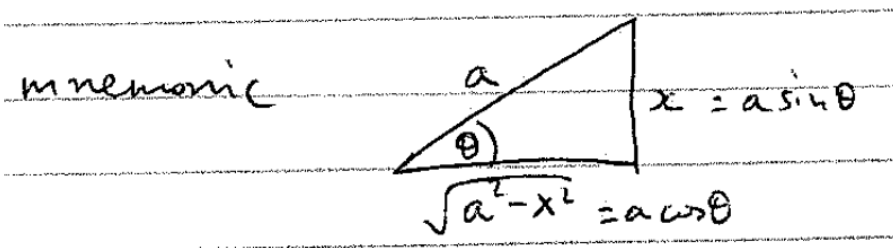
$$\int \tan^3(\theta)\sec^3(\theta)\mathrm{d}\theta = \int \tan(\theta)\sec(\theta)(\sec^2(\theta)-1)\sec^2(\theta)\mathrm{d}\theta \underbrace{\Rightarrow}_{u=\sec(\theta)} = \int (u^2-1)u^2\mathrm{d}u$$

Example 4

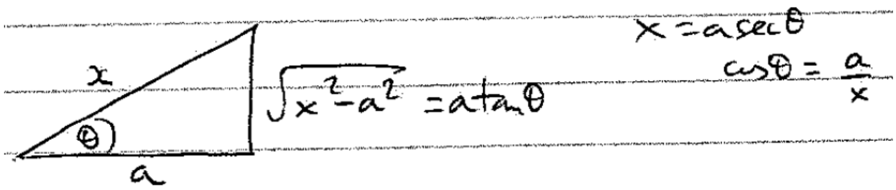
$$\int \cos(3x)\cos(5x)\mathrm{d}x = \frac{1}{2}\int(\cos(8x)+\cos(2x))\mathrm{d}x \text{ etc}$$

Trigonometric substitutions

(1) If $\sqrt{a^2-x^2}$ appears in an integral, try $x = a\sin(\theta)$, $\mathrm{d}x = a\cos(\theta)\mathrm{d}\theta$, $\sqrt{a^2-x^2} = a\cos(\theta)$ ($a > 0$, θ acute).



(2) If $\sqrt{x^2-a^2}$ occurs, try $x = a\sec(\theta)$, $\mathrm{d}x = a\tan(\theta)\sec(\theta)\mathrm{d}\theta$ and $\sqrt{x^2-a^2} = a\tan(\theta)$.



(3) If $\sqrt{a^2+x^2}$ or a^2+x^2 occur, try $x = a\tan(\theta)$, $\mathrm{d}x = a\sec^2(\theta)\mathrm{d}\theta$, $\sqrt{a^2+x^2} = a\sec(\theta)$. Also $x = a\sinh(\theta)$, $\mathrm{d}x = a\cosh(\theta)\mathrm{d}\theta$, $\sqrt{a^2+x^2} = a\cosh(\theta)$.

Example 5

$$\int \frac{x^2}{(1+x^2)^{3/2}}\mathrm{d}x \quad x = \tan(\theta)\ \mathrm{d}x = \sec^2(\theta)\mathrm{d}\theta$$
$$\Rightarrow \int \frac{\tan^2(\theta)\sec^2(\theta)}{\sec^3(\theta)}\mathrm{d}\theta = \int \tan(\theta)\sin(\theta)\mathrm{d}\theta$$

Better by parts:

$$\int \frac{x}{(1+x^2)^{3/2}} \cdot x\mathrm{d}x = -(1+x^2)^{-1/2} \cdot x + \int \frac{1}{\sqrt{1+x^2}}\mathrm{d}x$$
$$= -\frac{x}{\sqrt{1+x^2}} + \sinh^{-1}(x) + c$$

12.0.2 Recursion formulas

$$\text{Let } I_n = \int \sin^n(x)\mathrm{d}x.$$
$$\text{Then } I_n = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}I_{n-2}$$

Proof.

$$\begin{aligned}
I_n &= \int \sin^{n-1} x \sin(x)\mathrm{d}x \quad \text{integrate by parts} \\
&= -\cos(x)\sin^{n-1}(x) + \int (n-1)\sin^{n-2}(x)\cos^2(x)\mathrm{d}x \\
I_n &= -\cos(x)\sin^{n-1}(x) + (n-1)\int (-\sin^n(x) + \sin^{n-2}(x))\mathrm{d}x \\
\Rightarrow I_n &= -\cos(x)(\sin(x))^{n-1} - (n-1)I_n + (n+1)I_{n-2} \\
I_n &= -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}I_{n-2}
\end{aligned}$$

□

Example 6 Show that $\int_0^{\pi/2} \sin^5(x) = \frac{8}{15}$

Solution: Use recursion above, and keep track of the limits.

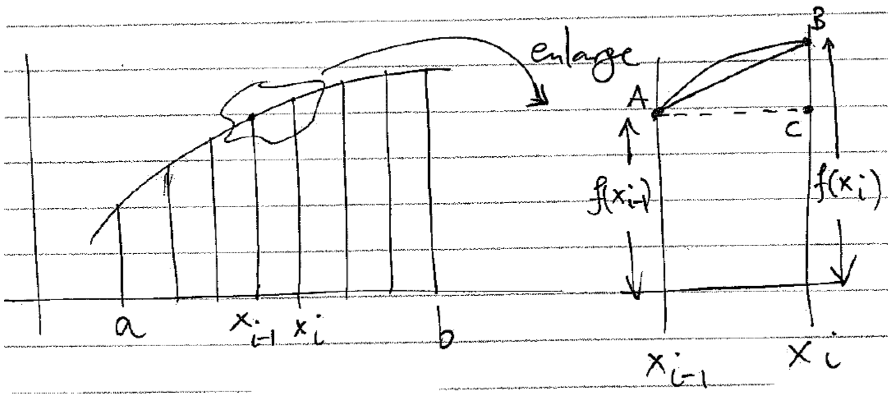
$$\begin{aligned}
I_5 &= -\frac{1}{5}\underbrace{\cancel{\sin^4(x)}}_{0}\cos(x)\Big|_0^{\pi/2} + \frac{4}{5}I_{5-2} \\
&= \frac{4}{5}\left[-\frac{1}{3}\underbrace{\cancel{\sin^2(x)}}_{0}\cos(x)\Big|_0^{\pi/2} + \frac{2}{3}I_1\right] \\
&= \frac{4}{5}\cdot\frac{2}{3}\int_0^{\pi/2}\sin(x)\mathrm{d}x = \frac{8}{15}
\end{aligned}$$

Chapter 13

Applications of Integration

13.1 Length of curves

Start with something we have already seen. Given $y = f(x)$, find the length of the graph of the function. Do something similar to Riemann sums but for the length. Partition $(x_0, x_1, \ldots, x_n)$, $x_0 = a$, $x_n = b$.



For the $\Delta ABC \quad (AB)^2 = (x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2$. But length of curve segment $AB \approx \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$.

$$\begin{aligned} \text{Total length } &\approx \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \\ &= \sum_{i=1}^{n} (x_i - x_{i-1}) \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)^2}. \end{aligned}$$

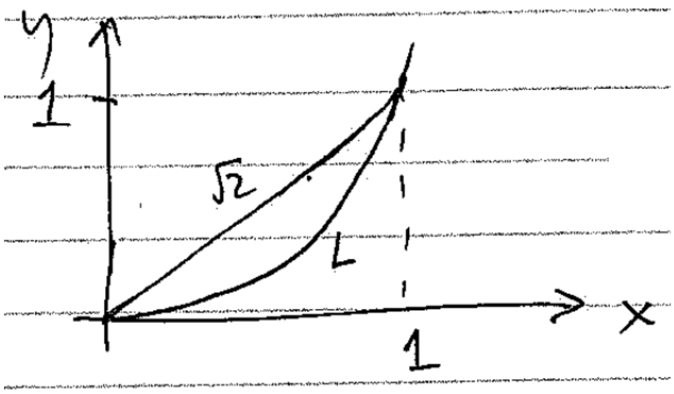
Now let $x_i - x_{i-1} = h = \frac{b-a}{n} := \Delta x$.

$$\begin{aligned} \text{Total length} &= \lim_{n\to\infty,\ (h\to 0,\ \Delta x \to 0)} \sum \Delta x \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} \right)^2} \\ L &= \int_a^b \left[1 + (f'(x))^2 \right]^{1/2} \mathrm{d}x \end{aligned}$$

In parametric form this is

$$L = \int_{t_0}^{t_1} \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 \right]^{1/2} \mathrm{d}t$$

Example Find the length of the parabola $y = x^2$, $0 \le x \le 1$.



Clearly $L > \sqrt{2} = 1.4142\ldots$, $y = f(x) = x^2$, $f'(x) = 2x$.

$$L = \int_0^1 (1+4x^2)^{1/2} \mathrm{d}x$$

$$\text{Substitution} \quad 2x = \tan(\theta) \begin{cases} x = 0, & \theta = 0 \\ x = 1, & \theta = a\tan(2) \end{cases}$$

$$2\mathrm{d}x = \sec^2(\theta)\mathrm{d}\theta \quad \text{see Fig 13.1 below}$$

$$L = \int_0^{a\tan(2)} \frac{1}{2}\sec^3(\theta)\mathrm{d}\theta$$

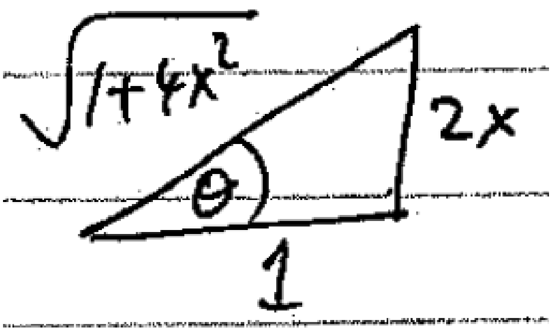


Figure 13.1:

Method 1

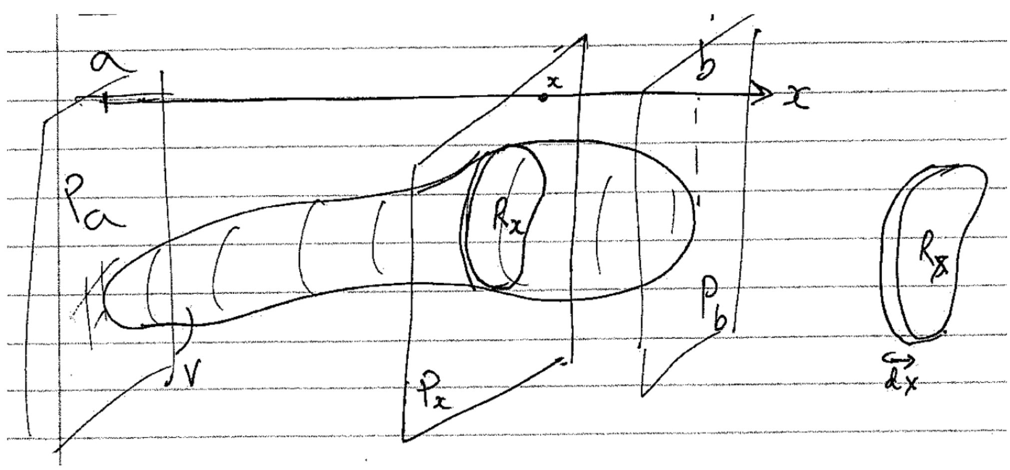
$$\begin{aligned}\int \sec^3(\theta)\mathrm{d}\theta &= \int \frac{\cos(\theta)}{\cos^4(\theta)}\mathrm{d}\theta = \int \frac{\cos(\theta)}{(1-\sin^2(\theta))^2}\mathrm{d}\theta \\ u=\sin(\theta) \quad &= \int \frac{\mathrm{d}u}{(1-u^2)^2} \text{ partial fractions} \\ &= \frac{A}{1+u} + \frac{B}{(1+u)^2} + \frac{C}{(1-u)} + \frac{D}{(1-u^2)} \\ &= \text{etc} \ldots \text{etc}\end{aligned}$$

or write in terms of x again.

Method 2

$$\begin{aligned}\int \sec^3(\theta)\mathrm{d}\theta &= \int \sec(\theta)\sec^2(\theta)\mathrm{d}\theta \quad \text{integrate by parts} \\ &= \tan(\theta)\sec(\theta) - \int \tan(\theta)\overbrace{\sec(\theta)\tan(\theta)}^{\frac{\mathrm{d}}{\mathrm{d}\theta}(\sec(\theta))}\mathrm{d}\theta \\ &= \tan(\theta)\sec(\theta) - \int \sec^3(\theta)(1-\cos^2(\theta))\mathrm{d}\theta \qquad \text{move } -\int \sec^3(\theta)\mathrm{d}\theta \text{ to other side} \\ 2\int \sec^3(\theta)\mathrm{d}\theta &= \tan(\theta)\sec(\theta) + \int \sec(\theta)\mathrm{d}\theta) \\ \textit{i.e.} \int \sec^3(\theta)\mathrm{d}\theta &= \frac{1}{2}\left[\tan(\theta)\sec(\theta) + \log(\sec(\theta)+\tan(\theta))\right] \quad \text{(see HW3)} \\ \int_0^1 (1+4x^2)^{1/2}\mathrm{d}x &= \frac{1}{4}\left[\tan(\theta)\sec(\theta) + \log(\sec(\theta)+\tan(\theta))\right]_0^{a\tan(2)} \\ &= \frac{1}{4}\left[2\times\sqrt{1+4x^2} + \log\left(2x+\sqrt{1+4x^2}\right)\right]_0^1 = \frac{1}{4}\left[2\sqrt{5} + \log\left(2+\sqrt{5}\right)\right] = 1.4789\end{aligned}$$

13.2 Volumes and Volumes of Revolution

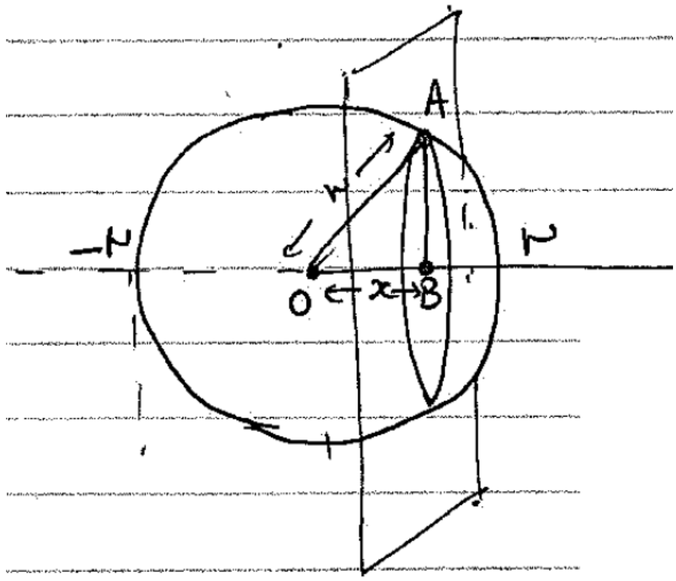


A plane cuts a solid V - cross sectional area is R_x say. Then the volume of a slide is $R_x \mathrm{d}x$. So if P_x is a family of parallel planes with common axis x, and the area of V cut by P_x is $A(x)$, then the volume of V is

$$\int_a^b A(x)\mathrm{d}x$$

where the solid V lies between planes P_a and P_b.

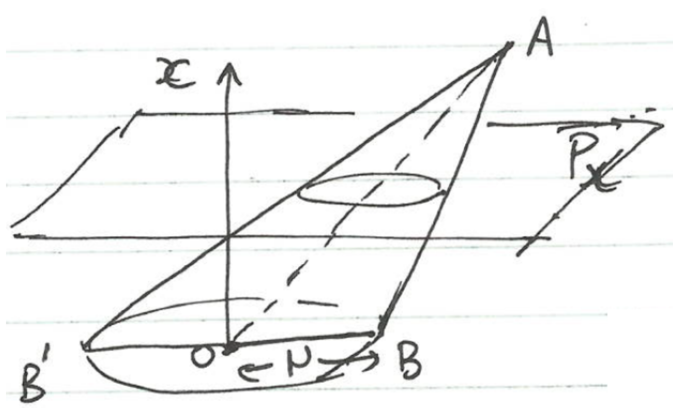
Example 1 Volume of a sphere of radius r. Pick planes along the x-axis.



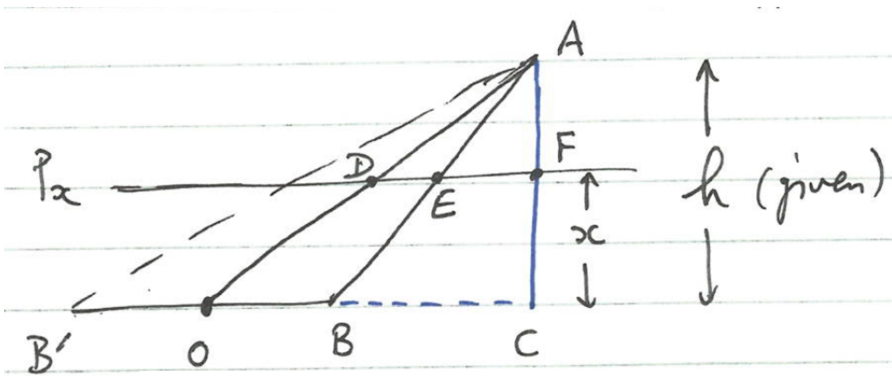
$$\Delta OAB \quad \begin{cases} OA &= r \\ OB &= x \\ AB &= (r^2 - x^2)^{1/2} \end{cases}$$

$$\text{So } A(x) = \pi(r^2 - x^2), \quad V = \int_{-r}^{r} \pi(r^2 - x^2)\mathrm{d}x \; = \frac{4}{3}\pi r^3$$

Example 2 Volume of conical solids, with circular base of radius r and height h.



Take x to be upwards as shown, so P_x cuts the cone in circular areas.
Geometry - $\Delta ABB'$ and ΔOAB.



Similar Δs - we are after DE, the radius of the circle cut by P_x.

$$\frac{DE}{OB} = \frac{AE}{AB} = \frac{AF}{AC} \to i.e. \to \frac{DE}{r} = \frac{h-x}{h}$$

$$\Rightarrow\ DE = \frac{h-x}{h} r$$

$$A(x) = \pi \frac{r^2}{h^2}(h-x)^2$$

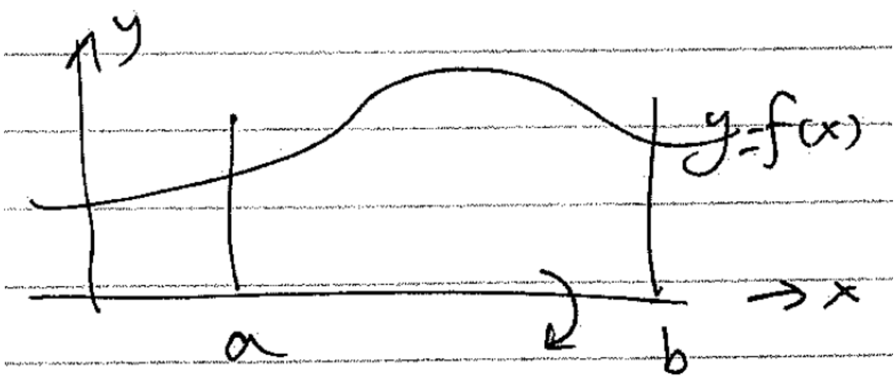
$$\text{Volume } = \int_0^h \pi \frac{r^2}{h^2}(h-x)^2 \mathrm{d}x$$

$$= -\pi \frac{r^2}{h^2} \frac{(h-x)^3}{3}\bigg|_0^h = \frac{1}{3}\pi r^2 h$$

Example 3 A sphere of radius r is cut into 3 pieces with the two cuts symmetrically placed about the centre. Where should the cuts be in order to get three equal volumes? *Complete for homework.*

13.2.1 Volumes of revolution

Given a area bounded by $x = a$, $x = b$, $y = f(x)$, $y = 0$.

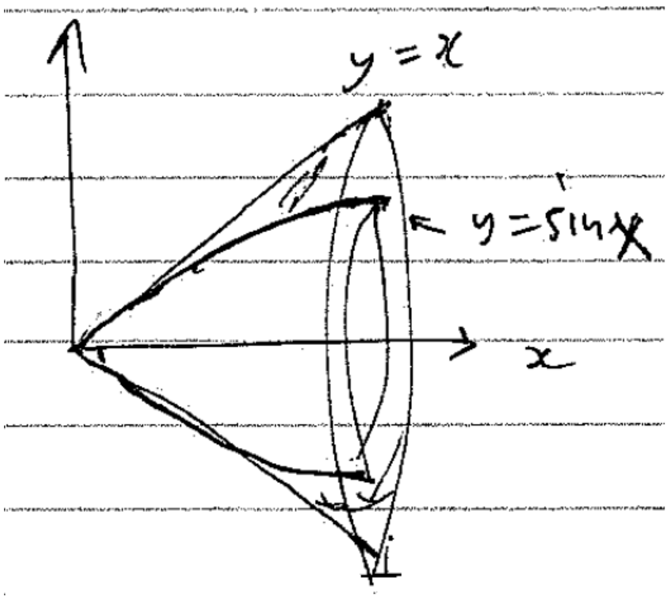


The volume of the solid produced by revolving $y = f$ about the x-axis (as shown) is given by

$$V = \int_a^b \pi(f(x))^2 \mathrm{d}x.$$

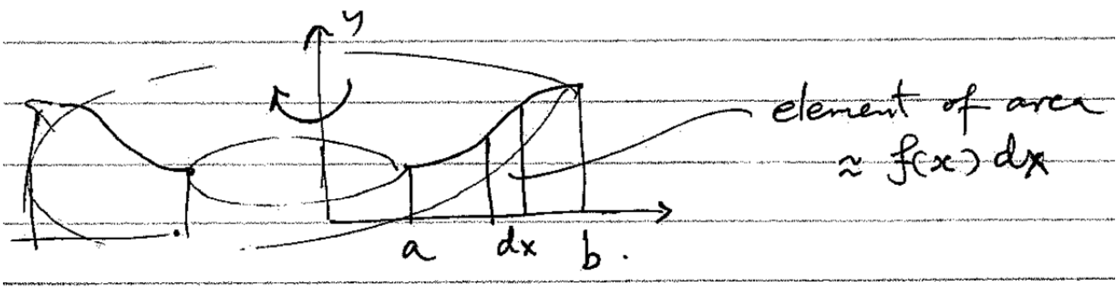
This follows immediately by the slice method seen earlier.

Example 4 The region between the graphs of $\sin(x)$ and x for $[0, \frac{\pi}{2}]$, is revolved about the x-axis. Sketch the resulting solid and find its volume.



$$V = \int_0^{\pi/2} \pi(x^2 - \sin^2(x))\mathrm{d}x = \frac{\pi^4}{24} - \frac{\pi^2}{4} \quad \textit{show this}$$

If we revolve about the y-axis, what is the volume? Consider a non-negative function $f(x)$ on $[a, b]$.



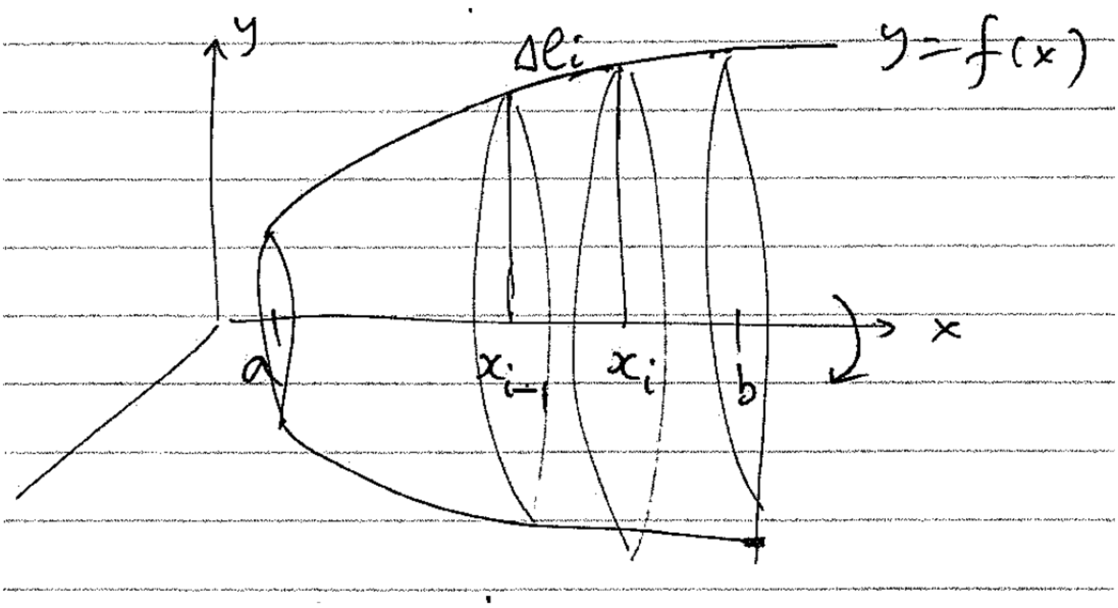
Revolving the element about the y-axis gives a shell of volume

$$\underbrace{2\pi x}_{\text{(a)}} \overbrace{f(x)}^{\text{(b)}} \underbrace{\mathrm{d}x}_{\text{(c)}}$$

$$\Rightarrow \ V = \int_a^b 2\pi x f(x) \mathrm{d}x.$$

Where (a) - circumference of cylindrical shell, (b) - radius of shell, (c) - thickness of shell. Note that this can be done by the slice method but with planes along the y-axis and parallel to the x-axis.

13.3 Surface Areas of Revolution



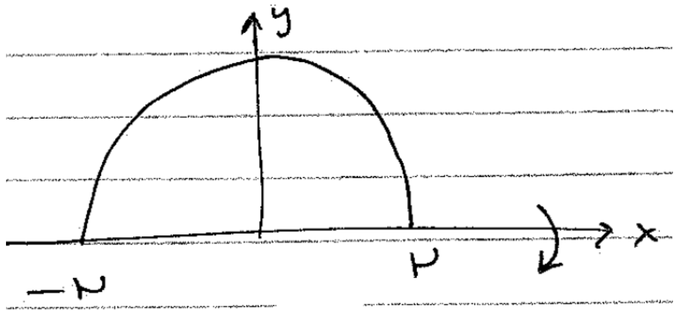
As we revolve about the x-axis, the area of the surface area swept out is a strip of length $\approx 2\pi f(x_i)$ and thickness Δl_i. Now

$$\begin{aligned} \Delta l_i &= \left[(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2\right]^{1/2} \\ &\approx \left[1 + (f'(x_i))^2\right]^{1/2} \Delta x \quad \text{as seen earlier.} \end{aligned}$$

Therefore, in the limit, area S is

$$S = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \mathrm{d}x$$

Example 5 Sphere radius r

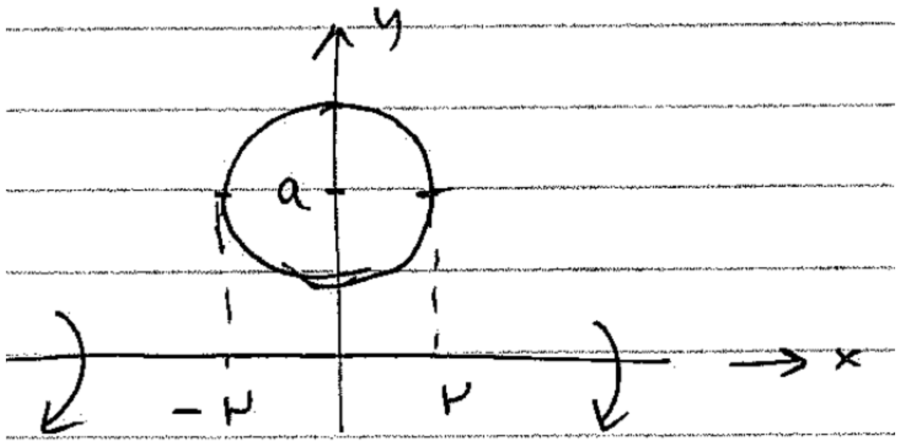


$$y = (r^2 - x^2)^{1/2} \text{ i.e. } f(x) = (r^2 - x^2)^{1/2}$$

$$f'(x) = -\frac{x}{(r^2 - x^2)^{1/2}} \Rightarrow S = \int_{-r}^{r} 2\pi (r^2 - x^2)^{1/2} \cdot \left[1 + \frac{x^2}{(r^2 - x^2)}\right]^{1/2} \mathrm{d}x$$

$$= 2\pi r \int -r^r \mathrm{d}x = 4\pi r^2$$

Example 6 Torus of cross-sectional radius r and radius $a > r$.



Revolve about the x-axis to get a torus.

$$\text{Circle equation is } x^2 + (y - a)^2 = r^2, \text{ i.e. } y = a \pm \sqrt{r^2 - x^2}$$

$$\text{Upper semi-circle: } f_+(x) = a + \sqrt{r^2 - x^2}$$

$$\text{Lower semi-circle: } f_-(x) = a - \sqrt{r^2 - x^2}$$

$$\Rightarrow \quad \mathcal{S} = S_+ + S_- = \int_{-r}^{r} 2\pi f_+(x)\left(1 + (f_+'(x))^2\right)^{1/2} \mathrm{d}x$$
$$+ \int_{-r}^{r} 2\pi f_-(x)\left(1 + (f_-'(x))^2\right)^{1/2} \mathrm{d}x$$

$$f_+' = -\frac{x}{\sqrt{-r^2 - x^2}} = -f_-'$$
$$\Rightarrow \; 1 + f_+'^2 = 1 + f_-'^2 = \frac{r^2}{(r^2 - x^2)}$$

Put together - don't integrate separately!

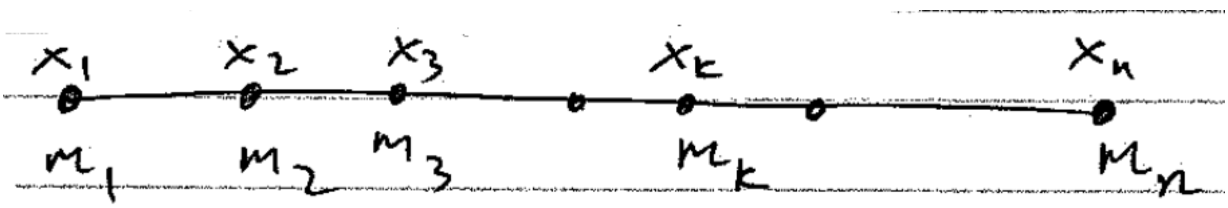
$$\mathcal{S} = 2\pi \int_{r}^{r} \left[a + \sqrt{r^2 - x^2} + 1 - \sqrt{r^2 - x^2}\right] \cdot \frac{r}{r^2 - x^2}^{1/2} \mathrm{d}x$$
$$= 4\pi a r \int_{-r}^{r} \frac{\mathrm{d}x}{\sqrt{r^2 - x^2}}$$

Put $x = r\sin(\theta)$ and show that $\int_{-r}^{r} \frac{\mathrm{d}x}{\sqrt{r^2 - x^2}} = \pi$.

$$\Rightarrow \; S_{\text{torus}} = 4\pi^2 a r = (2\pi a) \times (2\pi r)$$

13.4 Centres of Mass

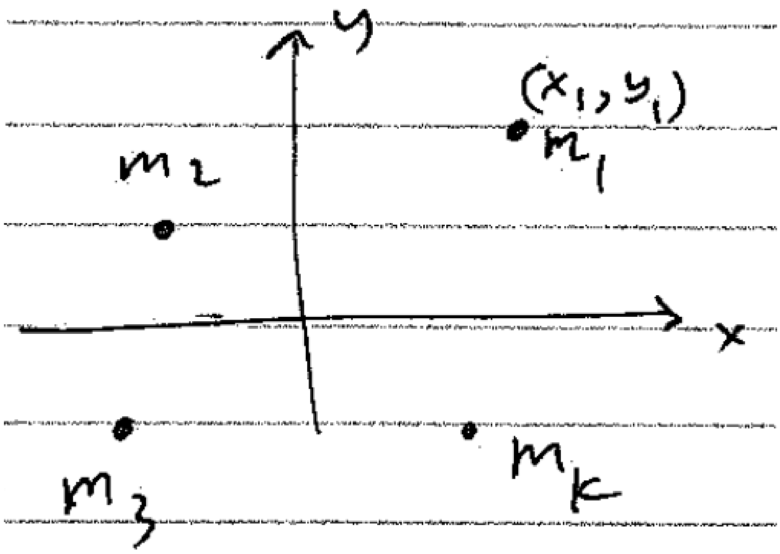
1D case - straightforward.



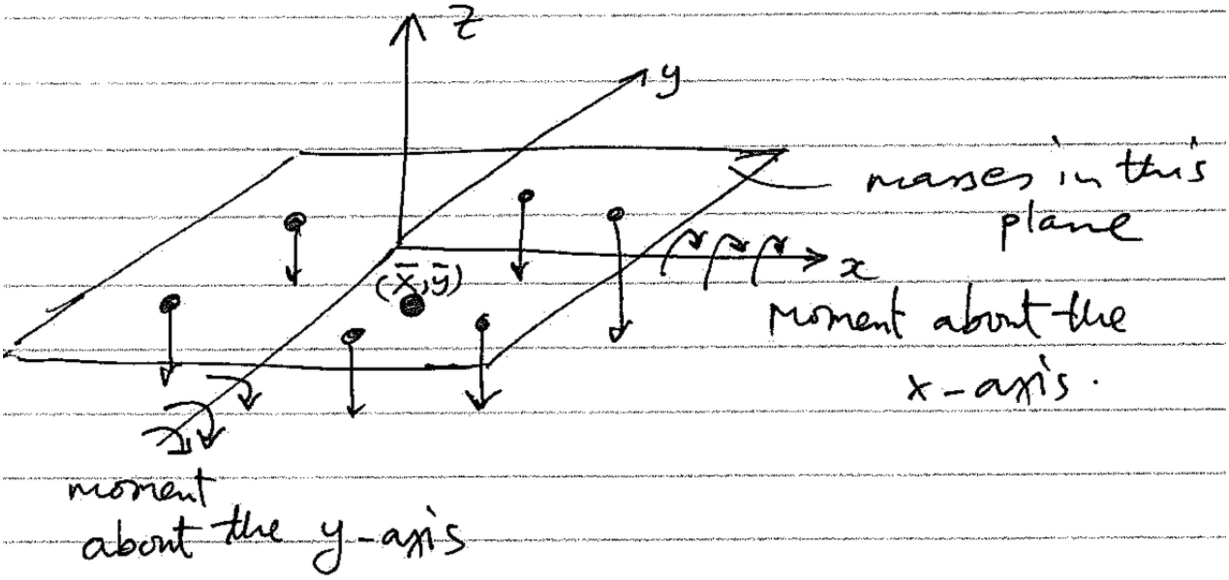
If centre of mass is at $x = \bar{x}$, then we must have a zero total moment. *i.e.*

$$\sum m_k(\bar{x} - x_k) = 0 \quad \textit{i.e.} \quad \bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$$

2D case - discrete masses.



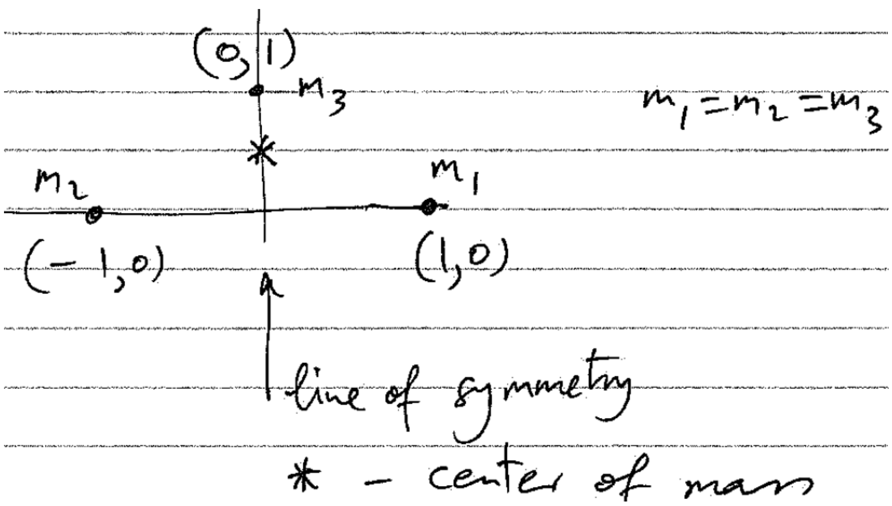
n masses of mass m_k and coordinates (x_k, y_k). Find the center of mass, assume it is $(\bar{x}, \bar{y})$. There are two degrees of freedom, so without loss of generality we need to have zero moments about the x-axis and the y-axis. What do I mean by this? Here is a schematic.



For balance I need:

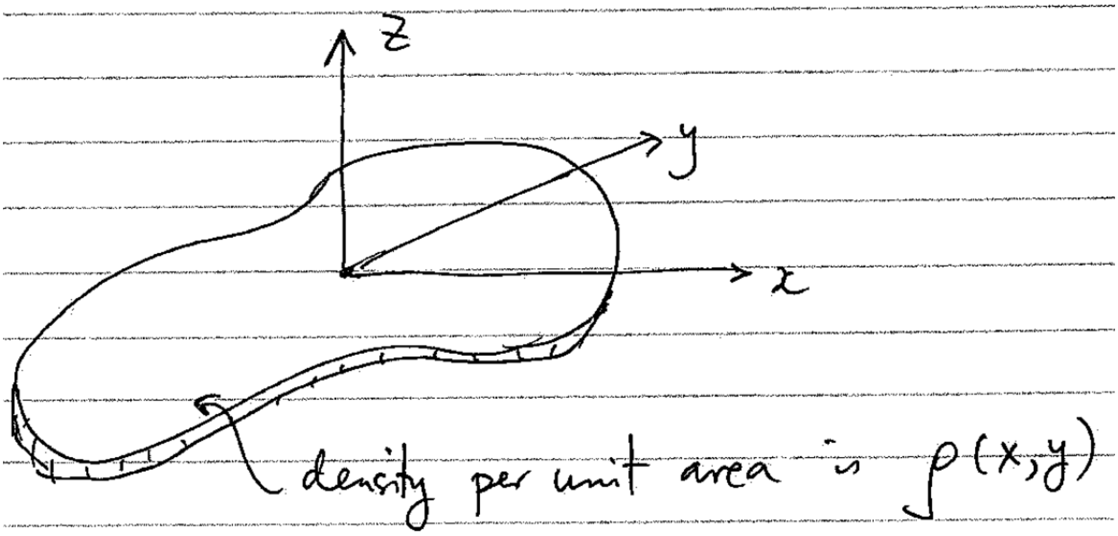
$$\left.\begin{array}{ll}(1) & \sum m_i(\bar{x} - x_i) = 0 \\ (2) & \sum m_i(\bar{y} - y_i) = 0\end{array}\right\} \Rightarrow (\bar{x}, \bar{y}) = \left(\frac{\sum m_i x_i}{\sum m_i}, \frac{\sum m_i y_i}{\sum m_i}\right)$$

Note: If the masses are places symmetrically and are equal, centre of mass is on the line of symmetry. *e.g.*

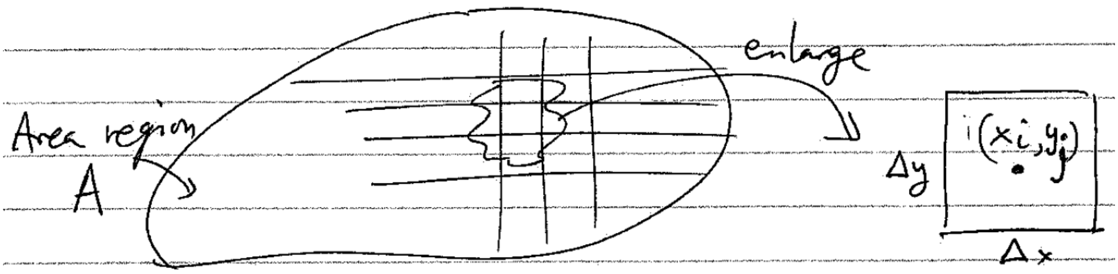


Exercise: if $m_1 = m_2 = m_3 = m$, find the centre of mass. What happens if $m_1 = m_2 \neq m_3$?

Now consider a continuous mass distribution, *i.e.* a place of a certain spatial density.



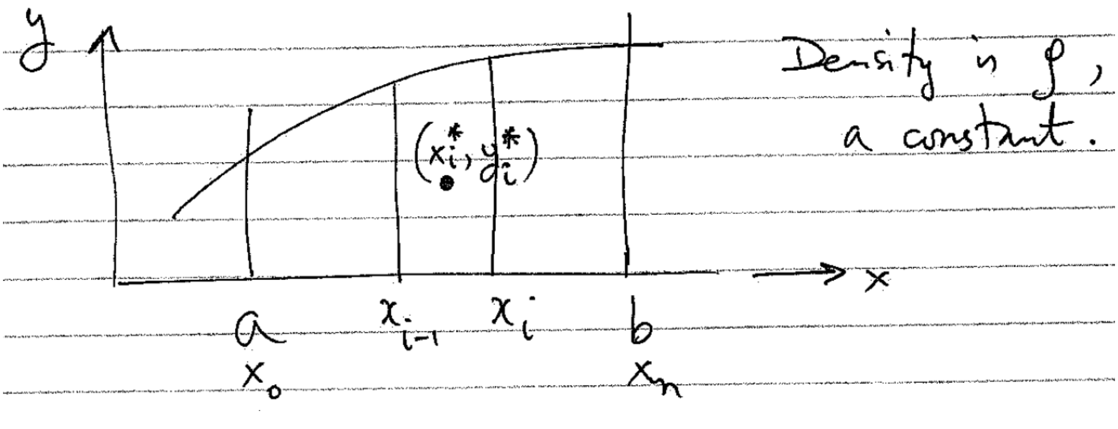
General theory - divide it into small rectangles



$$\begin{aligned}
\text{Moment about } y\text{-axis is} \quad & x_i \rho(x_i, y_i)\Delta x \Delta y \\
\text{Add all of them up (like a Riemann sum)} \quad & \sum_i \sum_j x_i \rho(x_i, y_i)\Delta x \Delta y \\
\text{Moment about the whole plate about the } y\text{-axis} \quad & \bar{x} \iint_{\text{Area}} \rho(x, y) \mathrm{d}x \mathrm{d}y \\
\text{In the limit} \quad & \iint_A x\rho(x, y) \mathrm{d}x \mathrm{d}y = \bar{x} \iint_A \rho \mathrm{d}x \mathrm{d}y \\
\text{Similarly} \quad & \iint_A y\rho(x, y) \mathrm{d}x \mathrm{d}y = \bar{y} \iint_A \rho(x, y) \mathrm{d}x \mathrm{d}y
\end{aligned}$$

You will see how to work with double integrals in term 2 and further. For the moment, we will consider the centre of mass of regions bounded by one or more graphs $y = f(x)$.

Case 1 Region $\{(x, y) : \ a \leq x \leq b, \ 0 \leq y \leq f(x)\}$



Consider a partition of $[a, b]$ as shown. For rectangle R_i, the centre of mass is *(by symmetry):*

$$x_i^* = \frac{1}{2}(x_{i-1} + x_i) \quad y_i^* = \frac{1}{2} f(x_i^*)$$

$$\text{Moment of } R_i \text{ about } y\text{-axis} \quad M_y(R_i) = \underbrace{\rho f(x_i^*)\Delta x}_{\text{mass}} \cdot \underbrace{x_i^*}_{\text{distance}}$$

$$\text{Moment of } R_i \text{ about } x\text{-axis} \quad M_x(R_i) = \rho f(x_i^*)\Delta x \cdot \frac{1}{2} f(x_i^*)$$

$$\text{Physics} \quad M(R_1 \cup R_2 \cup R_3 \cdots \cup R_n) = \sum_{i=1}^{n} M(R_i)$$

Moment of the union of rectangles = sum of moments of the individual rectangles *(Archimedes)*

$$\Rightarrow \quad M_y = \lim_{n\to\infty} \rho x_i^* f(x_i^*)\Delta x = \int_a^b \rho x f(x) \mathrm{d}x$$

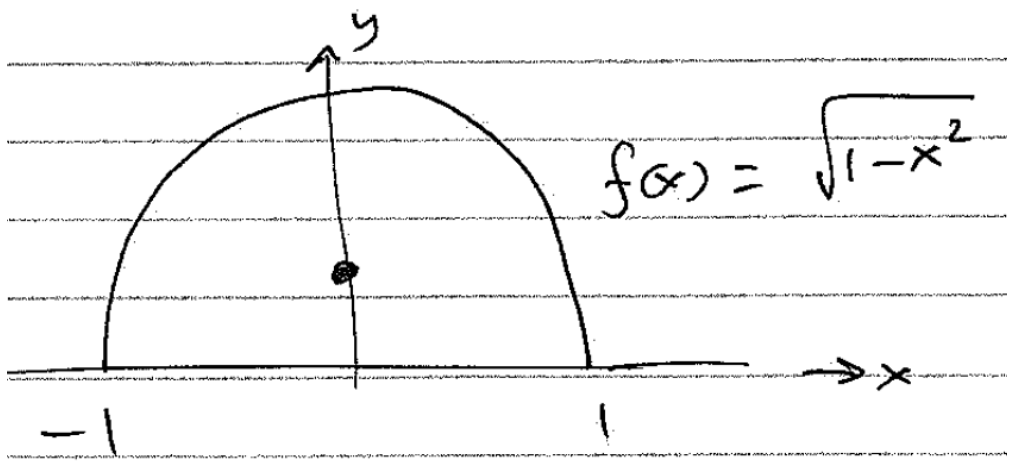
$$M_x = \lim_{n\to\infty} \rho \frac{1}{2} f(x_i^*)^2 \Delta x = \frac{1}{2}\int_a^b \rho (f(x))^2 \mathrm{d}x.$$

Now for a balance of moments, if the total mass of R is m (note $m = \int_a^b \rho f(x)\mathrm{d}x$). Then

$$\bar{x} = \frac{\int_a^b x f(x)\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x} \qquad \bar{y} = \frac{\frac{1}{2}\int_a^b (f(x))^2\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}.$$

Note: density ρ cancels out, so take $\rho = 1$ w.l.o.g.

Example 7 Half disk

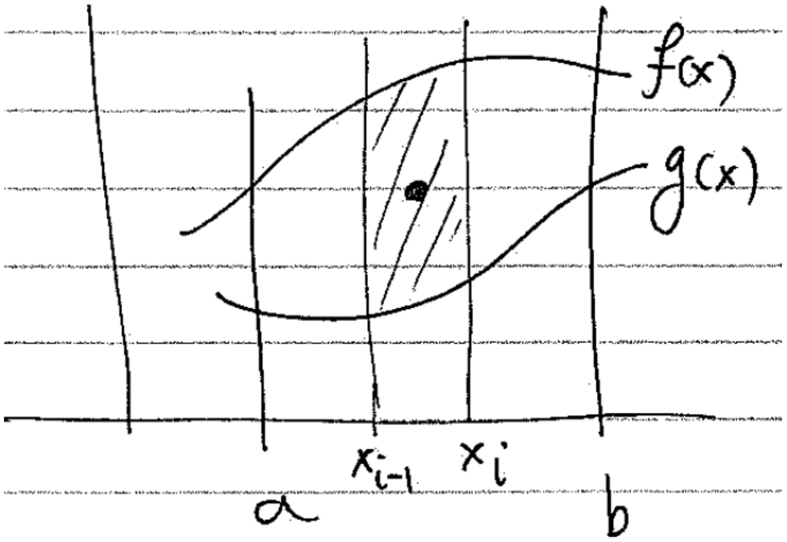


By symmetry $\bar{x} = 0$.

$$\bar{y} = \frac{\frac{1}{2}\int_{-1}^{1}(1-x^2)\mathrm{d}x}{\pi/2} = \frac{1}{\pi}(2 - \frac{2}{3}) = \frac{4}{3\pi} \approx 0.424$$

(Note: If we found it to be $> \frac{1}{2}$ then we know it's wrong! Why?)

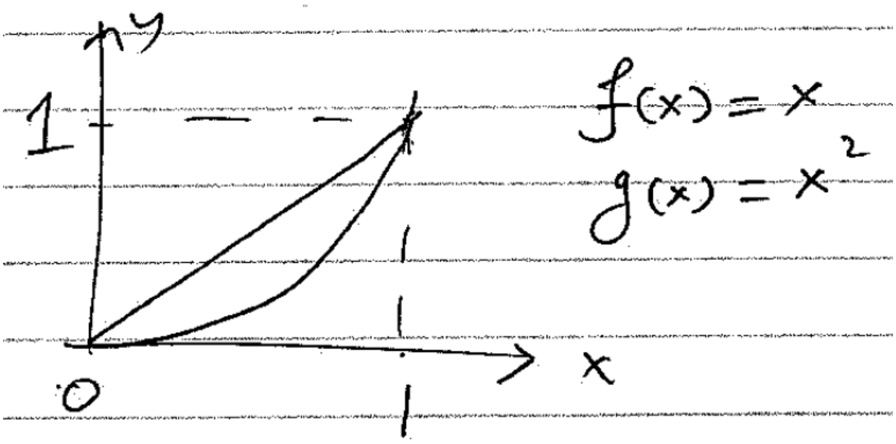
Case 2: $R = \{(x, y) : \ a \le x \le b, \ g(x) \le y \le f(x)\}$



Centre of mass of "rectangle" R_i is x_i^* as before, and $\frac{1}{2}(f(x_i^*) + g(x_i^*))$, area of rectangle $(f(x_i^*) - g(x_i^*))\Delta x$. So as before we find

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))\mathrm{d}x}{\int_a^b (f(x) - g(x))\mathrm{d}x} \qquad \bar{y}\frac{\frac{1}{2}\int_a^b (f(x)^2 - g(x)^2)\mathrm{d}x}{\int_a^b (f(x) - g(x))\mathrm{d}x}$$

Example 8 Region between $y = x$ and $y = x^2$, $0 \le x \le 1$. Find centre of mass



$$\bar{x} = \frac{\int 0^1 x(x - x^2)\mathrm{d}x}{\int_0^1 (x - x^2)\mathrm{d}x} = \frac{1/12}{1/6} = \frac{1}{2}$$

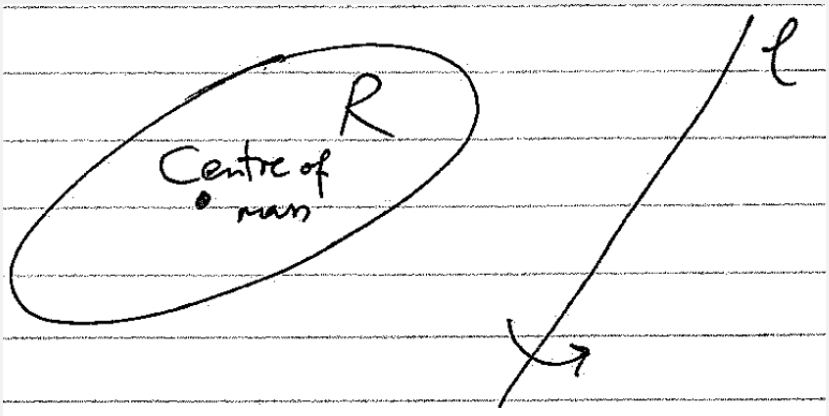
$$\bar{y} = \frac{\frac{1}{2}\int_0^1 (x^2 - x^4)\mathrm{d}x}{1/6} = \frac{1/15}{1/6} = \frac{2}{5}$$

$$\text{At} \quad x = \frac{1}{2} \begin{array}{ll} f(\frac{1}{2}) & = 1 \\ g(\frac{1}{2}) & = \frac{1}{4} \end{array} \quad \left.\begin{array}{l} 1 - \frac{2}{5} = \frac{3}{5} \\ \frac{2}{5} - \frac{1}{4} = \frac{3}{20} \end{array}\right\} \Rightarrow \text{closer to top curve}$$

Is this expected or not?

Note: If $f(x) = x^m$, $g(x) = x^m$, for some m, n, the centre of mass could be outside the region. (This is ok.)

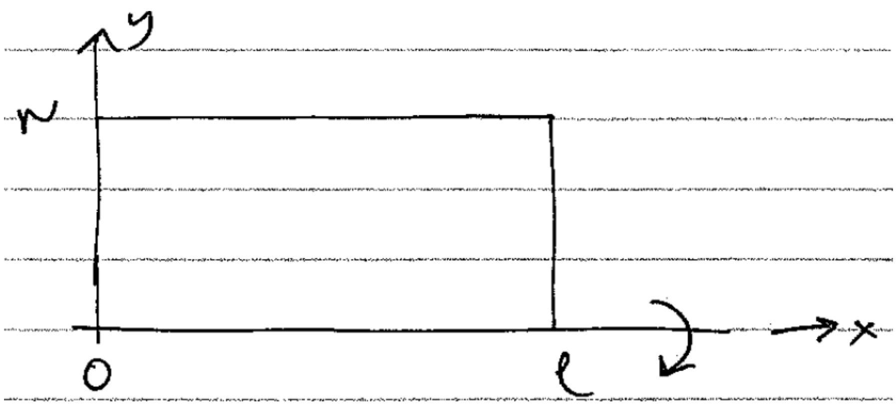
Theorem of Pappus
Let R be a region that lies on one side of a line l.



$A =$ area of R
$V =$ Volume obtained by rotating about l
$d =$ distance travelled by the centre of mass when R is rotated about l

Then $V = Ad$

Example 9 Volume of a cylinder radius r. Take the function $y = r$, $0 \leq x \leq l$.



Rotate about x-axis.

$$\text{Area } = rl \quad \bar{y} = \frac{1}{2}r \qquad \text{(by symmetry)}$$
$$\Rightarrow d = \frac{1}{2}r \cdot 2\pi = \pi r$$
$$V = rl \cdot \pi r = \pi r^2 l \quad \text{as known}$$

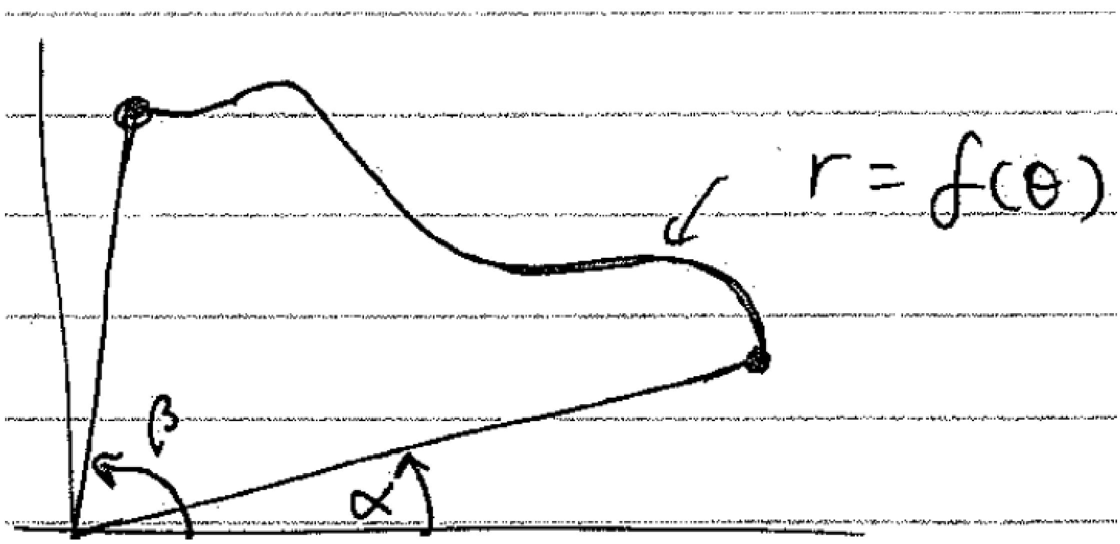
13.5 Length of curves and areas using polar coordinates

Recall

$$L = \int_a^b \left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right]^{1/2} \mathrm{d}t \quad \text{for parametric curved } (x(t), y(t)).$$

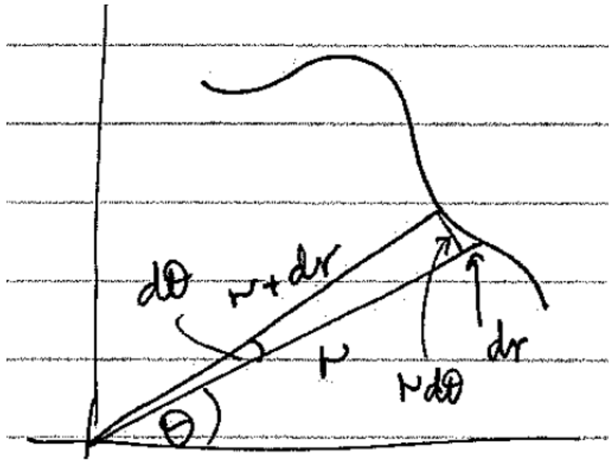
Now in polar coordinates we have curves $r = f(\theta)$ so we use θ as a parameter.

$$x = r\cos(\theta) = f(\theta)\cos(\theta) \quad y = r\sin(\theta) = f(\theta)\sin(\theta)$$



$$L = \int_{\theta=\alpha}^{\beta} \left[(f'\cos(\theta) - f\sin(\theta))^2 + (f'\sin(\theta) + f\cos(\theta))^2\right]^{1/2} \mathrm{d}\theta$$
$$= \int_{\alpha}^{\beta} \sqrt{(f'(\theta))^2 + (f(\theta))^2} \mathrm{d}\theta$$
$$= \int_{\alpha}^{\beta} \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\right]^{1/2} \mathrm{d}\theta$$

Another way using infinitesimals.

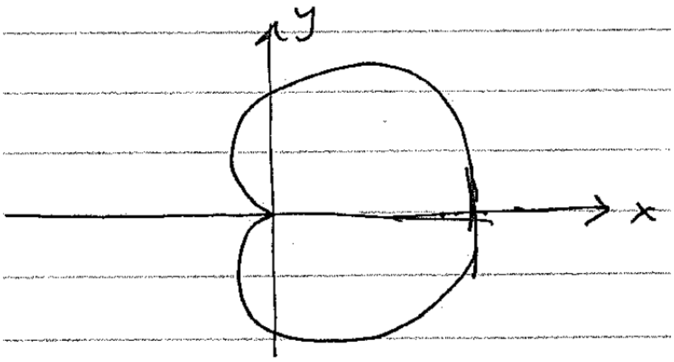


$$\text{Pythagoras:} \quad (\mathrm{d}r)^2 + r^2(\mathrm{d}\theta)^2 = (\mathrm{d}s)^2$$

$$\mathrm{d}s = \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\right]^{1/2} \mathrm{d}\theta$$

$$\Rightarrow \; L = \int_\alpha^\beta \left[\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2\right]^{1/2} \mathrm{d}\theta$$

Example 10 Find the length of the cardioid $r = 1 + \cos(\theta)$, $0 \leq \theta \leq 2\pi$.



$$L = \int_0^{2\pi} \sqrt{(1+\cos(\theta))^2 + \sin^2)\theta)}\mathrm{d}\theta = \int_0^{2\pi} \sqrt{2 + 2\cos(\theta)}\mathrm{d}\theta$$

$$\text{Now} \quad \cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right) - 1 \;\Rightarrow\; (1+\cos(\theta)) = 2\cos^2\left(\frac{\theta}{2}\right)$$

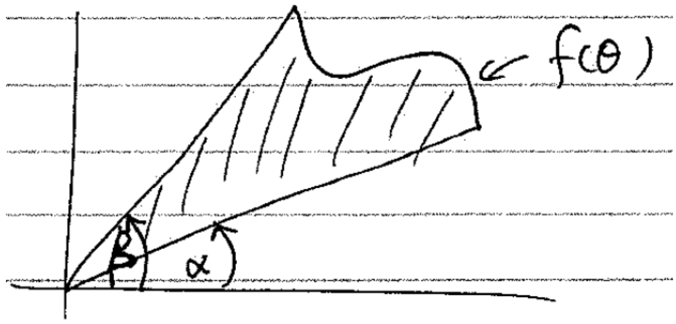
$$\Rightarrow \; \sqrt{2(1+\cos(\theta)} = 2\left|\cos\left(\frac{\theta}{2}\right)\right|$$

We need to do this because $1+\cos(\theta)$ can be positive but $\cos\left(\frac{\theta}{2}\right)$ is negative, *e.g.* for $\pi \leq \theta \leq 2\pi$. So need to write this integral as

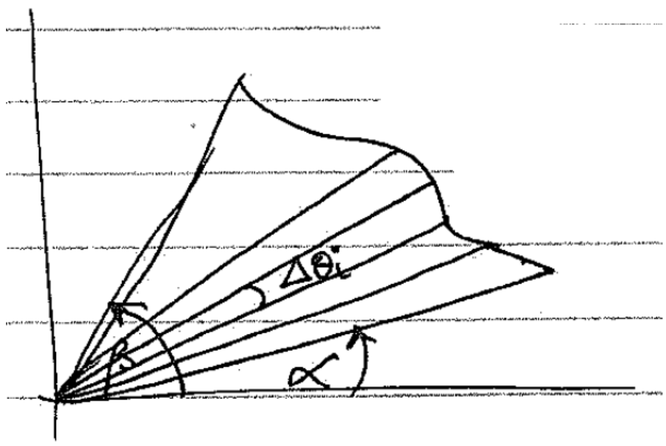
$$L = \int_0^{\pi} 2\cos\left(\frac{\pi}{2}\right) \mathrm{d}\theta + \int_{\pi}^{2\pi} \left(-2\cos\left(\frac{\theta}{2}\right)\right) \mathrm{d}\theta = 8$$

$$\text{Otherwise} \quad \int_0^{2\pi} 2\cos\left(\frac{\theta}{2}\right) \mathrm{d}\theta = 0 \quad \text{which is absurd!}$$

Area in polar coordinates in a region inside the graph of $f(\theta)$ on $[\alpha, \beta]$.



Use segments of angles $\Delta\theta_i$, and f constant.



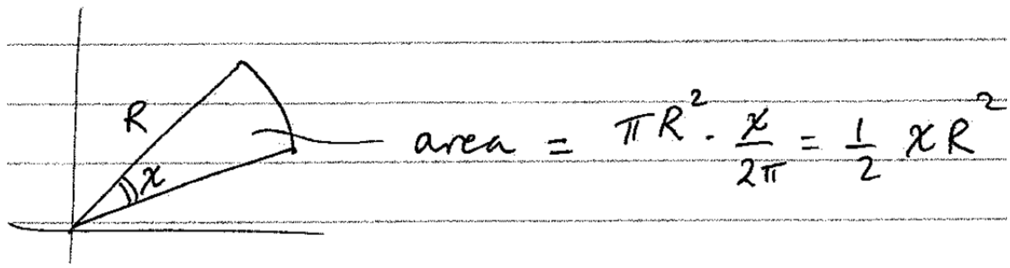
Approximate by $r = f(\theta_i)$ – constant.

$$\Delta A = \frac{1}{2}(f(\theta_i))^2 \Delta\theta_i$$

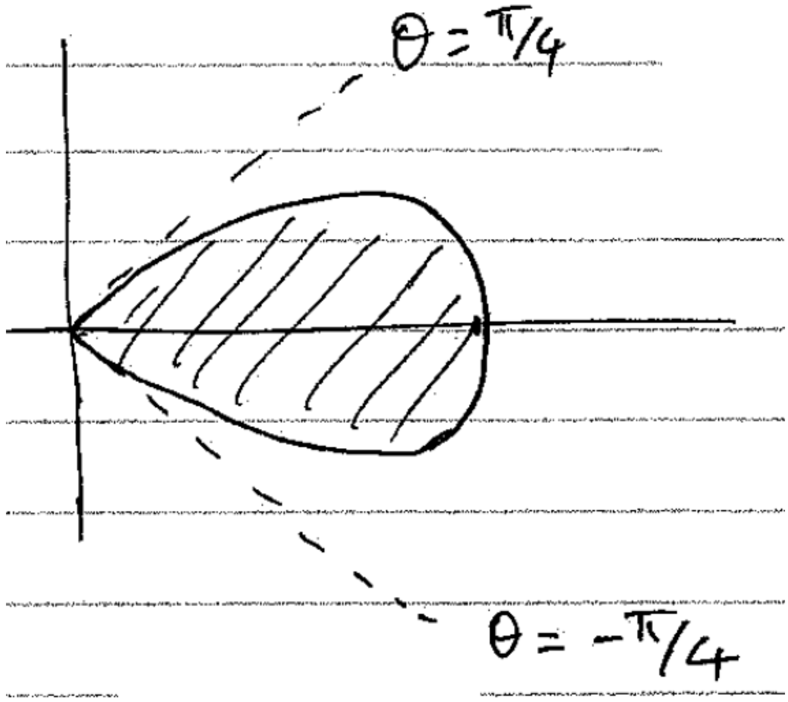
$$\Rightarrow \; A = \frac{1}{2}\int_{\alpha}^{\beta} f(\theta)^2 \mathrm{d}\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 \mathrm{d}\theta$$

In *Multi-variable Calculus* you will see a more general construction.

Note:



Example 11 Find the area enclosed by the four-petaled rose $r = \cos(2\theta)$.



$$r \geq 0 \Rightarrow -\frac{\pi}{2} \leq 2\theta \leq \frac{\pi}{2} \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$$

$$A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2(2\theta)\mathrm{d}\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos(4\theta)}{2}\mathrm{d}\theta$$

$$= \frac{1}{4}\frac{\pi}{2} = \frac{\pi}{8}$$

Part IV

Series, Power Series and Taylor's Theorem

Chapter 14

Series

Definition

*Given a sequence $\{a_n\}_{n\geq 1}$ of real numbers, define the sequence of **partial sums** by*

$$S_N = a_1 + a_2 + \cdots + a_N = \sum_{n=1}^{N} a_n.$$

*If $S_N \to S$ as $N \to \infty$, we say the **series converges to the sum** S. Write*

$$S = \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \sum_{n=1}^{\infty} a_n.$$

Example 1

The *geometric series* $\sum_{n=0}^{\infty} x^n$. $(x \neq 1)$

$$\begin{aligned} S_N &= 1 + (x + \cdots + x^N) \\ xS_N &= \quad (x + \cdots + x^N) + x^{N+1} \end{aligned}$$

$$\text{Subtract} \quad S_N = \frac{1 - xN + 1}{1 - x}$$

$$\text{If } |x| < 1, \quad \lim_{N\to\infty} S_N = \frac{1}{1-x} = \sum_{n=1}^{\infty} x^n,$$

hence the series converges. If $x \geq 1$, the series diverges.

Example 2

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} \quad S_N = \sum_{n=1}^{N} \frac{1}{n(n+1)} = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+1} \right)$$

Telescoping series $\Rightarrow S_N = 1 - \frac{1}{N+1} \to 1$ as $N \to \infty$. Series converges to 1.

14.1 Partial sums and geometric series

14.1.1 Series of positive terms

Note negative terms have the same theory.

Since terms are positive, the sequence S_N is an increasing sequence of numbers. Hence if the sequence of partial sums is *bounded above* then the series converges. If the sequence S_N is unbounded above, then $\sum \to \infty$.

Theorem 1
The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$.

Proof. It is enough to prove that the partial sums are not bounded above.

Consider

$$\begin{aligned} S_{2^K} &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^K} \\ &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{K-1}+1} + \cdots + \frac{1}{2^K}\right) \\ &\geq 1 + \frac{12}{+}\left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^K} + \cdots + \frac{1}{2^K}\right) \\ &= 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{K-1}}{2^K} = 1 + \frac{1}{2}K \end{aligned}$$

Partial sums unbounded for K large $\Rightarrow$ series diverges. □

Theorem 2
If $\alpha > 1$ is a rational number, then

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \quad \text{converges.}$$

Proof. Partial sums are increasing, so enough to prove that they are bounded above.

Compare $S_N \leq S_{2^N-1}$, note $N \leq 2^N - 1$.

$$\begin{aligned} S_N \leq S_{2^N-1} &= 1 + \frac{1}{2^\alpha} + \frac{1}{3^\alpha} + \cdots + \frac{1}{(2^N-1)^\alpha} \\ &= 1 + \left(\frac{1}{2^\alpha} + \frac{1}{3^\alpha}\right) + \left(\frac{1}{4^\alpha} + \frac{1}{5^\alpha} + \frac{1}{6^\alpha} + \frac{1}{7^\alpha}\right) + \cdots + \left(\frac{1}{2^{(N-1)\alpha}} + \cdots + \frac{1}{(2^N-1)^\alpha}\right) \\ &\leq 1 + \frac{2}{2^\alpha} + \frac{4}{4^\alpha} + \cdots + \frac{2^{N-1}}{2^{(N-1)\alpha}} \\ &= 1 + \frac{1}{2^{\alpha-1}} + \left(\frac{1}{2^{\alpha-1}}\right)^2 + \cdots + \left(\frac{1}{2^{\alpha-1}}^{N-1}\right) \\ &= \frac{1 - \left(\frac{1}{2^{\alpha-1}}\right)^N}{\left(1 - \frac{1}{2^{\alpha-1}}\right)} \leq \frac{1}{1 - \frac{1}{2^{\alpha-1}}} \quad \text{if} \quad \frac{1}{2^{\alpha-1}} < 1 \end{aligned}$$

i.e. $\alpha > 1$, which is the assumption of the theorem. □

Note: We will see a much easier proof later.

14.1.2 Elementary algebraic rules for series

If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, then $\sum_{n=1}^{\infty}(\alpha a_n + \beta b_n)$ also converges for any constants α, β.

Theorem 3 - Necessary condition for convergence.
If the series $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$ as $n \to \infty$.

Proof. Let the sum be S, i.e. $\sum_{1}^{\infty} a_n = S$. Then $S_N = \sum_{n=1}^{N} a_n \to S$ as $N \to \infty$, and $N \to \infty$, and $S_{N-1} = \sum_{n=1}^{N-1} a_n \to S$ as $N \to \infty$. Now $a_N = S_N - S_{N-1} \to S - S = 0$ as $N \to \infty$. □

Example 3

$$\sum_{n=1}^{\infty}(-1)^n = -1 + 1 - 1 + 1 \ldots \quad \text{diverges by the theorem above, } a_n \not\to 0 \text{ as } n \to \infty.$$

Note: Theorem 3 provides a necessary but not sufficient condition, *e.g.* $\sum_{n=1}^{\infty} \frac{1}{n}$ has $a_n \to 0$ but diverges.

Preposition: (follows from what we have shown).
If $\sum_{n=1}^{\infty} a_n$ converges, then for every N the series $\sum_{n=N}^{\infty} \to 0$ as $N \to \infty$. Intuitively, the "tail" of the series must go to zero if the series converges.

14.2 Cauchy sequences and convergence of series

Definition - Cauchy sequence
We say that the sequence of numbers $S_{k k=1,2,\ldots}$ is a Cauchy sequence, if given any $\epsilon > 0$ we can find an N such that for any $m > N$ and $n > N$

$$|S_m - S_n| < \epsilon$$

Intuition: as k increases, S_m and S_n get arbitrarily close. Cauchy sequences do not require all positive terms or any other special assumptions.

Connection with series: we have the following results for Cauchy sequences.

(1) Any convergent sequence is a Cauchy sequence.

(2) Any Cauchy sequence is bounded.

Theorem 4
Every Cauchy sequence converges. *(Proof by use of the Bolzano-Weierstrass theorem seen in Analysis).*

Theorem 5 (The alternating series test)
Suppose $\{a_n\}_{n\geq 1}$ is a decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. Then the series $\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + \dots$ converges.

Proof. We will show that the sequence S_k of partial sums is a Cauchy sequence, *i.e.* given any $\epsilon > 0$ we need to find N such that for all $n > m > N$, $|S_n - S_m| < \epsilon$.

Consider any $n > m$. Then since a_n is decreasing

$$0 \leq a_{m+1} - a_{m+2} + a_{m+3} - \cdots + a_n \leq a_{m+1}$$

Since $a_n \to 0$ as $n \to \infty$, given ϵ, I can find N such that for any $n > N$, $a_n < \epsilon$.

Now for any $n > m > N$.

$$\begin{aligned} |S_n - S_m| &= |(a_1 - a_2 + a_3 - a_4 + \dots a_n) - (a_1 - a_2 + \dots a_m)| \\ &= |a_{m+1} - a_{m+2} + a_{m+3} - \dots a_n| \\ &\leq a_{m+1} < \epsilon \text{ since } m > N \\ &\Rightarrow S_k \quad \text{is a Cauchy sequence and the Theorem follows.} \end{aligned}$$

□

Example 4

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \quad \text{converges since } |a_n| = \frac{1}{n} \to 0 \text{ and it is an alternating series.}$$

In fact, $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log(2) \quad \textit{(we will see this later)}$$

14.3 Convergence tests

Theorem 6 (Comparison test)
Let $\sum_{n=1}^{\infty} b_n$ be convergent with b_n non-negative. If $|a_n| \leq b_n$ $(n = 1, 2, \dots)$, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. Let $s_k = \sum_{j=1}^{k} a_j$, *i.e.* k-partial sum of $\sum a_n$. For $n > m$ we have

$$\begin{aligned}
|s_n - s_m| &= |a_{m+1} + a_{m+2} + \cdots + a_n| \\
&\le |a_{m+1}| + |a_{m+2}| + \cdots + |a_n| && \text{triangle inequality} \\
&\le b_{m+1} + b_{m+2} + \cdots + b_n && \text{by assumption} \\
&\le \sum_{i=m+1}^{\infty} b_i < \epsilon && \text{for } m \text{ large enough since } \sum b_i \text{ converges}
\end{aligned}$$

(More precisely, given $\epsilon > 0$, there is N such that $\sum_{i=m+1}^{\infty} b_i < \epsilon$ for all $m > N$).

Hence $\{s_k\}$ is Cauchy $\Rightarrow \sum_{i=1}^{\infty} a_n$ converges. □

Example 5 Show that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^{n+1}}$$

converges.

With $a_n = \frac{(-1)^n}{n3^n}$, compare with $\sum_{n=1}^{\infty} b_n$ with $b_n = \frac{1}{3^n}$.

$$n3^{n+1} > 3^n \Rightarrow |a_n| = \frac{1}{n3^n} < \frac{1}{3^n}$$

By comparison test $\sum_{n=1}^{\infty} a_n$ converges.

Example 6 Prove that if α is any positive number and $|x| < 1$, then the series

$$\sum_{n=1}^{\infty} n^{\alpha} x^n \quad \text{converges.}$$

First we note that $n^{\alpha} x^n \to 0$ as $n \to \infty$. In fact,

$$|n^{\alpha} x^n| = n^{\alpha} |x|^n = n^{\alpha} e^{n \log |x|}$$

and since $\log |x| < 0$, the exponential decay term dominates over any power of n.

Hence $n^{\alpha+2} x^n \to 0$ as $n \to \infty$ and the sequence $\{n^{\alpha+2} x^n\}$ is bounded. Hence there exists a constant C such that

$$|n^{\alpha+2} + x^n| \le C \quad i.e. \quad |n^{\alpha} x^n| \le \frac{C}{n^2} \quad n \ge 1$$

But $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, hence so does $\sum n^{\alpha} x^n$ by the comparison test. **Note:** it is much easier to use the Ratio Test (below).

14.3.1 Absolute and conditional convergence

A series $\sum_{n=1}^{\infty} a_n$ is said to be *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. A series that converges but does not do so absolutely is said to be *conditionally convergent.*

Theorem 7
Every absolutely convergent series is convergent.

Proof. Comparison test with $b_n = |a_n|$. □

Example 7

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is conditionally convergent.

Example 8 Discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n \sqrt{n}}{n+4}.$$

Series is *not* absolutely convergent since $\frac{\sqrt{n}}{n+4} = \frac{1}{\sqrt{n}+4/\sqrt{n}}$. Now for $n \geq 1$,

$$\frac{1}{\sqrt{n}+4/\sqrt{n}} \geq \frac{1}{5\sqrt{n}}$$

and since $\sum \frac{1}{\sqrt{n}}$ diverges, we are done.

For the alternating series test to apply we need to show that $\frac{\sqrt{n}}{n+4}$ is decreasing.

$$\text{If} \quad f(x) = \frac{\sqrt{x}}{x+4} \Rightarrow f'(x) = \frac{4-x}{2\sqrt{x}(x+4)^2} < 0 \quad \text{for} \quad x > 4.$$

So series terms decrease for $n > 4$. We only care about what happens beyond the 1st three terms - all the action is in the tail. Hence the series converges by the alternating series test, but it is not absolutely convergent.

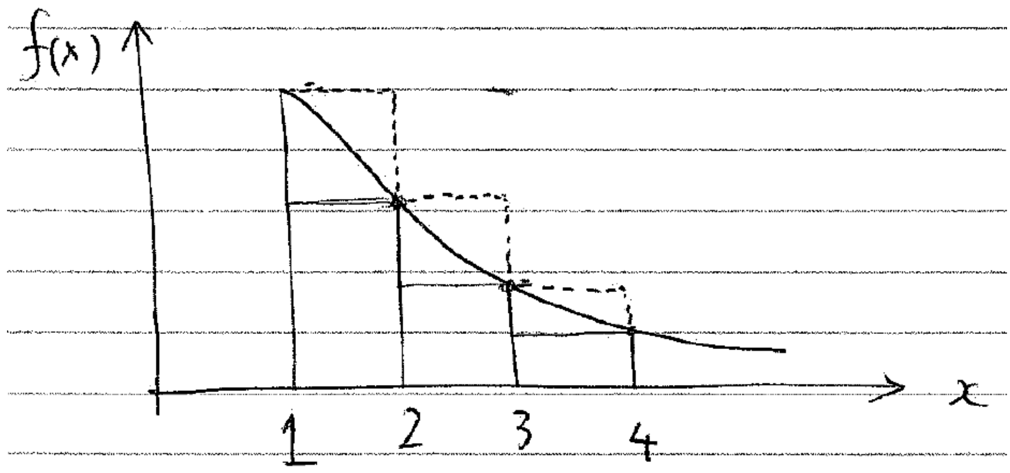
14.3.2 The Integral Test

Theorem 8
Let $f(x)$ be a function which is defined for all $x \geq 1$, and is *positive* and *decreasing.* Then the series

$$\sum_{n=1}^{\infty} f(n)$$

converges if and only if the improper integral $\int_1^{\infty} f(x)\mathrm{d}x$ converges.

To see this, consider the diagram:



For the partial sums $f(2) + f(3) + \cdots + f(n)$ we have

$$f(2) \leq \int_1^2 f(x)\mathrm{d}x, \quad f(3) \leq \int_2^3 f(x)\mathrm{d}x \quad \textit{etc.}$$
$$\Rightarrow \quad f(2) + f(3) + \cdots + f(n) \leq \int_1^n f(x)\mathrm{d}x$$

By assumption, *i.e.* that $\lim_{n\to\infty} \int_1^n f(x)\mathrm{d}x$ converges, $\sum_{k=2}^n f(k) \leq \int_1^\infty f(x)\mathrm{d}x$, *i.e* the partial sums s_k are bounded and so the series converges. Have proved this for one of the "ifs", *i.e.* when $\int_1^\infty f(x) < \infty$.

Conversely, assume that $f(1) + \cdots + f(n)$ approach a limit for large n. Consider the dashed rectangles in the diagram above.

$$f(1) \geq \int_1^2 f(x)\mathrm{d}x, \quad f(2) \geq \int_2^3 f(x)\mathrm{d}x \quad \text{etc}$$
$$\Rightarrow f(1) + f(2) + \cdots + f(n-1) \geq \inf_1^n f(x)\mathrm{d}x$$

So if the partial sums are bounded by L say, *(we know this is true since by assumption* $\sum_{n=1}^\infty f(n)$ *converges)* we have

$$\int_1^n f(x)\mathrm{d}x \leq L \quad (*)$$

Claim that this implies that $\int_1^\infty f(x)\mathrm{d}x$ exists. Give me any number b, however large you wish. Then I can find an integer $n > b$ so that

$$\int_1^b f(x)\mathrm{d}x \leq \int_1^n f(x)\mathrm{d}x \leq L \quad \text{by } (*)$$

Hence $\int_1^b f(x)\mathrm{d}x$ is bounded above for all b. Now send b to infinity.

Example 9 Show that

$$1+\frac{1}{2}+\cdots+\frac{1}{n}\geq\log(n+1)$$

and so obtain a new way of showing that $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Solution Take $f(x)=\frac{1}{x}$ in the integral test. Hence

$$1+\frac{1}{2}+\cdots+\frac{1}{n}\geq\int_1^{n+1}\frac{\mathrm{d}x}{x}=\log(n+1)$$

$$\Rightarrow\lim_{n\to\infty}\int_1^{n+1}\frac{\mathrm{d}x}{x}\quad\text{diverges, and by the integral test, so does }\sum_1^{\infty}\frac{1}{n}.$$

Example 10 For what values of p do the series

$$\sum_{n=1}^{\infty}\frac{1}{n^p}$$

converge/diverge?

Solution Let $f(x)=\frac{1}{x^p}$ and consider

$$\int_1^n\frac{\mathrm{d}x}{x^p}=\frac{n^{1-p}}{1-p}-\frac{1}{1-p}$$

(have shown this already when we did improper integrals.)

$$\text{Hence}\quad\lim_{n\to\infty}\int_1^n\frac{\mathrm{d}x}{x^p}\quad\text{exists if } p>1 \text{ and diverges if } p\leq 1.$$

$$\text{Hence}\quad\sum_{n=1}^{\infty}\frac{1}{n^p}\quad\text{converges for } p>1\text{, and diverges otherwise.}$$

Example 11 Show that $\int_{n=2}^{\infty}\frac{1}{n\sqrt{\log(n)}}$ diverges but $\sum_{n=2}^{\infty}\frac{1}{n(\log(n))^2}$ converges.

Solution

Use the integral test by considering

$$\int_2^{\infty}\frac{\mathrm{d}x}{x\sqrt{\log(x)}}=\lim_{b\to\infty}\int_2^b\frac{\mathrm{d}x}{x\sqrt{\log(x)}}=\lim_{b\to\infty}\int_2^b(\log(x))^{-1/2}\frac{1}{x}\mathrm{d}x$$

$$\text{(of the form }\int f'(g(x))g'(x)\mathrm{d}x\text{ since }\frac{\mathrm{d}}{\mathrm{d}x}(\log(x))=\frac{1}{x}\text{)}$$

$$=\lim_{b\to\infty}\left[2(\log(x))^{1/2}\right]_2^b=\lim_{b\to\infty}\left[2(\log(b))^{1/2}-2\sqrt{\log(2)}\right]=\infty$$

Note that we can do this generally, *i.e.*

$$\lim_{b\to\infty}\int_2^b \frac{\mathrm{d}x}{x(\log(x))^p} = \lim_{b\to\infty}\left[\frac{(\log(x))^{1-p}}{(1-p)}\right]_2^b$$
$$= \lim_{b\to\infty}\left[\frac{(\log(b))^{1-p}}{1-p} - \frac{(\log(2))^{1-p}}{1-p}\right]$$

converges if $p > 1$, so for $p = 2$ it converges. $p = 1$ must be done separately, *i.e.* consider.

$$\lim_{b\to\infty}\int_2^b \frac{\mathrm{d}x}{x\log(x)} = \lim_{b\to\infty}[\log(\log(x))]_2^b \longrightarrow \infty$$

Hence by the integral test, the series

$$\sum_{n=2}^{\infty}\frac{1}{n(\log(n))^p}$$

converges for $p > 1$ and diverges otherwise.

14.3.3 The Ratio Test

Theorem 9
Let $\sum n = 1^{\infty} a_n$ be a series satisfying

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

Then:

1. If $L < 1$ the series converges absolutely.
2. If $L > 1$ the series diverges.
3. If $L = 1$ the test is inconclusive.

Example 12 Prove that the series

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

converges for all values of x. By the ratio test:

$$a_n = \frac{x^n}{n!},\ \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!}\frac{n!}{|x|^n}$$
$$= \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$

In fact we will see that $\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$.

Proof. By assumption $\left|\frac{a_{n+1}}{a_n}\right|$ is close to L for large n.

Case (1), $L < 1$. Then pick $\epsilon > 0$ so small that $L + \epsilon < 1$, and for sufficiently large N we have

$$\left|\frac{a_{n+1}}{a_n}\right| < (L+\epsilon) \quad \text{if } n > N.$$

Now start with a_{N+K}. By the above bound

$$|a_{N+K}| < (L+\epsilon)|a_{N+K-1}| < (L+\epsilon)^2|a_{N+K-2}| < \cdots < (L+\epsilon)^K|a_N|.$$

Hence, $\displaystyle\sum_{j=1}^{\infty}|a_{N+j}| < |a_N|\sum_{j=1}^{\infty}(L+\epsilon)^j$

which is bounded since the series $\sum_{j=1}^{\infty}(L+\epsilon)$ is a geometric series ($(L+\epsilon) < 1$. Hence

$$\begin{aligned} & |a_{N+1}| + |a_{N+2}| + \dots \quad \text{converges} \\ \Rightarrow \quad & |a_1| + \cdots + |a_N| + |a_{N+1}| + |a_{N+2}| + \dots \\ & = \sum_{n=1}^{\infty}|a_n| \quad \text{also converges} \end{aligned}$$

(We only added a *finite* number of terms). This proves absolute convergence in case (1) $L < 1$.

In case (2), $L > 1$, pick $L + \epsilon > 1$ now and we have $|a_{N+K}| > (L+\epsilon)^N|a_N|$ which diverges now as a geometric series. To prove that if $L = 1$ the test is inconclusive, it is sufficient to pick an example, *i.e.*

$$\sum_{n=1}^{\infty}\frac{1}{n^p} \qquad \left|\frac{a_{n+1}}{a_n}\right|\left(\frac{n}{n+1}\right)^p = \left(\frac{1}{1+1/n}\right)^p$$

p is fixed, remember, so $\lim_{n\to\infty}\left(\frac{1}{1+1/n}\right)^p = 1^p = 1$. But $p > 1$, we have convergence but $p \leq 1$ divergence. Hence, test is inconclusive. □

Using this proof, we have a practical way of estimating errors in truncating series. Suppose

$$\left|\frac{a_n}{a_{n-1}}\right| < r < 1 \quad \text{for } n > N.$$

Then

$$\sum_{n=1}^{\infty}a_n - \sum_{n=1}^{N}a_n = \sum_{n=N+1}^{\infty}a_n$$

is the error. But $|a_{N+1}| < r|a_N|$ and generally $|a_{N+K}| < |a_N|r^K$. So

$$\sum_{k=1}^{\infty}|a_{N+K}| \leq |a_N|\sum_{k=1}^{\infty}r^k = |a_N|\frac{r}{1-r}.$$

So the error is $\leq |a_N|\frac{r}{1-r}$.

Example 12 What is the error made in approximating

$$\sum_{n=1}^{\infty} \frac{1}{n!} \quad \text{by} \quad \sum_{n=1}^{4} \frac{1}{n!}$$

Solution: We have $\frac{a_n}{a_{n-1}} = \frac{1}{n}$.

In the example, the truncation is $N = 4$, so if $n > 4$, $\left|\frac{a_n}{a_{n-1}}\right| < \frac{1}{5}$.

The error is $\le |a_4| \cdot \frac{1/5}{1-1/5} = \frac{1}{4!} \cdot \frac{1}{4} = \frac{1}{96} < 0.0105$

14.3.4 The Root Test

Theorem 10
For the given series $\sum_{n=1}^{\infty} a_n$, suppose that $\lim_{n\to\infty} |a_n|^{1/n} = L$. Then

1. If $L < 1$ the series converges absolutely.
2. If $L > 1$ the series diverges.
3. If $L = 1$ the test is inconclusive.

Proof. Similar to that for the ratio test.

Case (1), pick $\epsilon > 0$ so that $L + \epsilon < 1$ and for large N, $|a_n|^{1/n} < (L + \epsilon) < 1$ for $n > N$. Hence

$$|a_n| < (L + \epsilon)^n \quad \text{for } n > N.$$

Now compare

$$\sum_{n=N+1}^{\infty} |a_n| \qquad \text{with} \qquad \sum_{n=N+1}^{\infty} (L + \epsilon)^n.$$

The latter converges (geometric series with $L + \epsilon < 1$) hence $\sum_{n=N+1}^{\infty} |a_n|$ converges $\Rightarrow \sum_{n=1}^{\infty} |a_n|$ converges.

Case (2) is simply $\sum_{n=N+1}^{\infty} |a_n| > \sum_{n=N+1}^{\infty} (L+\epsilon)^n$ where now $L+\epsilon > 1$, *i.e.* diverges.

Case (3), consider $\sum_{n=1}^{\infty} a_n$ with $a_n = n$. Clearly series diverges but $\lim_{n\to\infty} n^{1/n} = 1$. *(Why!?)* □

Examples:

1.

$$\sum_{n=1}^{\infty} \frac{1}{n^n} \quad \text{converges}, \qquad |a_n|^{1/n} = \frac{1}{n} \to 0.$$

2.

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} \quad \text{diverges. } \textit{(We already know this by other methods.)}$$

$$i.e. \qquad a_n = \frac{3^n}{n^2} \not\to 0 \quad \text{as} \quad n \to \infty$$

$$|a_n|^{1/n} = \frac{3}{(n^{1/n})^2} \to 3 \quad \text{as} \quad n \to \infty.$$

3.

$$\sum_{n=1}^{\infty} \frac{n^n}{n!} \qquad \text{(again we have seen this before in HW3)}$$

$$\text{Use ratio test} \quad \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!}\frac{n!}{n^n} = (1+\frac{1}{n})^n$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = e > 1 \quad \text{diverges.}$$

4.

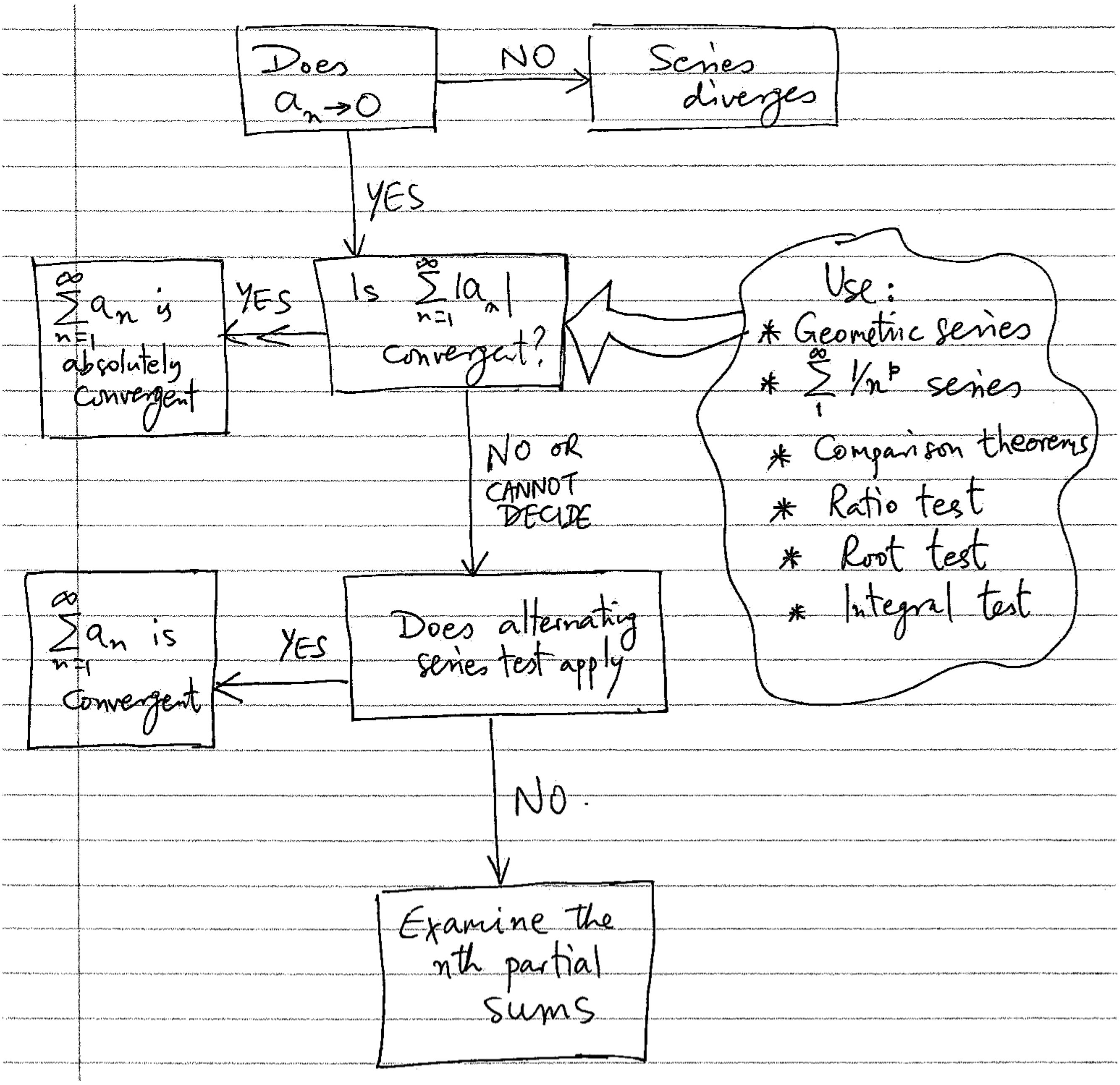
$$\sum_{n=1}^{\infty} \frac{1}{n^2 - \log(n)}.$$

$$\text{Intuition: large } n \text{ series } \approx \sum \frac{1}{n^2} < \infty$$

$$\frac{1}{n^2 - \log(n)} \quad \text{can be bounded below by} \quad \frac{1}{\alpha n^2} \quad \text{with} \quad 0 < \alpha < 1$$

$$\Rightarrow \quad \sum_1^{\infty} \frac{1}{n^2 - \log(n)} < \sum_1^{\infty} \frac{1}{\alpha n^2} < \infty \quad \textit{(Why!?)}$$

14.3.5 Testing convergence for $\sum_{n=1}^{\infty} a_n$



Chapter 15

Power Series

15.1 Convergence tests and radius of convergence

Definition

Let x be a real number (can extend to complex numbers also) and $\{a_n\}_{n\geq 0}$ be a sequence of numbers. Then we can form the ***power series*** $\sum_{n=0}^{\infty} a_n x^n$*. The partial sums* $s_N = \sum_{n=1}^{N} a_n x^n$ *are degree N polynomials.*

e.g. The geometric series $1+x+x^2+\dots$ converges for $|x| < 1$. Hence $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ if $|x| < 1$.

Theorem 1 Assume that there is a number $R > 0$ such that $\sum_{n=0} |a_n| R^n$ converges. Then for all $|x| < R$, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

Proof.

$$|a_n|\ |x|^n \leq |a_n| R^n$$

Hence, absolute convergence by the comparison test with the given series. □

Definition

The greatest value of R for which we get convergence is called the ***radius of convergence*** *and* $\sum_{n=0} a_n x^n$ *converges absolutely if $|x| < R$. $x = \pm R$ must be tested separately.*

Theorem 2 - Ratio Test for power series
Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and assume that $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L$ exists.
Let $R = \frac{1}{L}$. (If $L = 0$ let $R = \infty$, if $L = \infty$ let $R = 0$.)
Then

(i) If $|x| < R$ the series converges absolutely.

(ii) If $|x| > R$ the power series diverges.

(iii) If $x = \pm R$, could converge or diverge.

Proof. Ratio test for series of numbers

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| |x| \xrightarrow[n\to\infty]{} L|x| \quad \text{by hypothesis.}$$

For convergence, $L|x| < 1$, $\Rightarrow |x| < \frac{1}{L} = R$. Of course R is the radius of convergence defined earlier. □

Theorem 3 - Root test for power series
Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, and assume that $\lim_{n\to\infty} |a_n|^{1/n} = L$ exists. Then the radius of convergence of the power series is $R = 1/L$

Proof. For $\sum_{n=0}^{\infty} a_n x^n$ use the root test.

$$\lim_{n\to\infty} |a_n|^{1/n}|x| = L|x| \;\Rightarrow\; |x| < \frac{1}{L} = R \quad \text{for convergence.}$$

□

Examples:

(i) Determine the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{n^p}{(n+1)!} x^n \quad \text{where } p > 0 \text{ is given.}$$

Solution: Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^p}{(n+1)!}\frac{n!}{n^p} = \left(1 + \frac{1}{n}\right)^p \frac{1}{n_1} \to 0 \text{ as } n \to \infty.$$

Hence $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0$ and the radius of convergence $R = \infty$, *i.e.* $\sum_{n=0}^{\infty} \frac{n^p}{(n+1)!} x^p$ converges for all x.

(ii) $\sum_{n=0}^{\infty} \frac{x^n}{n}$ Ratio test - (will do it directly now, without the L intermediate step.)

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)} \frac{n}{x^n} \right| = \lim_{n\to\infty} \frac{n}{n+1}|x| = |x|.$$

Hence convergence if $|x| < 1$. Radius of convergence is $R = 1$.

$$\begin{aligned} x = 1 \quad & \sum \frac{1}{n} \text{ diverges} \\ x = -1 \quad & \sum \frac{(-1)^n}{n} \text{ diverges by alternating series test.} \end{aligned}$$

Note: Instead of $\sum_{n=0}^{\infty} a_n x^n$, could define power series centered at points other than 0, *i.e.*

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

Everything is the same, simply substitute $x - x_0 = y$.

(iii) For what x does the power series

$$\sum_{n=0}^{\infty} \frac{4^n}{\sqrt{2n+5}} (x+5)^n$$

converge? Use ratio test,

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left\{ 4 \sqrt{\frac{2n+5}{2(n+1)+5}} |x+5| \right\} \\ &= 4|x+5|. \end{aligned}$$

Convergence if $|x+5| < \frac{1}{4}$. *i.e.*

$$\begin{aligned} -\frac{1}{4} < x + 5 < \frac{1}{4} \\ -\frac{21}{4} < \quad x \quad < -\frac{19}{4} \end{aligned}$$

15.2 Differentiation and integration of power series

For polynomials of degree n, *i.e.* $a_0 + a_1 x + \cdots + a_n x^n := f_n(x)$, we can differentiate or integrate so that

$$\frac{\mathrm{d}f_n}{\mathrm{d}x} = \sum_{k=1}^{n} K a_K x^{K-1} \text{ and } \int f_n \mathrm{d}x = \sum_{K=0}^{n} \frac{a_K x^{K+1}}{K+1}.$$

Question is, can we do this for power series? The answer is **YES** if $|x| < R$, *i.e.* we are within the radius of convergence. We have the following very important theorems.

Theorem 4 Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series which converges absolutely for $|x| < R$. Then $f(x)$ is differentiable for $|x| < R$, and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

Theorem 5 Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series that converges absolutely for $|x| < R$. Then in the interval $|x| < R$, we have

$$\int f \mathrm{d}x = \sum_{n=0} \frac{a_n x^{n+1}}{n+1}.$$

Conclusion: For a power series *within its radius of convergence*, we can differentiate or integrate term by term.

Note $f(x) = \sum_{n=0}^{\infty} a_n x^n$ can be differentiated an infinite number of times as long as $|x| < R$, and the derivatives will exist. The function is smooth.

The way to show this is to consider each differentiated series as a new power series. For example

$$\begin{aligned}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum a_n x^n\right) &= \sum n(n-1)\ldots(n-(k-1))x^{n-k}a_n \\ &= \sum \frac{n!}{(n-k)}!a_n x^{n-k}.\end{aligned}$$

Ratio test

$$\begin{aligned}&\lim_{n\to\infty} \frac{(n+1)!}{(n+1-k)!}\frac{(n-k)!}{n!}\left|\frac{a_{n+1}}{a_n}\right||x| \\ &= \lim_{n\to\infty}\left(\frac{n+1}{n+1-k}\right)\left|\frac{a_{n+1}}{a_n}\right||x| = L|x|\end{aligned}$$

as for the undifferentiated power series.

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}\left(\sum a_n x^n\right)$$

converges for $|x| < R$. k is arbitrary so can be differentiated as many times as we want. Similarly, integrate as many times as needed.

Example Write down power series for

$$\frac{x}{1+x^2} \quad \text{and} \quad \log(1+x^2).$$

Solution Recall geometric series $1 + r + r^2 + \cdots = \frac{1}{1-r}$ for $|r| < 1$. If $r = -x^2$ we have

$$x(1 - x^2 + x^4 - x^6 + \dots) = \frac{1}{1+x^2} \cdot x.$$

Hence

$$\frac{x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \quad |x| < 1 \quad \text{for convergence.}$$

Now for the $\log(1+x^2)$ we observe that

$$\frac{\mathrm{d}}{\mathrm{d}x} \log(1+x^2) = \frac{2x}{1+x^2} \text{ so } \log(1+x^2) = 2 \int \frac{x}{1+x^2}.$$

If $|x| < 1$ we can integrate term by term, *i.e.*

$$\begin{aligned} \log(1+x^2) &= 2 \int (x - x^3 + x^5 \dots) \mathrm{d}x \\ &= x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \frac{x^8}{4} + \dots \end{aligned}$$

(Constant of integration is zero). Convergence for $|x| < 1$ and also $x = 1$ since it is alternating in the latter case $\Rightarrow \log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{14}{+} \dots$.

Theorem 6 - Algebraic operations Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R_1, and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ is another power series with radius of convergence R_2. Let

$$R = \min(R_1, R_2).$$

Then

(1) $f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n) x^n$ for $|x| < R$.

(2) $cf(c) = \sum_{n=0}^{\infty} c a_n x^n$ for $|x| < R_1$ ($c \neq 0$).

(3) $f(x)g(x) = \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} a_m b_{n-m}\right) x^n$ for $|x| < R$

Chapter 16

Taylor Series

This is a power series that represents a function $f(x)$ by using its derivatives at a single point. Intuitive construction: Assume the power series exists and identify the coefficients. Take a fixed point $x = x_0$. If

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

converges for $|x - x_0|$ small enough we can find the coefficients as follows:

Re-write as:

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots \\ f'(x) &= a_1 + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + \dots \\ f''(x) &= 2a_2 + 3 \cdot 2a_3(x - x_0) + \dots \\ f'''(x) &= 3 \cdot 2 \cdot 1a_3 + \dots \\ f^{(k)}(x) &:= \frac{\mathrm{d}^k f}{\mathrm{d}x^k} = k!a_k + \underbrace{\mathcal{O}(x - x_0).}_{(*)} \end{aligned}$$

(*) - This means "terms of order $(x - x_0)$" or smaller for $(x - x_0)$ small.

So we can see immediately that by putting $x = x_0$ in the formula of $f^{(k)}(x)$ we find

$$\begin{aligned} a_k &= \frac{1}{k!} f^{(k)}(x_0) \Rightarrow \\ f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n \quad \text{(Note } 0! = 1\text{).} \end{aligned}$$

This is the *Taylor series about the point* $x = x_0$. If $x_0 = 0$ we get the *Maclaurin series*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

In both formulas we have assumed that $f(x)$ is infinitely differentiable on some interval containing the point $x = x_0$. This of course can happen for many functions, *e.g.* $f(x) = \sin(x)$, $f(x) = e^x$, $f(x) = \cos(x)$ ...

Example 1 Maclaurin series for $f(x) = \sin(x) \Rightarrow f(0) = 0$

$$\begin{aligned} f'(x) &= \cos(x) & f'(0) &= 1 \\ f^{(2)}(x) &= -\sin(x) & f^{(2)}(0) &= 0 \\ f^{(3)}(x) &= -\cos(x) & f^{(3)}(0) &= -1 \\ f^{(4)}(x) &= \sin(x) & f^{(4)}(0) &= 0 \end{aligned}$$

repeats

$$\downarrow$$

$$\begin{aligned} f(x) = \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}. \end{aligned}$$

Example 2

$$f(x) = e^x \Rightarrow f^{(k)}(x) = e^x \ i.e.\ f^{(k)}(0) = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

16.1 Taylor's theorem with remainder

We have left an important detail out! In sending the sum to ∞, we assume that there is convergence, and the convergence is to the function $f(x)$. We have the following:

Theorem 1 (Taylor's) Let f be a function defined on a closed interval between two numbers x_0 and x. Assume that the function has $n + 1$ derivatives on the interval and that they are all continuous. Then

$$\begin{aligned} f(x) = f(x_0) + f'(x_0)(x - x_0) &+ \frac{f^{(2)}(x_0)}{2!}(x - x_0) + \dots \\ &+ \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + R_n \end{aligned}$$

where the *remainder* R_n is given by

$$R_n = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\, \mathrm{d}t.$$

Proof. Use integration by parts. From the fundamental theorem of calculus we have

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\mathrm{d}t.$$

Now

$$\int_{x_0}^{x} f'(t)\mathrm{d}t = \int_{x_0}^{x} f'(t)\,\mathrm{d}(-(x-t))$$

and use integration by parts.

$$\begin{aligned}\int_{x_0}^{x} f'(t)\mathrm{d}t &= \big[+(t-x)f'(t)\big]_{x_0}^{x} - \int_{x_0}^{x}(t-x)f''(t)\mathrm{d}t \\ &= (x-x_0)f'(x_0) + \int_{x_0}^{x}(x-t)f^{(2)}(t)\mathrm{d}t.\end{aligned}$$

One more

$$\begin{aligned}\int_{x_0}^{x}(x-t)f^{(2)}(t)\mathrm{d}t &= -\frac{(}{x} - t)^2 f^{(2)}(t)|_{x_0}^{x} + \int_{x_0}^{x}\frac{(x-t)^2}{2}f^{(3)}(t)\mathrm{d}t \\ &= \frac{(x-x_0)^2}{2}f^{(2)}(x_0) + \int_{x_0}^{x}\frac{(x-t)^2}{2}f^{(3)}(t)\mathrm{d}t.\end{aligned}$$

Repeat n times to get the result. □

Alternative form of the remainder

$$R_n = \int_{x_0}^{x}\frac{(x-t)^n}{n!}f^{(n+1)}(t)\mathrm{d}t.$$

Use the Integral MVT - *Problem 10, sheet 3.*

$$\begin{aligned}\text{Since } x-t \geq 0, \quad g(t) &:= \frac{(x-t)^n}{n!} \geq 0 \\ \Rightarrow R_n &= f^{(n+1)}(c)\int_{x_0}^{x}\frac{(x-t)^n}{n!}\mathrm{d}t \\ \Rightarrow R_n &= \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}\end{aligned}$$

Where c is a number between x_0 and x.

$$\begin{aligned}\text{Since } x-t \geq 0, \quad g(t) &:= \frac{(x-t)^n}{n!} \geq 0. \\ \Rightarrow R_n &= f^{(n+1)}(c)\int_{x_0}^{x}\frac{(x-t)^n}{n!}\mathrm{d}t.\end{aligned}$$

Convergence to $f(x)$ if $R_n \to 0$ as $n \to \infty$.

16.1.1 Summary and link with power series

(1) If $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ is a convergent power series on an open interval centered at x_0, then $f(x)$ is infinitely differentiable and

$$a_n = \frac{f^{(n)}(x_0)}{n!}.$$

i.e. We get Taylor's formula.

(2) If f is infinitely differentiable on an open interval centered at x_0, and if $R_n \to 0$ as $n \to \infty$ for all x in the interval, then the Taylor series of f converges an equals the function, *i.e.*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Here is a very useful alternative of Taylor's theorem that we use in Numerical Analysis. Put $x = x_0 + h$ (and after that $x_0 \to x$ if you want)

$$\begin{aligned} f(x_0 + h) &= f(x_0) + hf'(x_0) + \frac{h^2}{2!}f^{(2)}(x_0) + \cdots + \frac{h^n}{n!}f^{(n)}(x_0) + R_n(x_0, h) \\ R_n &= \int_{x_0}^{x_0+h} \frac{(x_0 + h - t)^n}{n!} f^{(n+1)}(t)\mathrm{d}t \\ &= \frac{h^{(n+1)}}{(n+1)!} f^{(n+1)}(c) \end{aligned}$$

where c is between x_0 and $x_0 + h$.

16.2 Examples, bounding the remainder, estimates

Will do this with $x_0 = 0$ (Maclaurin), other cases follow. Use form

$$R_n = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1}$$

c is a number between 0 and x. If $|f^{(n+1)}(x')| \leq M_{n+1}$ for all x' between 0 and x. Then

$$|R_n| \leq M_{n+1} \frac{|x|^{n+1}}{(n+1)!}$$

Example 3

$$\begin{aligned} f(x) = \sin(x) = x - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + R_{2n+3} \\ \text{where } |R_{2n+3}| = \left| \frac{f^{(2n+3)}(x)}{(2n+3)!} x^{2n+3} \right| \leq \frac{|x|^{2n+3}}{(2n+3)!}. \end{aligned}$$

Hence $\sin(0.1) \approx 0.1 - \frac{10^{-3}}{6}$ with an error which is less than

$$\frac{(0.1)^7}{7!} = \frac{10^{-7}}{5040} < 10^{-10}.$$

Example 4 Compute $\sin\left(\frac{\pi}{6} + 0.2\right)$ to an accuracy of 10^{-4}.

Solution Even though $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots$ will converge if we take enough terms, since $\frac{\pi}{6} + 0.2$ is not small, we will need **a lot of terms** to get to the required accuracy.

It is **much** better to expand about $\frac{\pi}{6}$ using the formula

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \cdots + \frac{h^n}{n!}f^{(n)}(x) + R_n$$
$$\text{where } R_n = \frac{f^{(n+1)}}{(n+1)!}h^{n+1}.$$

Now $h = 0.2$, $f^{(n+1)} = \sin$ or $\cos$.

$$\Rightarrow |R_n| \leq \frac{(0.2)^{n+1}}{(n+1)!} \quad \Rightarrow \quad R_3 \leq \frac{0.2^4}{24} = \frac{16\times 10^{-4}}{24} < 10^{-4}$$
$$\Rightarrow \sin\Big(\frac{\pi}{6}+0.2\Big) \approx \sin\Big(\frac{\pi}{6}\Big) + 0.2\cos\Big(\frac{\pi}{6}\Big)\frac{(0.2)^2}{2!}\Big(-\sin\Big(\frac{\pi}{6}\Big)\Big) + \frac{(0.2)^3}{3!}\Big(-\cos\Big(\frac{\pi}{6}\Big)\Big).$$

16.3 Exponentials and logarithms. Binomial theorem

16.3.1 The exponential e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \underbrace{\frac{e^c}{(n+1)!}x^{n+1}}_{\text{remainder}}$$

If $x < 0$ then $c < 0$ and $|R_n| \leq \frac{|x|^{n+1}}{(n+1)!}$.
If $x > 0$ and such that $x \leq b$, say. Then

$$|R_n| \leq e^b\frac{b^{n+1}}{(n+1)!} \to 0 \text{ as } n \to \infty$$

Example 5 Compute e to 3 decimals. Showed earlier (see chapter on logarithms) that $2 < e < 4$. From results above, $|R_n| \leq \frac{e}{(n+1)!} \leq \frac{4}{(n+1)!}$.

Need $\frac{4}{(n+1)!}$ to be less than 10^{-3}.

$$\text{Try } n = 4,5,6 \;\rightarrow\; \frac{4}{5!} = \frac{1}{5\times 3\times 2} = \frac{1}{30} > 10^{-3}$$
$$\frac{4}{6!} = \frac{1}{6\times 5\times 3\times 2} = \frac{1}{180} > 10^{-3}$$
$$|R_6| \leq \frac{4}{7!} = \frac{1}{7\times 6\times 5\times 3\times 2} = \frac{1}{1260} < 10^{-3}.$$

So

$$e \approx 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \underbrace{R_6}_{<10^{-3}}.$$

16.3.2 The Logarithm

Expansion of $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + R_{n+1}$. Need to show this. One way is Taylor's theorem - *exercise*. Another way is to use the identity *(telescoping product)*

$$(1 - t + t^2 - \dots + (-1)^{n-1}t^{n-1})(1+t) = 1 + (-1)^{n-1}t^n$$
$$\Rightarrow \frac{1}{1+t} = (1 - t + t^2 - \dots + (-1)^{n-1}t^{n-1}) + (-1)^n \frac{t^n}{(1+t)}. \quad (*)$$

In the interval $-1 < x \leq 1$ *(why?)*. Integrate (*) between 0 and x.

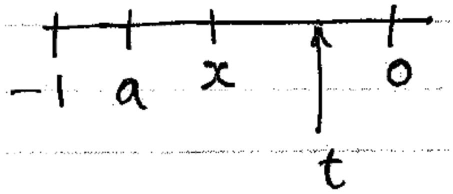
$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-1}\frac{x^n}{n} + R_n$$
$$\text{where } R_n = (-1)^n \int_0^x \frac{t^n}{1+t} \mathrm{d}t$$

Need to estimate this:

(i) Consider $0 \leq x \leq a \leq 1$. Here $1 + t \geq 1 \Rightarrow \frac{t^n}{1+t} \leq t^n$.

$$|R_n| \leq \frac{x^{n+1}}{n+1} \leq \frac{a^{n+1}}{n+1} \to 0 \text{ as } n \to \infty.$$

(ii) Now take $-1 < a < 0$ and consider x in the interval $a \leq x \leq 0$. Hence $1 + t \geq 1 + a > 0$ since t is in the interval $(x, 0)$.



$$\left|\frac{t^n}{1+t}\right| \leq \frac{(-t)^n}{1+a} \quad (t \leq 0 \text{ remember})$$

$$\text{So } |R_n| \leq \int_x^0 \frac{(-t)^n}{1+a}\mathrm{d}t = \frac{(-x)^{n+1}}{(n+1)(1_a)} \leq \frac{|a|^{n+1}}{(n+1)(1+a)} \to 0 \text{ as } n \to \infty$$

Exercise: Calculate $\log|\cdot|$ to 3 decimals.

16.3.3 Binomial Expansion

If $|x| < 1$ we have (for any real α)

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots$$
$$= \sum_{n=0}^{\infty} \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n.$$

Can prove $|R_n| \to 0$ as $n \to \infty$ for $|x| < 1$, hence convergence.

16.3.4 Alternatives to L'Hôpital

Sometimes easier to use Taylor's theorem instead of differentiating many times.

Example 6

$$\begin{aligned}\lim_{x\to\infty}\frac{\sin(x)-x}{\tan(x)-x} &= \lim_{x\to 0}\frac{\sin(x)\cos(x)-x\cos(x)}{\sin(x)-x\cos(x)}\\ &= \lim_{x\to 0}\frac{\left(x-\frac{x^3}{6}+\dots\right)\left(1-\frac{x^2}{2}+\dots\right)-x\left(1-\frac{x^2}{2}+\dots\right)}{x-\frac{x^3}{6}+\dots-x\left(1-\frac{x^2}{2}+\dots\right)}\\ &= \lim_{x\to 0}\frac{-\frac{x^3}{6}+\dots}{\frac{1}{3}x^3+\dots}=-\frac{1}{2}.\end{aligned}$$

Example 7

$$\begin{aligned}&\lim_{x\to 1}\frac{\log(x)}{e^x-e}\quad \text{put } x=1+y\\ &\qquad = \lim_{y\to 0}\frac{\log(1+y)}{e(e^y-1)}=\lim_{y\to 0}\frac{y-\frac{y^2}{2}+\frac{y^3}{3}+\dots}{e(1+y+\dots-1}\\ &\qquad = \frac{1}{e}.\end{aligned}$$

16.3.5 L'Hôpital's Rule derived from Taylor's Theorem

Consider $F(x)=\frac{f(x)}{g(x)}$ and consider $\lim_{x\to a}F(x)$ in cases where $f(a)=g(a)=0$. Assume that the first $(k-1)$ derivatives of f and g also vanish at $x=a$. *i.e.*

$$\begin{aligned}f^{(i)}(a) =& g^{(i)}(a)=0 \quad i=1,\ \dots,\ k-1\\ \text{Then}\quad f(a+h) &= f(a)+f'(a)h+\dots+f^{(k-1)}(a)\frac{h^{k-1}}{(k-1)!}+f^{(k)}(c_1)\frac{h^k}{k!}\\ g(a+h) &= g(a)+g'(a)h+\dots+g^{(k-1)}(a)\frac{h^{k-1}}{(k-1)!}+g^{(k)}(c_1)\frac{h^k}{k!}\end{aligned}$$

where c_1, c_2 are numbers between a and $a+h$.

$$\Rightarrow\quad F(a+h)=\frac{f^{(k)}(c_1)}{g^{(k)}(c_1)},\quad \text{send } h\to 0,\ c_1,c_2\to a \text{ so get result.}$$

An example of a function that does not have a Maclaurin series. Consider

$$f(x)=\begin{cases}e^{-1/x^2} & x\neq 0\\ 0 & x=0.\end{cases}$$

$f(x)$ and all its derivatives are continuous everywhere. *(You have shown this in the problem sheets)*. In addition, $f^{(n)}(0) = 0$ for all n. So the (Taylor) Maclaurin expansion is

$$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots$$

The approximating polynomial

$$P_n(x) = f(0) + xf'(0) + \cdots + \frac{f^{(n)}(0)}{n!}x^n.$$

But this is exactly zero for all n. Hence the remainder R_n cannot go to zero. In fact it must be equal to $\exp\left(-1/x^2\right)$! **Reason:** e^{-1/z^2} z complex is not analytic. In fact, $z = iy$ gives $f = e^{1/y^2} \to \infty$ as $y \to 0$. You will see more in Complex Analysis.

Part V

Fourier Series

Chapter 17

Orthogonal and orthonormal function spaces

Will see how to represent fairly arbitrary functions *(e.g. they can be discontinuous)* with approximations of smooth functions.

Definition 1

If f, g are real values functions that are Riemann integrable on $[a, b]$, then we define the inner produce of f and g, denotes by (f, g), by

$$(f, g) := \int_a^b f(x)g(x)\mathrm{d}x$$

$$\text{Note } (f, f)^{1/2} = \left(\int_a^b f^2 \mathrm{d}x\right)^{1/2} := ||f|| \geq 0.$$

Definition 2

Let $\mathcal{S} = \{\phi_0, \phi_1, \phi_2, \dots\}$ be a collection of functions that are Riemann integrable on $[a, b]$. If

$$(\phi_n, \phi_m) = 0 \quad \text{whenever } m \neq n$$

*then $\mathcal{S}$ is an **orthonormal** system on $[a, b]$. If in addition, $||\phi_n|| = 1$, i.e. $\int_a^b \phi_n^2 \mathrm{d}x = 1$, then $\mathcal{S}$ is said to be orthonormal on $[a, b]$.*

Note: Can easily go from orthogonal to orthonormal by considering $\frac{\phi_n}{||\phi_n||}$. The orthonormal *trigonometric system* will be used

$$\mathcal{S} = \{\phi_0, \phi_1, \phi_2, \dots\} \quad \text{where}$$

$$\phi_0(x) = \frac{1}{\sqrt{2\pi}}, \quad \phi_{2n-1} = \frac{\cos(nx)}{\sqrt{\pi}}, \quad \phi_{2n} = \frac{\sin(nx)}{\sqrt{\pi}} \quad (n = 1, 2, \dots)$$

i.e. the system is

$$\left\{\frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{\pi}}, \frac{\sin(x)}{\sqrt{\pi}}, \frac{\cos(2x)}{\sqrt{\pi}}, \frac{\sin(2x)}{\sqrt{\pi}}, \dots\right\}.$$

$\mathcal{S}$ defined above is orthonormal on any interval of length 2π, *e.g.* $[0, 2\pi]$, $[-\pi, \pi]$ *etc.*. *(You have already shown this in HW sheet 3).*

Chapter 18

Periodic functions and periodic extensions

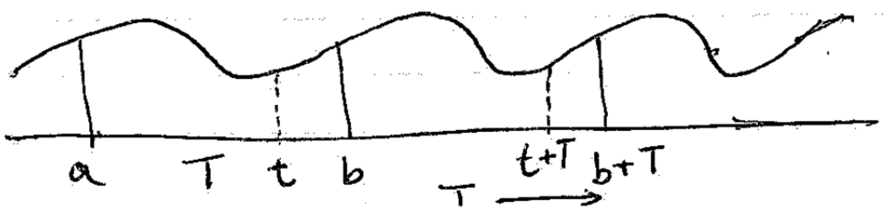
A function $f(x)$ is periodic with period T if

$$f(x+T) = f(x)$$

for all values of x. It follows that a T periodic function is also mT periodic for any integer m, *i.e.*

$$f(x \pm mT) = f(x).$$

e.g. $\sin(x)$ is 2π-periodic but also 4π, 6π etc. Geometrically $f(x)$ is T periodic if a shift by T units reproduces the shape of the function.



Start with any continuous function $f(x)$ in an interval $a \leq x < b$. Can extend this periodically to have period $T = b - a$.

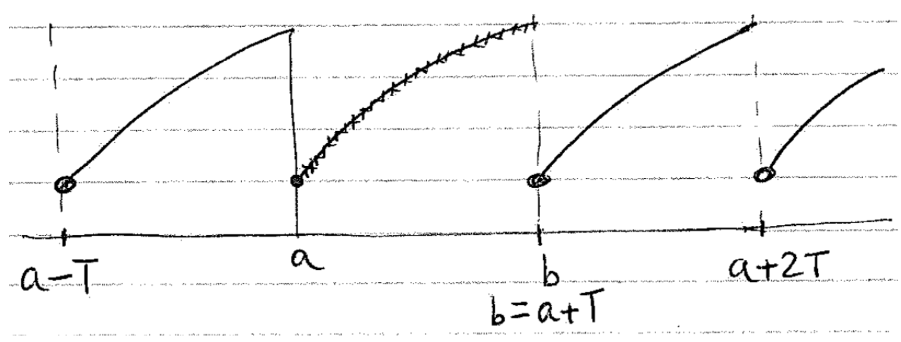
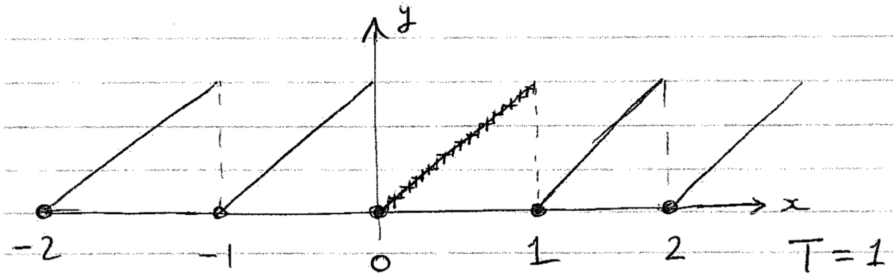


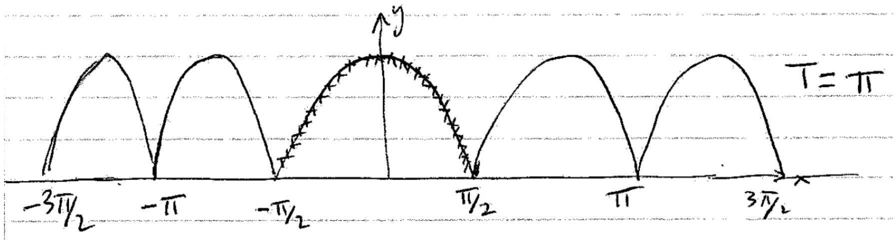
Figure 18.1: Function on $[a, b)$ extended periodically and the new function is discontinuous at $x = a + mT$ for any integer m.

Example 1 Extend $f(x) = x$ defined on $[0, 1)$ periodically.



This is known as $f(x) = [x] + x$.

Example 2 Extend periodically $f(x) = \cos(x)$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



From the examples, we see that the function can be discontinuous at some points $x = \xi$, *i.e.*

$$\lim_{x \to \xi_+} f(x) \neq \lim_{x \to \xi_-} f(x).$$

Definition

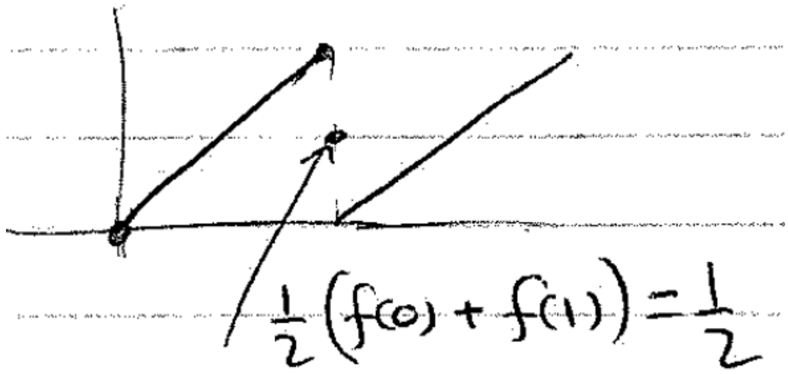
At points of discontinuity define

$$f(\xi) = \frac{1}{2}\left[f(\xi_+) + f(\xi_-)\right].$$

e.g. for $f(x) = [x] + x$, $f(n) = \frac{1}{2}$ *for all integers* n.

Conclusion: Given a function defined on a closed interval $[a, b]$, extend it periodically and at points of discontinuity prescribe the value $\frac{1}{2}\left(f(a) + f(b)\right)$.

$$e.g. \quad y = x \quad 0 \leq x \leq 1$$



18.0.1 Integrals over a period

For a periodic function $f(x)$ of period T and for arbitrary values of a, we have

$$\int_{-a}^{T-a} f(x)\mathrm{d}x = \int_0^T f(x)\mathrm{d}x.$$

In fact,

$$\int_{\alpha}^{\beta} f(x)\mathrm{d}x = \int_{\alpha+T}^{\beta+T} f(x)\mathrm{d}x.$$

Proof.

$$\underbrace{\int_{\alpha}^{\beta} f(x)\mathrm{d}x}_{\text{sub } x=y-T} = \int_{\alpha+T}^{\beta+T} f(y-T)\mathrm{d}y = \int_{\alpha+T}^{\beta+T} f(y)\mathrm{d}y = \int_{\alpha+T}^{\beta+T} f(x)\mathrm{d}x$$

□

Chapter 19

Trigonometric polynomials

Start with an oscillation $\sin(\omega x)$. Here ω is the frequency. The period is $T = \frac{2\pi}{\omega}$. This is a "pure" harmonic oscillation. Signals - *e.g.* sound, electromagnetic waves, water waves, are not pure oscillations, they contain *higher harmonics*.

Lets add another oscillation of frequency 2ω, *i.e.* $\sin(2\omega x)$, whose period is $T_2 = \frac{2\pi}{2\omega} = \frac{\pi}{\omega}$. This is called the *1st harmonic*.

Signal could be

$$S_2(x) = A_1 \sin(\omega x) + A_2 \sin(2\omega x).$$

$S_2(x)$ has period $T = \frac{2\pi}{\omega}$ overall. 1st harmonic has period $\frac{T}{2}$. Can add more and more higher frequencies and in fact can produce a wave *(oscillation)* that is a *trigonometric polynomial* defined by

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} \left[a_k \cos(k\omega x) + b_k \sin(k\omega x)\right].$$

The constant $\frac{1}{2}a_0$ is included ($\frac{1}{2}$ is useful as we will see later.)

Note: Went from $\omega_1 = \omega$ to $\omega_2 = 2\omega$ *etc. i.e.* all the frequencies have ratios that are rational. If $\frac{\omega_1}{\omega_2}$ is irrational, we get *quasi-periodic* oscillations.

19.1 Euler's relation

Useful to use Euler's relation

$$\cos(\theta) + i\sin(\theta) = e^{i\theta}$$

and since

$$\begin{aligned}
\cos(\theta) - i\sin(\theta) &= e^{-i\theta} \quad \text{(take complex conjugate)}\\
\cos(\theta) &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)\\
\sin(\theta) &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right).
\end{aligned}$$

So, can represent everything as complex and find real expressions by taking real or imaginary parts. *e.g.*

$$\frac{\mathrm{d}}{\mathrm{d}x} a e^{i\omega(x-\phi)} = a i \omega e^{i\omega(x-\phi)}.$$

Can also integrate

$$\begin{aligned}\int e^{inx}\mathrm{d}x = \int (\cos(nx) + i\sin(nx))\,\mathrm{d}x &= \left[\frac{\sin(nx)}{n} - \frac{i\cos(nx)}{n}\right] \\ &= \frac{1}{in}e^{inx}.\end{aligned}$$

19.1.1 Orthogonality

$$\int_{-\pi} \pi e^{inx}\mathrm{d}x = \begin{cases} 0 & n \neq 0 \\ 2\pi & n = 0 \end{cases}.$$

For any integers m, n we have

$$\int_{-\pi}^{\pi} e^{inx}e^{-imx} = \begin{cases} 0 & n \neq m \\ 2\pi & n = m \end{cases} \qquad \text{(easier than HW3.)}$$

19.2 Complex notation for trigonometric polynomials

Start with the polynomial (have set $\omega = 1$).

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right).$$

Use the relations found earlier

$$\begin{aligned} S_n(x) &= \frac{1}{2}a_0 + \sum_{k=1}^{n} a_k \left(\frac{e^{ikx} + e^{-ikx}}{2}\right) + b_k \left(\frac{e^{ikx} - e^{-ikx}}{2i}\right) \\ &= \frac{1}{2}a_0 + \sum_{k=1}^{n} \frac{1}{2}\left(a_k - ib_k\right)e^{ikx} + \sum_{k=1}^{n} \frac{1}{2}\left(a_k + ib_k\right)e^{-ikx}. \end{aligned}$$

Can now write this as a single complex series as follows

$$S_n(x) = \sum_{k=-n}^{n} \gamma_k e^{ikx} \tag{19.1}$$

where

$$\left.\begin{aligned} \gamma_0 &= \tfrac{1}{2}a_0 \\ \gamma_k &= \tfrac{1}{2}\left(a_k - ib_k\right) \\ \gamma_{-k} &= \tfrac{1}{2}\left(a_k + ib_k\right) \end{aligned}\right\} \; k = 1, 2, \ldots, n.$$

Notice that $\gamma_k = \gamma_{-k}^*$, (or $\gamma_k^* = \gamma_{-k}$), where $*$ denotes complex conjugate. This is *not accidental*. $S_n(x)$ in equation (19.1) is real. Hence it must equal its complex conjugate. Calculate

$$S_n(x) = \sum_{k=-n}^{n} \gamma_k e^{ikx}, \; S_n(x)^* = \sum_{k=-n}^{n} \gamma_k^* e^{-ikx}.$$

Change indexing, put $k = -l$ to find

$$\begin{aligned} S_n(x)^* = \sum_{l=n}^{-n} \gamma_{-l}^* e^{ilx} &= \sum_{l=-n}^{n} \gamma_{-l}^* e^{ilx} \\ &= \sum_{k=-n}^{n} \gamma_{-k}^* e^{ikx}. \end{aligned}$$

In the last step above, I just changed the dummy l to k. Comparing the two, we see that they are equal iff

$$\gamma_k = \gamma_{-k}^* \quad \text{i.e.} \quad \gamma_k^* = \gamma_{-k}, \quad \text{identical statements.}$$

Conversely, if we are given a complex form

$$f(x) = \sum_{k=-n}^{n} \gamma_k e^{ikx},$$

then $f(x)$ is real if and only if $\gamma_k = \gamma_{-k}^*$, *i.e.*

$$\begin{aligned} &\gamma_k + \gamma_{-k} = \gamma_k + \gamma_k^* = \text{real} \\ \text{and} \quad &\gamma_k - \gamma_{-k} = \gamma_k - \gamma_k^* = \text{pure imaginary.} \end{aligned}$$

Example 1 Take $S_n(x) = \cos(x) + \frac{1}{2}\sin(x) + 3\cos(2x)$. Express as a complex trigonometric series

$$\begin{aligned} S_n &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right) - \frac{i}{4}\left(e^{ix} - e^{-ix}\right) + \frac{3}{2}\left(e^{2ix} + e^{-2ix}\right) \\ &= \left(\frac{1}{2} - \frac{i}{4}\right) e^{ix} + \left(\frac{1}{2} + \frac{i}{4}\right) e^{-ix} + \frac{3}{2} e^{2ix} + \frac{3}{2} e^{-2ix} \\ &= \sum_{k=-2}^{2} \gamma_k e^{ikx} \end{aligned}$$

where

$$\gamma_0 = 0 \qquad \begin{aligned} \gamma_1 &= \frac{1}{2} - \frac{i}{4} \qquad & \gamma_{-1} &= \frac{1}{2} + \frac{i}{4} = \gamma_1^* \\ \gamma_2 &= \frac{3}{2} & \gamma_{-2} &= \frac{3}{2} = \gamma_2^*. \end{aligned}$$

Chapter 20

Fourier series

Consider the trigonometric polynomial

$$f(x) = S_N(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx).$$

There are $2N+1$ coefficients to determine. Use orthogonality on the interval $[-\pi, \pi]$ for $\sin(mx)$, $\cos(nx)$ etc.

Find for all n (including $n = 0$) *(See HW3)*

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x.$$

Big question is: starting with fairy arbitrary functions $f(x)$ (*e.g.* they are discontinuous), can we represent them by S_N by letting $N \to \infty$?

Orthogonality properties:
If m, n are integers, then

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\mathrm{d}x = \int_{-\pi} \pi\cos(mx)\cos(nx)\mathrm{d}x = \begin{cases} 0 & m \neq n \\ \pi & \text{if } m = n \neq 0 \end{cases}$$
$$\int_{-\pi}^{\pi} \sin(mx)\cos(nx) = 0.$$

Complex form

$$\int_{-\pi}^{\pi} e^{imx}e^{-inx}\mathrm{d}x = \begin{cases} 0 & m \neq n \\ 2\pi & m = n. \end{cases}$$

Theorem 1
The Fourier series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx))$ (or $\sum_{n=-\infty}^{\infty}\alpha_n e^{inx}$), formed by the Fourier coefficients $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\mathrm{d}x$ and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x$ converges to the value $f(x)$ for any piecewise continuous function $f(x)$ of period 2π which has piecewise continuous derivatives of first and second order.[a] At any discontinuities, the value of the function must be defined by $f(x) = \frac{1}{2}\left[f(x^+)f(x^-)\right]$.

[a]Can relax the assumption of the second derivative. It is enough to have $f'(x)$ be piecewise continuous, *i.e.* the function is piecewise smooth. If $f(x)$ is continuous, the convergence is absolute and uniform. If it is discontinuous, absolute and uniform convergence everywhere except at the discontinuity.

For the proof we will need some additional Lemmas.

20.1 Fourier series theorem, Riemann-Lebesgue Lemma

20.1.1 A trigonometric formula

We will prove the following - needed later

$$\begin{aligned} c_n(x) &= \frac{1}{2} + \cos(x) + \cos(2x) + \cdots + \cos(nx) \\ &= \frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\left(\frac{1}{2}x\right)}. \end{aligned}$$

Clearly $\frac{1}{2}x \neq 0,\ \pm\pi,\ \pm2\pi,\ \ldots$ *i.e.* $x = 0,\ \pm2\pi,\ \pm4\pi,\ \ldots$ If we define $c_n(x)$ at these points by $n+\frac{1}{2}$, then the function is continuous everywhere *(show this?)*.

Use $\cos(kx) = \frac{1}{2}\left(e^{ikx} + e^{-ikx}\right)$ to re-write

$$c_n(x) = \frac{1}{2}\sum_{k=-n}^{n} e^{ikx} = \frac{1}{2}\left(e^{-inx} + e^{ix}e^{-inx} + (e^{ix})^2 e^{-inx} + \cdots + e^{inx}\right)$$

i.e. a geometric progression with ratio $r = e^{ix} = \cos(x) + i\sin(x)$. Now $r = 1$ only if $x = 0, \pm2\pi, \ldots,$ *i.e.* the exceptional points that we excluded (treated separately). Sum it up to find

$$c_n(x) = \frac{1}{2}e^{-inx}\frac{1-\mu^{2n+1}}{1-\mu} = \frac{1}{2}\left[e^{-inx} - e^{i(n+1)x}\right]\frac{1}{1-e^{ix}}.$$

Multiply top and bottom by $e^{-\frac{1}{2}ix}$

$$\begin{aligned} \Rightarrow c_n(x) &= \frac{1}{2}\frac{\left[e^{-i(n+\frac{1}{2})x} - e^{i(n+\frac{1}{2})x}\right]}{e^{-\frac{1}{2}ix} - e^{\frac{1}{2}ix}} \\ \Rightarrow c_n(x) &= \frac{\sin\left((n+\frac{1}{2})x\right)}{2\sin\left(\frac{1}{2}x\right)}. \end{aligned}$$

Integrate from 0 to π we find

$$\int_0^{\pi} \frac{\sin\big((n+\frac{1}{2})t\big)}{2\sin\big(\frac{1}{2}t\big)}\mathrm{d}t = \int_0^{\pi}\left(\frac{1}{2} + \sum_{k=1}^{n}\cos(kt)\right)\mathrm{d}t = \frac{\pi}{2}.$$

Lemma 1 (Riemann-Lebesgue)
If the function $g(x)$ is integrable on $[a, b]$, (*e.g.* it is piecewise continuous), then

$$I_\lambda = \int_a^b g(x)\sin(\lambda x)\mathrm{d}x$$

tends to zero as $\lambda \to \infty$.

Proof. (Will do it when g' is also piecewise continuous. For the general case see HW problems).

Can use integration by parts

$$\begin{aligned} I_\lambda &= \int_a^b g(x)\sin(\lambda x)\mathrm{d}x = \left[-\frac{\cos(\lambda x)}{\lambda}g(x)\right]_a^b + \int_a^b \frac{\cos(\lambda x)}{\lambda}g'(x)\mathrm{d}x \\ &= \frac{1}{\lambda}\left[g(a)\cos(\lambda a) - g(b)\cos(\lambda b) + \int_a^b \cos(\lambda x) + g'(x)\mathrm{d}x\right] \\ \Rightarrow \quad |I_\lambda| &\le \frac{1}{\lambda}M \end{aligned}$$

for some constant M, and the result follows. □

Lemma 2

$$\int_0^{\infty}\frac{\sin(z)}{z}\mathrm{d}z = \frac{\pi}{2}.$$

Proof. Show improper integral exists, *i.e.*

$$I = \lim_{M\to\infty}\int_0^M \frac{\sin(z)}{z}\mathrm{d}z \quad \text{exists.}$$

(Note $z = 0$ is not a problem. Why?) Consider $0 < M < N$ and calculate

$$\begin{aligned} I_N - I_M = \int_M^N \frac{\sin(z)}{z}\mathrm{d}z &= -\frac{\cos(z)}{z}\bigg|_M^N - \int_M^N \frac{\cos(z)}{z^2}\mathrm{d}z \\ &= \frac{\cos(M)}{M} - \frac{\cos(N)}{N} - \int_M^N \frac{\cos(z)}{z^2}\mathrm{d}z. \end{aligned}$$

$$\Rightarrow \quad |I_N - I_M| \le \frac{1}{M} + \frac{1}{N} + \int_M^N \frac{\mathrm{d}z}{z^2} = \frac{2}{M}$$

hence convergence since $|I_N - I_M|$ can be made arbitrarily small *(Cauchy).*

In fact, letting $N \to \infty$, we see that $|I - I_M| \le \frac{2}{M}$ so I_M approaches its limit algebraically. Now take $p > 0$ arbitrary and pick $M = \lambda p$.

$$\begin{aligned} I_M = I_{\lambda p} &= \int_0^{\lambda p} \frac{\sin(z)}{z} \mathrm{d}z \overbrace{=}^{z=\lambda x} \int_0^p \frac{\sin(\lambda x)}{\lambda x} (\lambda \mathrm{d}x) \\ &= \int_0^p \frac{\sin(\lambda x)}{x} \mathrm{d}x \end{aligned}$$

where we have now *fixed* the integration range to $[0, p]$. As $M \to \infty$, $\lambda p \to \infty$ *i.e.* $\lambda \to \infty$, and by the estimate above

$$\left| I - \int_0^p \frac{\sin(\lambda x)}{x} \mathrm{d}x \right| \le \frac{2}{M} = \frac{2}{\lambda p}$$

$$\textit{i.e.} \quad \lim_{\lambda \to \infty} \int_0^p \frac{\sin(\lambda x)}{x} \mathrm{d}x = I \tag{20.1}$$

for all p sufficiently big. Cannot apply Riemann-Lebesgue directly. Consider the function

$$h(x) = \begin{cases} \frac{1}{x} - \frac{1}{2\sin(x/2)} & x \ne 0 \\ 0 & x = 0. \end{cases}$$

Fact: $h(x)$ is continuous and also has a continuous first derivative for $0 \le x < 2\pi$. *(Proof see HW5).* Now we use the Riemann-Lebesgue Lemma 1 to see that for $0 \le p < 2\pi$

$$\int_0^p \sin(\lambda x) \left(\frac{1}{x} - \frac{1}{2\sin(x/2)} \right) \mathrm{d}x \quad \to 0 \text{ as } \lambda \to 0.$$

Note: The convergence is uniform for $0 \le p \le \pi$ since $|h(x)|$ and $|h'(x)|$ are both bounded in this interval. From (20.1) we have immediately,

$$\lim_{\lambda \to \infty} \int_0^p \frac{\sin(\lambda x)}{2\sin(x/2)} \mathrm{d}x = I.$$

Pick $\lambda = n + \frac{1}{2}$ and $p = \pi$, we have shown already that

$$\int_0^\pi \frac{\sin\big((n + \frac{1}{2})x\big)}{2\sin(x/2)} \mathrm{d}x = \frac{\pi}{2}$$

independent of n. Hence we have proved:

$$I = \int_0^\infty \frac{\sin(z)}{z} \mathrm{d}z = \frac{\pi}{2}.$$

□

20.1.2 Proof of Theorem 1

Start with nth "Fourier polynomial"

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$$

and substitute the formulas for a_k, b_k, change order of summation and integration *(finite sum, so ok)*, to find

$$\begin{aligned} S_n(x) &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}(\cos(kt)\cos(kx) + \sin(kt)\sin(kx))\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{k=1}^{n}\cos(k(t-x))\right]\mathrm{d}t \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin\left[(n+\frac{1}{2})(t-x)\right]}{2\sin\left(\frac{1}{2}(t-x)\right)}\mathrm{d}t \\ \text{substitute } \xi = t-x \Rightarrow \quad &= \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} \frac{f(x+\xi)\sin\left((n+\frac{1}{2})\xi\right)}{2\sin\left(\frac{1}{2}\xi\right)}\mathrm{d}\xi \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\xi)\frac{\sin\left((n+\frac{1}{2})\xi\right)}{2\sin\left(\frac{1}{2}\xi\right)}\mathrm{d}\xi \end{aligned} \tag{20.2}$$

by using properties of integrals of periodic functions discussed earlier. Note that x is a fixed number. We will prove that *(and this proves the Theorem)*:

$$\lim_{n\to\infty} \frac{1}{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x+\xi)\frac{\sin\left((n+\frac{1}{2})\xi\right)}{2\sin\left(\frac{1}{2}\xi\right)}\mathrm{d}\xi = f(x).$$

At all points $x \in [-\pi, \pi]$, even points of discontinuity, we have

$$f(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right].$$

We have proven already that

$$\int_0^{\pi} \frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin\left(\frac{1}{2}t\right)}\mathrm{d}t = \frac{\pi}{2}$$

and by change of variables $t = t'$ we also find

$$\int_{-\pi}^{0} \frac{\sin\left((n+\frac{1}{2})t'\right)}{2\sin\left(\frac{1}{2}t'\right)}\mathrm{d}t' = \frac{\pi}{2}.$$

Hence

$$f(x) = \frac{1}{\pi}\int_0^{\pi} f(x^+)\frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin\left(\frac{1}{2}t\right)}\mathrm{d}t + \frac{1}{\pi}\int_{-\pi}^{0} f(x^-)\frac{\sin\left((n+\frac{1}{2})t\right)}{2\sin\left(\frac{1}{2}t\right)}\mathrm{d}t.$$

Using this identity gives

$$\begin{aligned} S_n(x) - f(x) =& \frac{1}{\pi}\int_0^{\pi} \left[f(x+\xi) - f(x^+)\right] \frac{\sin\left((n+\frac{1}{2})\xi\right)}{2\sin\left(\frac{1}{2}\xi\right)} \mathrm{d}\xi \\ &+ \frac{1}{\pi}\int_{-\pi}^{0} \left[f(x+\xi) - f(x^-)\right] \frac{\sin\left((n+\frac{1}{2})\xi\right)}{2\sin\left(\frac{1}{2}\xi\right)} \mathrm{d}\xi \end{aligned}$$

What is left to do is to prove the following: *(see HW)*

(i) Prove

$$\frac{f(x+\xi) - f(x^+)}{\sin\left(\frac{1}{2}\xi\right)}$$

is piecewise continuous and so is the 1st derivative, on $0 \le \xi \le \pi$.

(ii) Prove

$$\frac{f(x+\xi) - f(x^-)}{\sin\left(\frac{1}{2}\xi\right)}$$

is piecewise constant along with its 1st derivative on $-\pi \le \xi \le 0$.

Then by Riemann-Lemma, $S_n \to f(x)$ as $n \to \infty$, *i.e.* convergence *(uniform away from discontinuities).*

20.2 Examples, sine and cosine series

Will consider $f(x)$ to be 2π-periodic.

(i) If $f(x)$ is even, *i.e.* $f(-x) = f(x)$, then

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = b_n = 0.$$

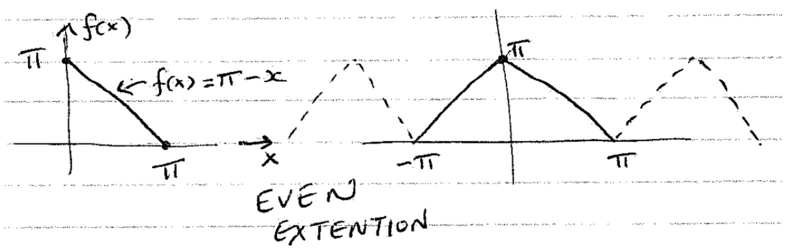
So f(x) has only a *cosine series.* If $f(x)$ is odd, $f(-x) = -f(x)$ then $a_n = 0$ and $f(x)$ has a sine series.

(ii) If a function is defined on $[0, \pi]$ by an expression $f(x)$, then it can be extended as an *even* or *odd* function on $[-\pi, \pi]$. *e.g.*

$$f(x) = \pi - x \quad 0 \le x \le \pi$$

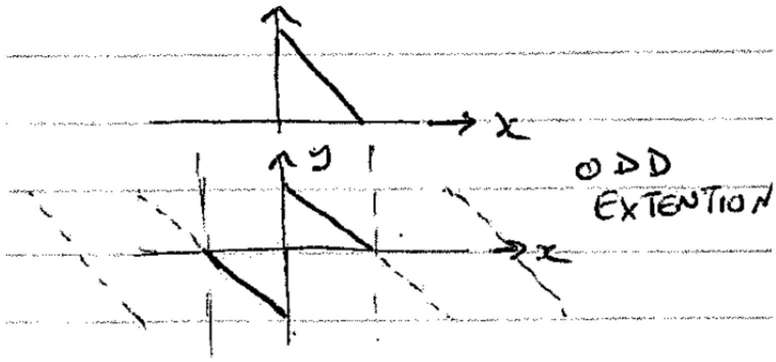
(A) Extend to an even function

$$f(x) = \begin{cases} \pi - x & 0 \le x \le \pi \\ \pi + x & -\pi \le x \le 0. \end{cases}$$



(B) Extend to an odd function, $f(-x) = -f(x)$

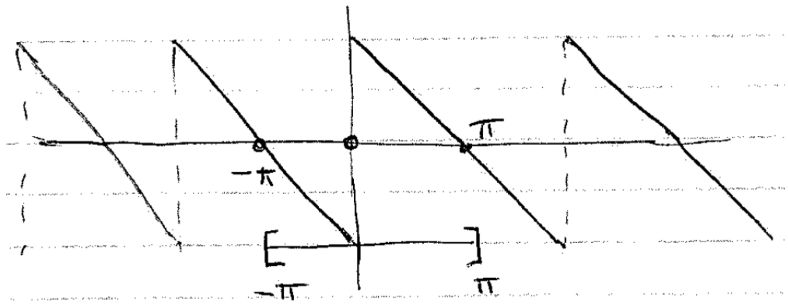
$$f(x) = \begin{cases} \pi - x & 0 < x < \pi \\ -\pi - x & -\pi < x < 0 \\ 0 & x = 0, \pi, -\pi. \end{cases}$$



Consider the second function on $[-\pi, \pi]$. It is

$$f(x) = \begin{cases} \pi - x & x > 0 \\ 0 & x = 0 \\ -\pi - x & x > 0. \end{cases} \tag{20.3}$$

Here it is periodically extended:



Aside: General result for odd functions $f(x)$:

$$a_n = 0 \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\mathrm{d}x \quad \textit{(show this!)}$$

Similarly for $f(x)$ even:

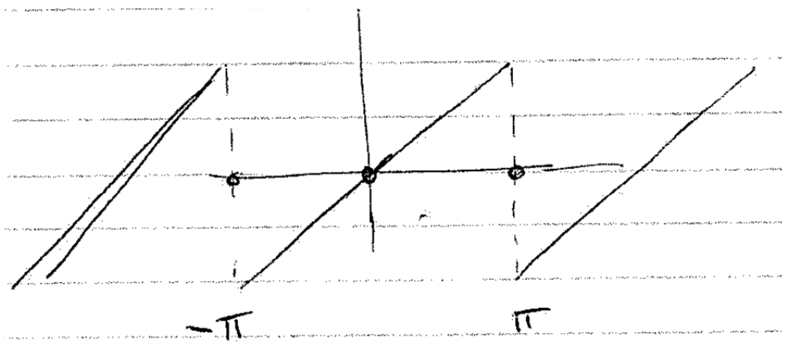
$$b_n = 0 \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\mathrm{d}x.$$

Hence for (20.3),

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi (\pi - x)\sin(nx)\mathrm{d}x \\ &= \frac{2}{\pi}\left\{\left[(\pi - x)\left(-\frac{\cos(nx)}{n}\right)\right]_0^\pi - \int_0^\pi \frac{\cos(nx)}{n}\mathrm{d}x\right\} \\ &= \frac{2}{\pi}\left\{\left[-(\pi - x)\frac{\cos(nx)}{n} - \frac{\sin(nx)}{n^2}\right]_0^\pi\right\} \\ &= \frac{2}{\pi}\left[\frac{\pi}{n}\right] = \frac{2}{n} \quad \text{convergence is uniform as long as } \epsilon < |x| \leq \pi \\ \Rightarrow \quad f(x) &= 2\left(\sin(x) + \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} + \dots\right). \end{aligned}$$

This now gives us for free the Fourier series of the function

$$\phi(x) = x \quad -\pi < x < \pi.$$



This is also odd of course, but we get $\phi(x)$ from $f(x)$ by (i) shifting the latter to the right by π (ii) reflecting about $x = 0$.

$$\begin{aligned} \Rightarrow \quad \phi(x) &= f(-(x - \pi)) = f(\pi - x) \\ &= 2\left[\sin(\pi - x) + \frac{\sin(2(\pi - x))}{2} + \frac{\sin(3(\pi - x))}{3} + \dots\right] \\ &= 2\left[+\sin(x) - \frac{\sin(2x)}{2} + \frac{\sin(3x)}{3} - \dots\right] \\ &= 2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin(kx)}{k}. \end{aligned}$$

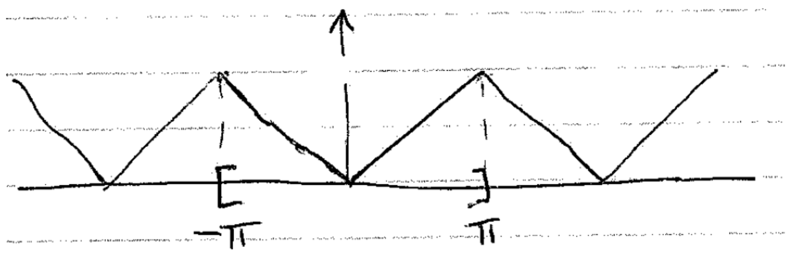
Convergence is uniform as long as $|x| < \pi - \epsilon$ for any small $\epsilon > 0$. In particular, putting $x = \frac{\pi}{2}$ we recover the Leibnitz series

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right)$$

Note: Cannot differentiate $\frac{\mathrm{d}}{\mathrm{d}x}2\sum_{k=1}^{\infty}(-1)^{k+1}\frac{\sin(kx)}{k}$ and get a convergent series. **Reason:** derivative of $\phi(x)$ does not satisfy conditions of Fourier Theorem.

Example 2:

$$f(x) = |x| \quad -\pi \leq x \leq \pi$$



Function is now continuous but has discontinuous derivatives at a set of finite points $\pm n\pi$.

$$\text{Even function} \ \Rightarrow \quad f(x) = \frac{1}{2}a_0 + \sum_{n=1} a_n \cos(nx)$$

and

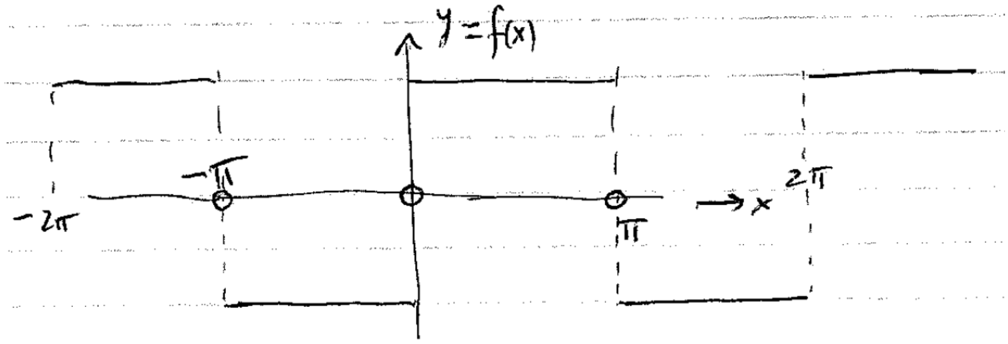
$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi x\cos(nx)\mathrm{d}x \\
&= \frac{2}{\pi}\left\{ x\left.\frac{\sin(nx)}{n}\right|_0^\pi + \left.\frac{\cos(nx)}{n^2}\right|_0^\pi \right\} \\
&= \frac{2}{\pi}(\cos(n\pi)-1)\frac{1}{n^2} \\
\Rightarrow a_n &= \begin{cases} 0 & n \text{ even } n \neq 0 \\ -\frac{4}{\pi n^2} & n \text{ odd} \end{cases} \\
a_0 &= \frac{2}{\pi}\int_0^\pi x \ \mathrm{d}x = \pi \\
\Rightarrow |x| &= \frac{1}{2}\pi - \frac{4}{\pi}\left(\cos(x) + \frac{\cos(3x)}{3^2} + \frac{\cos(5x)}{5^2} + \dots\right).
\end{aligned}$$

Convergence is uniform at all x. Put $x = 0$ we find a formula for π^2, *i.e.*

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Example 3

$$f(x) = \mathrm{sgn}(x) = \begin{cases} -1 & \text{for } -\pi < x < 0 \\ 0 & x = 0 \\ +1 & 0 < x < \pi \end{cases}$$



Clearly $f(x)$ is odd $\Rightarrow f(x) = \sum b_n \sin(nx)$

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi \sin(nx)\mathrm{d}x = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^\pi \\ &= \frac{2}{\pi}\frac{(1-\cos(n\pi))}{n} = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd} \end{cases} \\ \Rightarrow \operatorname{sgn}(x) &= \frac{4}{\pi}\left(\sin(x) + \frac{\sin(3x)}{3} + \dots\right). \end{aligned}$$

Check: function $f(x) = 0$ at $x = n\pi$, uniform convergence elsewhere. Putting $x = \frac{\pi}{2}$ again gives

$$\frac{\pi}{2} = 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right). \quad \textit{Leibnitz}$$

Note: $\frac{\mathrm{d}}{\mathrm{d}x}|x| = \operatorname{sgn}(x)$ two series agree everywhere except at discontinuities. **Question:** why can I differentiate $|x|$ series but not $\phi(x)$? Former is sectionally or piecewise continuous.

20.3 Complex form of Fourier series

Have already shown that for $f(x)$ real

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos(nx) + b_n\sin(nx)) \\ &= \sum_{n=-\infty}^{\infty}\gamma_n e^{inx}, \quad -\pi < x < \pi \end{aligned}$$

where

$$\left.\begin{aligned} \gamma_n &= \tfrac{1}{2}(a_n - ib_n) \\ \gamma_{-n} &= \tfrac{1}{2}(a_n + ib_n) \end{aligned}\right\} \text{ for } n = 1, 2, \dots$$

$$\begin{aligned} \gamma_n = \frac{1}{2}(a_n - ib_n) &= \frac{1}{2}\cdot\frac{1}{\pi}\int_{-\pi}^{\pi}(f(x)\cos(nx) - if(x)\sin(nx))\,\mathrm{d}x \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x. \end{aligned}$$

Similarly

$$\gamma_{-n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{+inx}\mathrm{d}x.$$

(Clearly $\gamma_n^* = \gamma_{-n}$ since $f(x)$ is real).

Hence,

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma)ne^{inx} \qquad -\pi < x < \pi$$

where

$$\gamma_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\mathrm{d}x \quad n = 0, \pm1, \pm2, \ldots$$

Note: If the period is $2L$ instead of 2π

$$f(x) = \sum_{-\infty}^{\infty} \gamma_n e^{in\pi x/L}|x| < L, \; \gamma_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\mathrm{d}x \quad n = 0, \pm1, \pm2, \ldots$$

20.4 Fourier series on $2L$−periodic domains

The set of functions

$$\frac{1}{\sqrt{2L}}, \quad \frac{1}{\sqrt{L}}\cos\left(\frac{n\pi x}{L}\right), \quad \frac{1}{\sqrt{L}} \qquad n = 1, 2, \ldots$$

are orthonormal on $[-L, L]$ (and in fact on any interval $[a, a+2L]$ since the function is periodic). In addition,

$$\int_{-L}^{L}\sin\Big(\frac{n\pi x}{L}\Big)\sin\Big(\frac{m\pi x}{L}\Big)\mathrm{d} = \int_{-L}^{L}\cos\Big(\frac{n\pi x}{L}\Big)\cos\Big(\frac{m\pi x}{L}\Big)\mathrm{d}x = \begin{cases} L & m = n \\ 0 & m \neq n \end{cases}$$

and $\displaystyle\int_{-L}^{L}\sin\Big(\frac{n\pi x}{L}\Big)\cos\Big(\frac{m\pi x}{L}\Big)\mathrm{d}x = 0$

$$\Rightarrow \; f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\Big[a_n\cos\Big(\frac{n\pi x}{L}\Big) + b_n\sin\Big(\frac{n\pi x}{L}\Big)\Big]$$

where $\displaystyle a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\Big(\frac{n\pi x}{L}\Big)\mathrm{d}x$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\Big(\frac{n\pi x}{L}\Big)\mathrm{d}x.$$

The complex form is

$$f(x) = \sum_{n=-\infty}^{\infty}\gamma_n e^{in\pi x/L} \quad |x| \leq L \tag{20.4}$$

$$\gamma_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\mathrm{d}x \quad n = 0, \pm1, \pm2, \ldots \tag{20.5}$$

20.5 Parseval's theorem

If $f(x)$ is represented by its Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx), \quad -\pi \leq x \leq \pi$$

then we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2 \mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2).$$

Proof. Easier to use complex notation

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{-inx} \quad \text{where} \quad \begin{aligned} \gamma_n &= \frac{1}{2}(a_n - ib_n) \\ \gamma_{-n} &= \frac{1}{2}(a_n + ib_n) = \gamma_n^* \\ \gamma_0 &= \frac{1}{2}a_0 \end{aligned}$$

$$(f(x))^2 = \left(\sum_{n=-\infty}^{\infty} \gamma_n e^{-inx}\right)\left(\sum_{m=-\infty}^{\infty} \gamma_m e^{-imx}\right).$$

Integrate and use orthogonality - see earlier

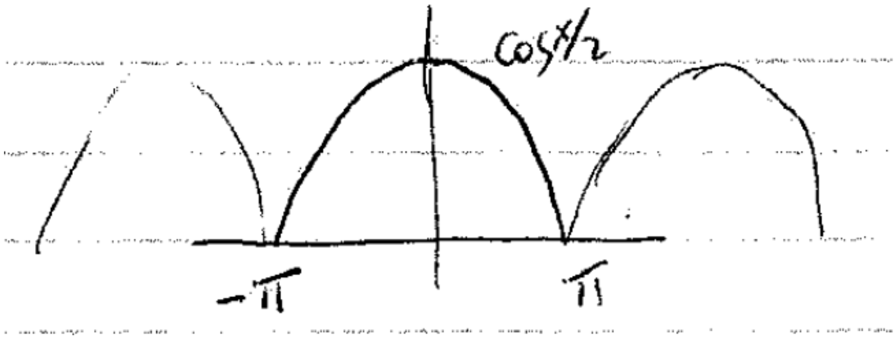
$$\int_{-\pi}^{\pi} [f(x)]^2 \, \mathrm{d}x = 2\pi \sum_{-\infty}^{\infty} \gamma_n \gamma_{-n} = 2\pi \sum_{-\infty}^{\infty} |\gamma_n|^2$$

$$\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 \mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$$

as needed. □

Example 4 Compute the Fourier series of $\cos(x/2)$ over $(-\pi,\pi]$. Use Parseval's theorem to deduce the value of

$$\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}.$$



Function is even $\Rightarrow$

$$\begin{aligned}
\cos\Big(\frac{x}{2}\Big) &= \frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos(nx) \qquad -\pi\le x\le\pi\\
a_0 &= \frac{2}{\pi}\int_{-\pi}^{\pi}\cos\Big(\frac{x}{2}\Big)\mathrm{d}x=\frac{4}{\pi}\\
a_n &= \frac{2}{\pi}\int_0^{\pi}\cos\Big(\frac{x}{2}\Big)\cos(nx)\mathrm{d}x\\
&= \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\left[\cos\Big((n+\frac{1}{2})x\Big)+\cos\Big((n-\frac{1}{2})x\Big)\right]\mathrm{d}x\\
&= \frac{1}{\pi}\left[\frac{\sin\big((n+\frac{1}{2})\pi\big)}{n+\frac{1}{2}}+\frac{\sin\big((n-\frac{1}{2})\pi\big)}{n-\frac{1}{2}}\right]\\
&= \frac{1}{\pi}\left[\frac{\cos(n\pi)}{n+\frac{1}{2}}-\frac{\cos(n\pi)}{n-\frac{1}{2}}\right]=\frac{(-1)^n}{\pi}\left[\frac{2}{2n+1}-\frac{2}{2n-1}\right]\\
&= \frac{(-1)^n}{\pi}\frac{-4}{4n^2-1}.
\end{aligned}$$

By Parsevel's theorem

$$\begin{aligned}
\frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(\frac{x}{2})\mathrm{d}x &= \frac{1}{2}a_0^2+\sum_{n=1}^{\infty}a_n^2\\
&= \frac{8}{\pi^2}+\frac{16}{\pi^2}\sum_{1}^{\infty}\frac{1}{(4n^2-1)^2}
\end{aligned}$$

$$LHS=1 \quad\Rightarrow\quad \sum_{n=1}^{\infty}\frac{1}{(4n^2-1)^2}=\frac{\pi^2-8}{16}.$$

20.6 Fourier transforms as limits of Fourier series

We discussed 2π periodic functions in detail. Consider now $f(x)$ periodic on $[-L, L]$ with L arbitrary. We have shown that

$$f(x) = \sum_{n=-\infty}^{\infty} \gamma_n e^{in\pi x/L} \quad -L \le x \le L$$

$$\text{where} \quad \gamma_n = \frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\pi t/L}\mathrm{d}t \quad n = 0, \pm 1, \pm 2, \ldots$$

Put γ_n into the sum to find

$$f(x) = \sum_{n=-\infty}^{\infty} \left\{\frac{1}{2L}\int_{-L}^{L} f(t)e^{-in\pi t/L}\mathrm{d}t\right\} e^{in\pi x/L}.$$

This is exact, we want to send $L \to \infty$.

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} h\left(\int_{-L}^{L} f(t)e^{-inht}\mathrm{d}t\right) e^{inhx}$$

where $h = \frac{\pi}{L}$. In the limit $L \to \infty$, $h \to 0$ but $nh := \omega_n = \mathcal{O}(1)$.

$$f(x) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} h\left(\int_{-L}^{L} f(t)e^{-i\omega_n t}\mathrm{d}t\right) e^{i\omega_n x}.$$

This is of the form $\sum_{n=-\infty}^{\infty} G(\omega_n)h$. Now $h = \omega_{n+1} - \omega_n = (n+1)h - nh := \delta\omega$

$$\Rightarrow \quad \text{Riemann sum} \quad \sum_{n=-\infty}^{\infty} G(\omega_n)\delta\omega \to \sum_{-\infty}^{\infty} G(\omega)\mathrm{d}\omega.$$

This gives, sending $L \to \infty$,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(t)e^{i\omega t}\mathrm{d}t\right\} e^{i\omega x}\mathrm{d}\omega$$

where $f(x)$ is defined on $\mathbb{R}$.

This gives the *Fourier Transform pair*

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}\mathrm{d}k$$

$$\hat{f}(K) = \int_{-\infty}^{\infty} f(x)e^{-ikx}\mathrm{d}x$$

Very useful in many applications. You will use them a lot to solve differential equations.

APPENDIX: List of Asynchronous Recordings mapped to chapters and sections

1. Recording 1 - Chapter 1
2. Recording 2 - Chapter 2.1
3. Recording 3 - Chapter 2.2
4. Recording 4 - Chapter 2.3
5. Recording 5 - Chapter 3
6. Recording 6 - Chapter 4
7. Recording 7 - Chapter 5.1, 5.2
8. Recording 8 - Chapter 5.3, 5.4
9. Recording 9 - Chapter 5.5
10. Recording 10 - Chapter 6
11. Recording 11 - Chapter 7
12. Recording 12 - Chapter 8
13. Recording 13 - Chapter 9
14. Recording 14 - Chapter 10
15. Recording 15 - Chapter 12
16. Recording 16 - Chapter 13.1
17. Recording 17 - Chapter 13.2
18. Recording 18 - Chapter 13.3
19. Recording 19 - Chapter 13.4
20. Recording 20 - Chapter 13.4
21. Recording 21 - Chapter 13.5
22. Recording 22 - Chapter 14.1
23. Recording 23 - Chapter 14.2
24. Recording 24 - Chapter 14.3
25. Recording 25 - Chapter 14.3
26. Recording 26 - Chapter 15.1
27. Recording 27 - Chapter 15.2
28. Recording 28 - Chapter 16.1
29. Recording 29 - Chapter 16.2

30. Recording 30 - Chapter 16.3
31. Recording 31 - Chapter 17
32. Recording 32 - Chapter 18
33. Recording 33 - Chapter 19.1, 19.2
34. Recording 34 - Chapter 20.1
35. Recording 35 - Chapter 20.2
36. Recording 36 - Chapter 20.2
37. Recording 37 - Chapter 20.3, 20.4
38. Recording 38 - Chapter 20.5
39. Recording 39 - Chapter 20.1
40. Recording 40 - Chapter 20.6
41. Recording 41 - Chapter 20.7