Mathematics Year 1, Calculus and Applications I

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Problem Sheet 2

1. (a) If $f(u) = u^3 + 2$ and $g(x) = (x^2+1)^2$, what is $h = f \circ g$.
   (b) Let $f(x) = \sqrt{x}$ and $g(x) = x^3 - 5$. Find $f \circ g$ and $g \circ f$.
   (c) Write $\sqrt{x^2+1}/[2 + (1+x^2)^3]$ as a composition of simpler functions.

2. Find a formula for $\frac{d^2}{dx^2}(f \circ g)$ in terms of first and second derivatives of f and g.

3. Differentiate $\left(1 + \left(1 + \left(1 + x^2\right)^8\right)^8\right)^8$.

4. A point in the plane moves in such a way that it is always twice as far from $(0,0)$ as it is from $(0,1)$.
   (a) Find the equation describing the particle's trajectory.
   (b) At the moment when the point crosses the segment between $(0,0)$ and $(0,1)$, what is dy/dt if the parametric description of the curve is $x(t)$, $y(t)$.
   (c) Find the point(s) when $\frac{dx}{dt} = \frac{dy}{dt}$ (assume that dx/dt and dy/dt are not zero simultaneously).

5. (a) Give a rule for determining when the tangent line to a parametric curve $x = f(t)$, $y = g(t)$ is horizontal and when it is vertical.
   (b) When is the tangent line to the curve $x = t^2$, $y = t^3 - t$ horizontal and when is it vertical?
   (c) At which points is the tangent line to the curve parallel to the line $y = x$?
   (d) Sketch the curve.

6. Consider the function

$$f(x) = \begin{cases} x^n \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   where n is a positive integer.
   (a) For $n = 2$ prove that the function is differentiable for all x but $f'(x)$ is not continuous at $x = 0$.
   (b) Find the smallest n that ensures that $\frac{d^2 f}{dx^2}$ exists and is continuous at $x = 0$.

7. Find the domain where the function $f(x) = x + \sin x$ has an inverse given by $x = g(y)$. Find $g'(0)$, $g'(2\pi)$ and $g'(1 + \frac{\pi}{2})$.

8. Let $f(x) = x^{\frac{1}{\sin(x-1)}}$. How should $f(1)$ be defined in order to make f continuous.

9. Consider only values of $x \geq 0$, and let

$$f_1(x) = x - \sin x \qquad f_2(x) = -1 + \frac{x^2}{2} + \cos x$$
$$f_3(x) = -x + \frac{x^3}{3 \cdot 2} + \sin x \qquad f_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4 \cdot 3 \cdot 2} - \cos x$$
$$f_5(x) = x - \frac{x^3}{3 \cdot 2} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2} - \sin x$$

(a) Determine whether $f_1(x)$ is increasing or decreasing. Using the value of $f_1(0)$, show that $\sin x \le x$.

(b) Determine which of the other given functions are increasing or decreasing. Using the value of each function at $x = 0$, prove the following inequalities

$$x - \tfrac{x^3}{3\cdot 2} \le \sin x \le x - \tfrac{x^3}{3\cdot 2} + \tfrac{x^5}{5\cdot 4\cdot 3\cdot 2}$$
$$1 - \tfrac{x^2}{2} \le \cos x \le 1 - \tfrac{x^2}{2} + \tfrac{x^4}{4\cdot 3\cdot 2}$$

(c) Show how the above procedure can be continued to get further inequalities for $\sin x$ and $\cos x$. Give the general formula.

10. In this example we contemplate a calculus proof of the *arithmetic-geometric mean inequality* which states that

$$\frac{a+b}{2} \ge \sqrt{ab} \qquad \text{for every} \qquad a > 0,\ b > 0.$$

In other words, the arithmetic mean $(a+b)/2$ of a and b is greater than their geometric mean $\sqrt{ab}$.

(a) Prove the arithmetic-geometric mean inequality using algebra. Hint: Use the fact $(\sqrt{a} - \sqrt{b})^2 \ge 0$.

(b) Now prove it using calculus as follows: Given a number $a > 0$, find the minimum value of the function $(a + x)/\sqrt{ax}$ where $x > 0$.

11. (a) Consider a circle of radius a inscribed between two parallel lines that are at a distance $2a$ apart as shown in figure 1. A line is drawn through the point O and meets the circle at A and the top line at B as shown in the diagram. The angle the line OA makes with the horizontal is θ. The line AP is also horizontal and P is placed vertically below B.

Find the locus of the point P as a parametric equation involving the angle θ. Find also its Cartesian $x - y$ form.

(b) Now modify the problem slightly. Instead of a circle of radius a, an ellipse of semi-minor axis a and semi-major axis b is inscribed between the two horizontal lines as above. A line A at an angle θ to the horizontal is constructed as above and the point P is identified analogously.

Find the Cartesian $x - y$ form of the locus of the point P as the angle θ varies.

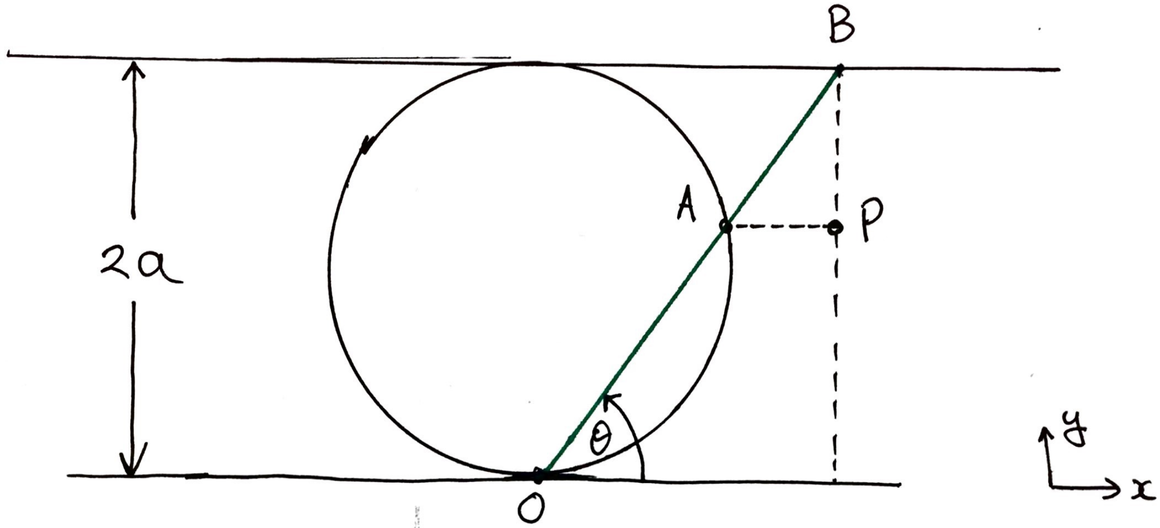

Figure 1: Sketch of the setup in problem 11.