Mathematics Year 1, Calculus and Applications I

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Problem Sheet 6

Starred problems 3, 8, 9, 15 and 16 are possible candidates for questions to be discussed in tutorials

1. Let $\{r_n\}$ denote the rational numbers in the interval $(0, 1)$ arranged in the sequence whose first few terms are $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \ldots$. Prove that the series $\sum_1^\infty r_n$ diverges.

2. Determine the convergence or divergence of the following infinite series:

$$(a)\ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} \quad (b)\ \sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!} 5^n \quad (c)\ \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} \quad (d)\ \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2} 4^n$$

$$(d)\ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}} \qquad (e)\ \sum_{n=1}^{\infty} \frac{1}{n}\left(\sqrt{n+1} - \sqrt{n}\right) \qquad (f)\ \sum_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$$

$$(g)\ \sum_{n=1}^{\infty} \frac{2^n}{(2n+1)!}, \qquad (h)\ \sum_{1}^{\infty} \frac{2^{n^2}}{n!}, \qquad (i)\ \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{\sqrt{n}}\right)$$

3. *

   (a) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots = 1.$$

   Use the result to prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, and obtain upper and lower bounds for this sum.

   (b) Find the sum of the series $\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$.

   (c) Find the sum $\sum_{n=1}^{\infty} \frac{1+n}{2^n}$. [Hint: Differentiate a certain power series, justifying any operations.]

4. Suppose that $\{a_n\}$ is a decreasing sequence of positive terms such that $\sum_{n=1}^{\infty} a_n$ converges. Prove that $na_n \to 0$ as $n \to \infty$. [Hint - consider the sum $a_{n+1} + a_{n+2} + \ldots + a_{2n}$.]

5. (a) For what values of α do the following series converge or diverge

$$(i)\ \sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}} \qquad (ii)\ \sum_{n=3}^{\infty} \frac{1}{n \log n (\log \log n)^{\alpha}}$$

   (b) Show that the following series converges

$$\sum_{n=2}^{\infty} \frac{\log(n+1) - \log n}{(\log n)^2}.$$

6. For what values $p > 0$ does the series $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n^p}\right)^n$ converge.

7. This problem follows closely the derivation in class for the power series expansion for $\log(1+x)$.

   (a) Write down the sum of the geometric series $\sum_{k=0}^{n} r^k$.

   (b) Use (a) to show that

   $$\frac{1}{1+t^2} = 1 - t^2 + t^4 - \ldots + (-1)^{n-1} t^{2n-2} + (-1)^n \frac{t^{2n}}{1+t^2}.$$

   (c) Use (b) to show that

   $$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} + R_n, \tag{1}$$

   where R_n is the remainder which you should express as an integral involving x.

   (d) Show that the power series for $\tan^{-1} x$ converges absolutely for x in the closed interval $[-1, 1]$.

   (e) Use the power series to show that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$. How many terms do we have to keep in this series in order to estimate π with accuracy to 10 decimal places, i.e. with error less than 10^{-10}?

8. * Following up from the calculation of π above, here is a much more efficient way.

   (a) Starting from the addition formula for the tangent

   $$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \, \tan y},$$

   introduce the inverse functions $x = \tan^{-1} u$ and $y = \tan^{-1} v$ to show that

   $$\tan^{-1} u + \tan^{-1} v = \tan^{-1}\left(\frac{u+v}{1-uv}\right). \tag{2}$$

   (b) Show that choosing $(u+v)/(1-uv) = 1$ in expression (2), we have the following formula for π,

   $$\frac{\pi}{4} = \tan^{-1} u + \tan^{-1} v, \tag{3}$$

   and that restricting u and v to be in the interval $(0, 1)$ we can express them as the one-parameter family

   $$u = \frac{1-p}{1+p}, \qquad v = p, \qquad 0 < p < 1, \tag{4}$$

   or equivalently

   $$u = \frac{n-m}{n+m}, \qquad v = \frac{m}{n}, \qquad 0 < m < n, \tag{5}$$

   where we picked p to be the rational number $p = m/n$.

   Use your earlier findings regarding the power series for $\tan^{-1} x$ (equation (1)) to explain why the choices (4)-(5) are useful.

(c) Hence show that (first derived and used by Euler)

$$\frac{\pi}{4} = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}. \tag{6}$$

Noting that $\frac{\frac{1}{3}+\frac{1}{7}}{1-\frac{1}{21}} = \frac{1}{2}$, show that $\tan^{-1}\frac{1}{2} = \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$, which when combined with (6) gives the formula (used by Jurij Vega, 1754-1802, a Slovenian mathematician who got 140 digits accuracy to π using this formula)

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}, \tag{7}$$

and on use of $\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{40}} = \frac{1}{3}$ and previous results we also have

$$\frac{\pi}{4} = 2\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{8}. \tag{8}$$

(d) If we use the expressions (6), (7) and (8), respectively, how many terms in the expansion (1) do we need to compute π to 10 decimals accuracy? Compare with your answer to question 8(e).

9. *

Binomial Theorem. Let $f(x) = (1+x)^s$ where s is a real number. Use induction arguments to show that $f^{(n)}(x) = s(s-1)\ldots(s-n+1)(1+x)^{s-n}$ and hence write down the Taylor series for $f(x)$ including the remainder term. Hence show that $(1+x)^s$ converges uniformly (i.e. it is analytic) for $|x| < 1$.

(b) Use the Binomial Theorem to compute $(126)^{1/3}$ and $\sqrt{96}$ to 4 decimals.

(c) Write out the Maclaurin series for $1/\sqrt{1+x^2}$ using the binomial series. What is $\frac{d^{20}}{dx^{20}}\left(\frac{1}{\sqrt{1+x^2}}\right)\Big|_{x=0}$?

(d) Find the Maclaurin series for $g(x) = \sqrt{1+x} + \sqrt{1-x}$, and hence calculate $g^{(20)}(0)$ and $g^{(2001)}(0)$.

10. Find the radius of convergence of the following series:

$$(1)\ \sum_{n=1}^{\infty}\frac{(2n)!}{(n!)^2}x^n \quad (2)\ \sum_{n=1}^{\infty}\frac{n^n}{(n!)}x^n \quad (3)\ \sum_{n=1}^{\infty}\frac{(n!)^3}{(3n)!}x^n \quad (4)\ \sum_{n=1}^{\infty}\frac{n^{5n}}{(2n)!\,n^{3n}}x^n$$

$$(5)\ \sum_{n=1}^{\infty}\frac{(3n)!}{(n!)^2}x^n \quad (6)\ \sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{2^n}x^n \quad (7)\ \sum_{n=1}^{\infty}\frac{\log n}{2^n}x^n \quad (8)\ \sum_{n=1}^{\infty}\frac{1+\cos 2\pi n}{3^n}x^n$$

$$(9)\ \sum_{n=1}^{\infty} n\,x^n \quad (10)\ \sum_{n=1}^{\infty}\frac{\sin(2\pi n)}{n!}x^n \quad (11)\ \sum_{n=1}^{\infty} n^2 x^n \quad (12)\ \sum_{n=1}^{\infty}\frac{\cos n^2}{n^n}x^n$$

$$(13)\ \sum_{n=1}^{\infty}\frac{n}{\log n}x^n \quad (14)\ \sum_{n=1}^{\infty}\frac{(-1)^n}{n!-1}x^n \quad (15)\ \sum_{n=1}^{\infty}\frac{n!}{n^n}x^n \quad (16)\ \sum_{n=1}^{\infty}\frac{(-1)^n+1}{n!}x^n$$

You may use Stirling's formula

$$n! = (2\pi n)^{1/2}\, n^n\ e^{-n}\, e^{\theta/12n}, \qquad 0 \le \theta \le 1,$$

in its appropriate form for large n.

[**Answers:** (1) 1/4, (2) $1/e$, (3) 27, (4) $4/e^2$, (5) 0, (6) 2, (7) 2, (8) 3, (9) 1, (10) ∞, (11) 1, (12) ∞, (13) 1, (14) ∞, (15) e, (16) ∞.]

11. Find the Taylor series of the function $f(x) = \int_1^x \log t\, dt$ for x near 1. Do the same for the function $x \log x$ and compare the two. What do you conclude?

12. Find the first four non-vanishing terms of the Maclaurin series for the following functions:

$$(a)\ \ x\cot x \qquad (b)\ \ e^{\sin x}, \qquad (c)\frac{\sqrt{\sin x}}{\sqrt{x}}$$
$$(d)\ \ e^{e^x}, \qquad (e)\ \ \sec x, \qquad (f)\ \ \log\sin x - \log x$$

13. Consider the function $h(x)$ defined on the interval $[-\pi, \pi]$ and given by

$$h(x) = \begin{cases} \frac{1}{x} - \frac{1}{2\sin(x/2)} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Use a Maclaurin expansion to show that $h(x)$ is continuous and has a continuous first derivative at $x = 0$.

14. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = f(x)/(1-x)$.

    (a) By multiplying the power series of $f(x)$ and $1/(1-x)$, show that $g(x) = \sum_{n=0}^{\infty} b_n x^n$, where $b_n = a_0 + \ldots + a_n$ is the nth partial sum of the series $\sum_{n=0}^{\infty} a_n$.

    (b) Suppose that the radius of convergence of $f(x)$ is greater than 1 and that $f(1) \neq 0$. Show that $\lim_{n\to\infty} b_n$ exists and is not equal to zero. What does this tell you about the radius of convergence of $g(x)$?

    (c) Let $\frac{e^x}{1-x} = \sum_{n=0}^{\infty} b_n x^n$. What is $\lim_{n\to\infty} b_n$?

15. *

    (a) Write the Maclaurin series for the functions $1/\sqrt{1-x^2}$ and $\sin^{-1} x$. For what values of x do they converge?

    (b) Find the terms up to and including x^3 in the series for $\sin^{-1}(\sin x)$ by substituting the series for $\sin x$ into the series for $\sin^{-1} x$.

    (c) Use the substitution method from part (b) to obtain the first five terms of the series for $\sin^{-1} x$ by using the relation $\sin^{-1}(\sin x) = x$ and solving for a_0 to a_5.

    (d) Find the terms up to and including x^5 of the Maclaurin series for the inverse function $g(s)$ of $f(x) = x^3 + x$. [Hint: Use the relation $g(f(x)) = x$ and solve for the coefficients in the series for g.]

16. * (This problem will guide you through an example of the use of power series to solve differential equations.)

    Consider the differential equation

$$\frac{d^2y}{dx^2} + y = 0. \tag{9}$$

    (i) Verify that $y = A\sin x + B\cos x$ where A, B are arbitrary constants, is the general solution of (9).

(ii) Look for a solution of (9) in the form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

and by equating different powers of x determine all possible values of a_n.

(iii) Use your results to (i) and (ii) to find power series expansions for $\sin x$ and $\cos x$.