Mathematics Year 1, Calculus and Applications I

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Problem Sheet 3

1. Sketch the functions $y = x\exp(-x)$, $y = x^2\exp(-x^2)$, $y = \frac{\exp(x)}{x}$.

2. Consider the function $f(x) = \exp(1/x)$, $x \neq 0$.

   (a) What are the limits

   $$\lim_{x\to 0+} f(x), \qquad \lim_{x\to 0-} f(x), \qquad \lim_{x\to +\infty} f(x), \qquad \lim_{x\to -\infty} f(x).$$

   (b) Now define $f(0) = 0$. Is the function differentiable?

   (c) Calculate $\lim_{x\to 0-} \frac{d^n f}{dx^n}$ for any positive integer n.

   (d) Sketch $y = f(x)$.

3. Sketch the function $y = x\exp(1/x)$.

4. Show that the equation $\mathrm{e}^x = ax$ has at least one solution for any number a, except when $0 \leq a < \mathrm{e}$.

5. Consider the function

   $$f(x) = \begin{cases} \exp(-1/x^2) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   (a) Show that $f(x)$ has a derivative at $x = 0$ and that $f'(0) = 0$.

   (b) Does f' have a derivative everywhere? If yes, what is it?

   (c) Do any further derivatives of $f(x)$ exist?

   (d) Sketch the function.

6. Find the derivative of the function $f(x) = x^x$, $x > 0$. Does the derivative at $x = 0+$ exist? Explain. Sketch the curve of $f(x)$.

7. Calculate $\frac{d}{dx}\left(x^{x^x}\right)$.

8. Is the logarithm to base 2 of an irrational number ever rational? If yes, give an example.

9. (a) Find $\lim_{a\to 0} \frac{1}{a}\log\left(\frac{e^a - 1}{a}\right)$.

   (b) Find $\lim_{a\to \infty} \frac{1}{a}\log\left(\frac{e^a - 1}{a}\right)$.

10. Find the following limits

    $$\lim_{x\to 1} x^{1/(1-x^2)} \qquad \lim_{x\to 0}(\tan x)^x$$

    $$\lim_{x\to\infty}[\log x - \log(x-1)] \qquad \lim_{x\to 1}\frac{\log x}{e^x - 1}$$

    $$\lim_{x\to 0}\frac{\cos x - 1 + x^2/2}{x^4}$$

11. Suppose that f is continuous at $x = x_0$, that $f'(x)$ exists for x in an interval about x_0, $x \neq x_0$, and that $\lim_{x\to x_0} f'(x) = m$. Prove that $f'(x_0)$ exists and equals m. [Hint. Use the mean value theorem.]

12. *[This problem is an application from fluid dynamics concerning flow in a channel with parallel walls driven by a pressure gradient parallel to the walls and the sliding motion of one boundary.]*

    Take x to be along the direction of the flow and y to be perpendicular to the channel boundaries that are located at $y = 0$ and $y = h$. A constant *transpiration* velocity V is imposed in the $y-$direction (the walls support suction/injection as can be found in control problems on airplane wings, for example), and the two-dimensional velocity field is $\boldsymbol{u} = (u(y), V)$. A constant pressure gradient $-P < 0$ acts to drive the flow, and the *Navier-Stokes* equations of fluid motion reduce to the simple problem

$$V\tfrac{du}{dy} = \tfrac{P}{\rho} + \nu\tfrac{d^2u}{dy^2}, \tag{1}$$

$$u = 0 \text{ at } y = 0, \qquad u = U \text{ at } y = h. \tag{2}$$

    Here ρ is the fluid density, ν its kinematic viscosity and U the constant sliding speed of the upper plane, and they are all constants. The boundary conditions (2) prescribe the velocity u at the two boundaries and are know as *no-slip conditions*.

    (a) *Verify* that the following expression satisfies (1)-(2)

$$u(y) = \frac{P}{\rho V}y + U\left(1 - \frac{Ph}{\rho UV}\right)\left(1 - \mathrm{e}^{Vy/\nu}\right)/(1 - \mathrm{e}^{R}), \tag{3}$$

    where $R = Vh/\nu$ is a constant called the *Reynolds number*.

    (b) Now *derive* the solution (3). Write $q(y) = du/dy$ to cast (1) into a simpler equation. Integrate to find $q(y)$ and then integrate again to find $u(y)$; use the boundary conditions (2) to fix the two constants of integration.

    (c) Starting with the solution (3) show that in the limit $V \to 0$, i.e. when the walls are impermeable, the velocity field is

$$u(y) = \frac{U}{h}y + \frac{1}{2}\frac{Ph^2}{\rho\nu}\left(\frac{y}{h} - \left(\frac{y}{h}\right)^2\right). \tag{4}$$

    (d) The flow rate or mass flux along the channel per unit depth is given by $Q = \int_0^h \rho u\, dy$. Find Q for the flow (4).
    If in turn $U = 0$ and P is a constant (e.g. the amount you can blow into a tube), what happens to Q if the channel height is halved? [This explains why it is much harder to blow up a balloon through a smaller and smaller tube!]

    (e) Now consider the case when $V \gg 1$ and such that $V \gg \nu/h$, $V \gg Ph/\rho U$. Show from (3) that the solution is zero everywhere except in a small region near $y = h$ and find an approximation of the solution in this region. [This solution is called the asymptotic suction boundary layer.]