Mathematics Year 1, Calculus and Applications I

1. In the following examples you do not need to prove things but you should state which limit properties you are using.

   (a) Using the fact that $\lim_{x\to 0}\left(\frac{\tan x}{x}\right) = 1$, find $\lim_{x\to 0}\exp\left(\frac{3x}{\tan x}\right)$.

   (b) Using the fact that $\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = 1$, find $\lim_{x\to 0}\cos\left(\frac{\pi \sin x}{4x}\right)$.

2. Find the following limits (do not use L'Hopital's rule, Binomial expansions etc.).

   (a) $\lim_{x\to 27}\frac{x^{1/3}-3}{x-27}$ (b) $\lim_{x\to 0}\frac{(3+x)^2-9}{x}$

   (c) $\lim_{x\to 1+}\frac{x(x+3)}{(x-1)(x-2)}$ (d) $\lim_{x\to 0+}\frac{(x^3-1)|x|}{x}$

   (e) $\lim_{x\to \frac{1}{2}-}\frac{2x-1}{\sqrt{(2x-1)^2}}$ (f) $\lim_{x\to\infty}\sqrt{x}\left(\sqrt{ax+b}-\sqrt{ax+b/2}\right)$, $(a, b > 0)$

3. (a) Establish the Comparison Test 2 given in the notes, using the $\varepsilon - A$ definition of the limit.

   (b) Use (a) above to find $\lim_{x\to\infty}\frac{1}{x}\sin\left(\frac{1}{x}\right)$.

4. (a) Use the $B-\delta$ definition of limits to show that if $\lim_{x\to x_0} f(x) = \infty$ and $g(x) \geq f(x)$ for x close to x_0, $x \neq x_0$, then $\lim_{x\to x_0} g(x) = \infty$.

   (b) Use (a) above to show that $\lim_{x\to 1}\frac{1+\cos^2 x}{(1-x)^2} = \infty$.

5. (a) Graph the function $y = f(x)$ where

   $$f(x) = \begin{cases} |x|/x & x \neq 0 \\ 1 & x = 0 \end{cases}$$

   What is $\lim_{x\to 0} f(x)$? Is the function continuous? (Justify your answer.)

   (b) Graph the function $y = g(x)$ where

   $$g(x) = \begin{cases} x+1 & x < 0 \\ 2x-1 & x \geq 0 \end{cases}$$

   What is $\lim_{x\to 0} g(x)$? Is the function continuous? (Justify your answer.)

   (c) Graph $y = f(x) + g(x)$ with $f(x)$ as in (a) above and $g(x)$ as in (b). Find $\lim_{x\to 0}[f(x) + g(x)]$? What can you conclude regarding the statement "*the sum of discontinuous functions can be continuous*".

   (d) Give an example of two distinct functions which are singular at $x = 0$ but whose difference is a continuous function.

6. Suppose that a function f is defined on an open interval I containing the point x_0, and that there are numbers m and K such that we have the inequality

   $$|f(x) - f(x_0) - m(x - x_0)| \leq K(x - x_0)^2 \qquad \forall\ x \in I.$$

   Prove that f is differentiable at x_0 with derivative $f'(x_0) = m$.

7. How close to 3 does x have to be to ensure that $|x^3 - 2x - 21| < \frac{1}{1000}$? Do not use your phone. The answer is not unique. Clearly 3 ± 10^{-n} with $n = 100$ works and it also does for $n = 10$. What you need to do is find an estimate for the largest interval around 3 that will give the desired smallness.

8. Let the set of rational numbers be denoted by $\mathbf{Q}$, and consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q} \\ 0 & \text{if } x \notin \mathbf{Q} \end{cases}$$

Show that $\lim_{x \to 0} f(x)$ does not exist.

[Hint: Suppose that a limit exists and look for a contradiction.]