Linear Algebra and Groups 1 - Concise Notes

MATH40002

Term 1 Content

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Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

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Contents

2 System of Linear Equations

2.1 Introductions

Definition 2.1.1.

Given a system of linear equations in n unknowns we can write this in matrix form as follows:

$$AX = B$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$ and $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ are column matrices, and $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$ is an $m \times\ n\ matrix$

We can also use an **Augmented Matrix** to represent the system of linear equations:

$$\left(\begin{array}{cccc|c} a_1 1 & a_1 2 & \dots & a_1 n & b_1 \\ a_2 1 & a_2 2 & \dots & a_2 n & b_2 \\ \vdots & & & & \vdots \\ a_m 1 & a_m 2 & \dots & a_m n & b_m \end{array}\right)$$

2.2 Matrix Algebra

Given $A = [a_{ij}]_{m \times n}, B = [b_{ij}]_{m \times n}$

- **Matrix Sum:** $C = A + B,\ c_{ij} = a_{ij} + b_{ij}$
- **Scalar Multiplication:** $\lambda A = [\lambda a_{ij}]$
- **Matrix Multiplication:** $A = [a_{ij}]_{p \times q}, B = [b_{ij}]_{q \times r} \Rightarrow C = AB = [c_{ij}]_{p \times r}$ where $c_{ij} = \sum_{k=1}^{q} a_{ik} b_{kj}$

Theorem 2.2.4 Associativity of Matrix Multiplication.

$$\text{Let } A, B, C \text{ be matrices, and } \alpha \in \mathbb{R} \implies (AB)C = A(BC)$$

Proof
For $A(BC)$ to be defined, we require the respective sizes of the matrices to be $m \times n$, $n \times p$, $p \times q$ in which case the product $A(BC)$ is also defined. Calculating the $(i, j)^{\text{th}}$ element of this product, we obtain,

$$[A(BC)]_{ij} = \sum_{k=1}^{n} a_{ik} [BC]_{kj} = \sum_{k=1}^{n} a_{ik})(\sum_{t=1}^{p} b_{kt} c_{tj}) = \sum_{k=1}^{n} \sum_{t=1}^{p} a_{ik} b_{kt} c_{tj}$$

If we now calculate the $(i, j)^{\text{th}}$ element of $(AB)C$ we obtain the same result:

$$[(AB)C]_{ij} = \sum_{t=1}^{p} [AB]_{it} c_{tj} = \sum_{t-1}^{p} (\sum_{k=1}^{n} a_{ik} b_{kt}) c_t j = \sum_{t=1}^{p} \sum_{k=1}^{n} a_{ik} b_{kt} c_{tj}$$

$\implies A(BC) = (AB)C$

2.3 Row Operations

Definition Elementary Row operation Are performed on an augmented matrix
There are three allowable operations:

- **Multiply** a row by any non zero number
- **Add to any row a multiple of another row**
- **Interchange** two rows

Remark 2.3.3

1. Performing row operations preserves the solutions of a linear system
2. Each row operation has an inverse row operation

Definition 2.3.5
Two systems of linear equations are equivalent if either :

1. They are both inconsistent
2. The augmented matrix of the first system can be obtained using row operations from the augmented matrix of the second system and vice versa

Remark 2.3.6
Equivalently, by Remark 2.3.3 two systems of linear equations are equivalent $\iff$ they have the same set of solutions
If a row consists of mainly 0s and 1s it becomes easier to read off the solutions to the equation

Definition 2.3.8
We say a matrix is in **echelon form**if it satisfies the following:

1. All of the zeros are at the bottom
2. The first non-zero entry in each row is 1
3. The first non-zero entry in row i is strictly to the left of the first non-zero entry in row $i+1$

We say a matrix is in **row reduced echelon form** if it is in echelon form and:

- The first non-zero entry in row i appears in column j, then every other element in column j is zero

2.4 Elementary Matrices

Definition 2.4.1
Any matrix that can be obtained from an identity matrix by means of one elementary row operation is an **elementary matrix**

There are three types of elementary matrix:

- The general form of the elementary matrix which multiplies a row by any non-zero number, α is of the form

$$E_r(\alpha) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \alpha & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

- The general form of the elementary matrix which adds a multiple of a row by any non-zero number α to another is of the form

$$E_r s(\alpha) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & \dots & \alpha & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

  where all elements of row s are multiplied by α and added to row r

- The general form of the elementary matrix which interchanges two rows is of the form E_{rs} where r and s are the rows to interchange.**I'm not typing out another bloody matrix fuck you**

Theorem 2.4.4
Let A be a $m \times n$ matrix and let E be an elementary $m \times n$. The matrix multiplication EA applies the same elementary row operation on A that was performed on the identity matrix to obtain E

2.5 More Matrices

Definition 2.5.1
We say a matrix is square if it has the same number of rows as it does columns(i.e, it's a member of $M_{n\times n}(\mathbb{F})$ for some field $\mathbb{F}$)

Definition 2.5.2
A square matrix $A = a_{ij} \in M_{n\times n}(F)$ is said to be:

1. **upper triangular** if $a_{ij} = 0$ wherever $i > j$. A has zeros for all its elements below the diagonal
2. **lower triangular** if $a_{ij} = 0$ wherever $i < j$. A has zeros for all its elements above the diagonal.
3. **diagonal** if $a_{ij} = 0$ wherever $i \neq j$. This is to say A has zeros for all its elemetns except those on the main diagonal.

Definition 2.5.4
The $n \times n$ **identity matrix** is denoted by I_n. An identity matrix has all of its diagonal entries equal to 1 and all other entries equal to 0. It is called the identity matrix because it is the multiplicative identity for $n \times n$ matrices

Definition 2.5.5
If, for a square matrix B, if there exists another square matrix B^{-1} such that $BB^{-1} = I = B^{-1}B$, then we say that B is invertible and B^{-1} **is an inverse of** B

Definition 2.5.6
A matrix without an inverse is called a **singular** matrix.

Theorem 2.5.8. The inverse of a given matrix is unique.
If $\exists A, B, C \in M_n(F)$ s.t $AB = I = CA \implies B = C$
Proof
Suppose that $AB = BA = I$ and $AC = CA = I$ then

$$B = BI = B(AC) = (BA)C = IC = C$$

This theorem shows that if a matrix A is invertible, we can talk about the inverse of A, denoted by A^{-1}. In some circumstances, we can say that a matrix is invertible, and we can find an expression for its inverse without knowing exactly what the matrix is.

Definition 2.5.10.
If $A = [a_{ij}]_{m\times n}$ then the **Transpose of A** is $A^T = [a_{ij}]_{n\times m}$

Theorem 2.5.13
Given an invertible square matrix A, then A^T is also invertible, and $(A^T)^-1 = (A^-1)^-T$

Proof
From the definition of the inverse

$$AA^{-1} = I(AA^{-1})^T = I^T(A^{-1})^T A^T = I$$
Also
$$A^{-1}A = I(A^{-1}A)^T = I^T A^T (A^{-1})^T = I$$

Equation 8 and 8 prove that $(A^{-1})^T$ is the unique inverse of A^T, as required

2.6 Inverse Row Operations

Theorem 2.6.1
Every elementary matrix is invertible and the inverse of also an elementary matrix.
Proof
Matrix multiplication can be used to check that

$$E_r(\alpha)E_r(\alpha^{-1} = E_r(\alpha^{-1})E_r(\alpha) = I$$
$$\mathrm{E}_{rs}(\alpha)E_{rs}(\alpha^{-1} = E_{rs}(\alpha^{-1})E_{rs}(\alpha) = I$$
$$\mathrm{E}_{rs}(\alpha)E_{rs}(\alpha) = I$$

Alternatively, the results can be checked by the corresponding inverses.

Theorem 2.6.2
If the square matrix A an be row reduced to an identity matrix by a sequence of elementary row operations, then A is invertible and the inverse of A is found by applying the same sequence of elementary row operations to I

Proof
Let A be a square matrix, then A can be row-reduced to I by a sequence of elementary row operations. Let $E_1, E_2, \ldots, E_r$ be the elementary matrices corresponding to the elementary row operation, so that

$$E_r \ldots E_2 E_1 A = 1$$

But Theorem 2.6.1 states that matrices representing elementary row operations are invertible. Thus the above equation can be rearranged to give

$$A^{-1} = (E_1^{-1} E_2^{-1} \ldots E_r^{-1})^{-1}$$
$$(E_r \ldots E_2 E_1) I$$

2.7 Geometric Interpretations

As you have seen in the introductory module, vectors in $\mathbb{R}^2/\mathbb{R}^3$ can be represented as points in 2 or 4 dimensional space. In this section we will at geometric interpretations of some of the things we have seen so far.

A system of linear equations in n unknowns specifies a set in n-space
Definition 2.7.4
Let T be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$ then we say T is a **linear transformation** if for every $\nu_1, \nu_2 \in \mathbb{R}^n$ and every $\alpha, \beta \in \mathbb{R}$ we have:

$$T(\alpha\nu_1 + \beta\nu_2) = \alpha T(\nu_1) + \beta T(\nu_2)$$

Proposition 2.7.4
Let $A \in M_{n\times m}(\mathbb{R})$ then it can be seen as a map from $\mathbb{R}^n$ to $\mathbb{R}^m$. A is a linear transformation.
Proof

$$A(\alpha\nu_1 + \beta\nu_2) = A(\alpha\nu_1) + A(\beta\nu_2)$$
$$= \alpha A(\nu_1) + \beta A(\nu_2)$$

By distributivity of matrix multiplication

Proposition 2.7.5
Let $A \in M_{n\times m}(\mathbb{R})$. The following are equivalent:

(i) A is invertible with inverse $A^-1 = A^T$

(ii) $A^T A = I_n = AA^T$

(iii) A preserves inner products(i.e. for all $x, y \in= \mathbb{R}^n$
$(Px) \cdot (Py) = x \cdot y)$

Proof:
(i) $\iff$ (ii) is just by definition
(ii) $\iff$ (iii)
First note that for $x, y \in \mathbb{R}^n$ $x \cdot y$ as defined in the intro to maths course is just $x^T y$ as a matrix multiplication. So A preserves inner products if and only if:

$$(Px)\cdot(Py) = x \cdot y, y \in \mathbb{R}^n$$
$$\iff (Px)^T(Py) = x^T y, y \in \mathbb{R}^n$$
$$\iff x^T P^T P y = x^T I_n y, y \in \mathbb{R}^n$$
$$\iff x^T(P^T P - I_n) y = 0, y \in \mathbb{R}^n$$

(ii) $\implies$ (iii) now trivial (iii) $\implies$ (ii): Let $x_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ i.e column vector with 0's everywhere except the $i^t h$ row where there is a 1. Then we know for each x_i $(x_i)^T(P^T P - I_n)y = 0$ so we can conclude that

$$(P^TP - I_n)y = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Similarly taking y_i to be the column vector with 0's everywhere except the i^{th} row where there is a 1 we get $(P^TP - I_N) = 0$ so $P^TP = I_n$

Definition 2.7.6

A matrix $A \in M_{n \times n}$ is called **Orthogonal** if it is such that $A^-1 = A^T$

2.8 Fields

So far, for both matrices and linear equations we have only been using entries in $\mathbb{R}$. However, we could have taken entries from any field.

Every field has distinguished elements 0 - **(additive identity)** and 1 - **(multiplicative identity)**.

Theorem 2.8.3

Let $\mathbb{F}_p = 0, 1, \ldots, p-1$, consider $\mathbb{F}_p$ with addition defined by addition modulo p and multiplication as multiplication modulo p. Then the structure $((F)_p, +_{\text{mod}p}, \times_{\text{mod}p}$ is a field)

Proof:

A 1-4 are obvious frop properties of addition in $\mathbb{Z}$

M1-3 are obvious from properties of addition in $\mathbb{Z}$

M4: inverse: obviously for $0 \leq x < p$ we have $gcd(x, p) = 1$ by Intro to Uni Maths we have:

$$\exists s, t \in \mathbb{Z} \text{ such that } 1 = sx + tp \text{ then take } x^{-1} = s(\text{mod } p)$$

D1 obvious from properties of addition in $\mathbb{Z}$

3 Vector Spaces

3.1 Introduction to Vector Spaces

Definition 3.1.1.

Let $\mathbb{Z}$ be a field. A **vector space** over $\mathbb{F}$ is a non-empty set V together with the following maps:

1. **Addition**

$$\oplus : V \times V \mapsto V$$
$$(\nu_1, \nu_2) \mapsto \nu_1 \oplus \nu_2$$

2. **Scalar multiplication**

$$\odot : F \times V \mapsto V$$
$$(f, \nu_2) \mapsto f \odot \nu_2$$

$\oplus$ and $\odot$ must satisfy and following Vector Space axioms:

For Vector Addition:

(A1) *Associative law:* $(u \oplus v) \oplus w = u \oplus (v \oplus w)$

(A2) *Commutative law*

(A3) *Additive identity:* $0_V \oplus v = v$

(A4) *Additive inverse*

For Scalar Multiplication:

(A5) *Distributive law*

(A6) *Distributive law v2*

(A7) *Associative law*

(A8) *Identity for scalar mult.*

Definition 3.1.2.

Let V be a vector space over $\mathbb{F}$:

- Elements of V are called vectors
- Elements of $\mathbb{F}$ are called scalars
- We sometimes refer to V as an $\mathbb{F}$-vector space

3.2 Subspaces

Definition 3.2.1.

A subset W of a vector space V is subspace of V if

(S1) W is not empty(i.e $e \in W$)

(S2) for $v, w \in W$, then $v \oplus w \in W$ closed under vector addition

(S3) close under scalar multiplication.

Remark 3.2.3
Any subspace of V that is not V or the zero vector space is called a **proper subspace** of V

Proposition 3.2.3.
Every subspace of an F-vector space V must contain the zero vector

Theorem 3.1.

Let U, W be subspaces of V. Then $U \cap W$ is a subspace of V. In general, the intersection of any set of subspaces of a vector space V is a subspace of V.

3.3 Spanning

Definition 3.3.1.

Let V be an $\mathbb{F}$-vector space. Let $u_1, \ldots, u_m \in$ V then:

- A **Linear Combination** of $u_1, \ldots, u_m \in$ is a vector of the form $\alpha_1 u_1 + \ldots + \alpha_m u_m$ for scalars $\alpha_1, \ldots, \alpha_m \in \mathbb{F}$. Note we can also write $\alpha_1 u_1 + \ldots + \alpha_m u_m$ as $\sum_{i=1}^{m} \alpha_i u_1$
- **span** of $u_1, \ldots, u_m \in$ is the set of linear combinations of $u_1, \ldots, u_m \in$

Lemma 3.3.2
Let V be an $\mathbb{F}$ vector space, and $u_1, \ldots, u_m \in V$ then $\text{Span}(u_1, \ldots, u_m)$ is a subspace of V.

Definition 3.3.2.

Let V an $\mathbb{F}$ vector space and suppose $S \subset V$ is such that *Span(S)* $= V$ then we say S *spans* V, or equivalently S is a *spanning set* set for

3.4 Linear Independence

Definition 3.4.1.

Let V an $\mathbb{F}$ vector space. We say $u_1, \ldots, u_m \in$ V are *linearly independent* if whenever

$$\alpha_1 u_1 + \ldots + \alpha_m u_m = 0_V$$
then it must be that
$$\alpha_1 = \cdots = \alpha_m = 0$$

We say $u_1, \ldots, u_m \in$ is a *linearly independent set*

Alternatively, a set $u_1, \ldots, u_m \in$ is *linearly dependent* if $\alpha_1 u_1 + \ldots + \alpha_m u_m = 0_V$ where at least one $\alpha_i \neq 0$ and a set is linearly independent if it is not linearly dependent.

Lemma 3.4.4 Let ${}_n u_1, \ldots, {}_n u_n$ be linearly independent in an $\mathbb{F}$-vector space V. Let v_{n+1} be such that ${}_n u_{n+1} \notin Span({}_n u_1 . \ldots , {}_n u_n)$. Then ${}_n u_1 . \ldots , {}_n u_{n+1}$ is linearly independent

3.5 Bases

Definition 3.5.1.

- Let V be and $\mathbb{F}$-vector space. A basis of V is a linearly independent spanning set of V.
- If V has a finite basis then we say V is a *finite dimensional* vector space

Proposition 3.5.4 Let V be an $\mathbb{F}$-vector space, let $S = u_1, \ldots, u_m \in \subseteq V$ Then S is a basis of $V \iff$ every vector in V has a unique expression as linear combination of elements of S

Remark 3.5.5 Let B $= u_1, \ldots, u_m \in$ be a basis for an $\mathbb{F}$-vector space V. By proposition 3.5.4 we see that we have a bijective map f from V to $\mathbb{F}^m$, for ${}_nu = \alpha_1 u_1 + \ldots + \alpha_m u_m$ we define $f({}_nu) = (\alpha_1, \ldots, \alpha_m)$ we ca;; $(\alpha_1, \ldots, \alpha_m)$ the coordinates of ${}_nu$

Proposition 3.5.6 Let V be a non-trivial(i.e. not 0) $\mathbb{F}$-Vector space and suppose V has finite spanning set S then S contains a linearly independent spanning set.

3.6 Dimensions

Lemma 3.6.1 *Steinitz Exchange Lemma*
Let B be a vector space over $\mathbb{F}$. Take $X \subseteq V$ and suppose $u \subseteq \mathit{Span}(X)$ but $u \notin \mathit{Span}(X\{_{nu})$ for some ${}_nu \in X$. Let $Y = (X\{_{nu}) \cup u$. Then $\mathit{Span}(\mathrm{X}) = \mathit{Span}(Y)$.

Theorem 3.2.

Let V be a finite dimensional vector space over $\mathbb{F}$. Let S, T be finite subsets of V. If S is LI and T spans V then $|S| \leq |T|$. That is, LI sets are at most big as spanning sets. The proof is simple but the way char* !* writes it is absolutely dogshirt.

Definition 3.6.1.

V a finite dimensional vector space.Let S, T be bases of V then S and T are both finite and $|S| = |T|$
Lemma 3.6.8 Suppose $\mathrm{dim}V = n$:

1. Any spanning set of size n is a basis
2. Any linearly independent set of size n is a basis
3. S is a spanning set $\iff$ it contains a basis(as a subset)
4. S is linearly independent $\iff$ it is contained in a basis
5. Any subset of V of size $> n$ is linearly dependent

3.7 More subspaces

Definition 3.7.1.

Let V be a vector space U and W be subspace of V.

- The *intersection* of U and W is:
  $U \cap W = \nu \in V : \nu \in W and_n u \in U$
- The *sum* of U and W is: $U + W = u + w : u \in U, W \in W$

Proposition 3.7.5
Let V be a vector space over $\mathbb{F}$. Let U and W be subspaces of V, suppose additionally:

- $U = \mathrm{Span}\{u_1, \ldots, u_s\}$
- $W = \mathrm{Span}\{w_1, \ldots, w_r\}$

Then $U + W = \mathit{Span}\{u_1 \ldots, u_s, w_1, \ldots, w_r\}$ Proof is ez

Theorem 3.3.

Let V be a vector space over $\mathbb{F}$, U and W subspaces of V. Then

$$dim(U + W) = dimU + dimW - dim(U \cup W)$$

3.8 Rank of Matrix

Definition 3.8.1.

Let A be an $m \times n$ matrix with entries from a field $\mathbb{F}$. Define:

- The *Row Space* of A ($RSp(A)$) as the span of of the rows of A. This is a subspace of $\mathbb{F}^n$
- The Row Rank of A is $dim(RSp(A))$
- The Column Space of A ($Csp(A)$) as the span of the columns of A. This is a subspace of $\mathbb{F}^m$
- The Column Rank of A is $dim(CSp(A))$

Definition 3.8.2.

For any matrix A the row rank of A is equal to the column rank of A.

Definition 3.8.3.

Let A be a matrix. The rank of A written rank(A) or $rk(A)$, is the row rank of A

Proposition 3.8.11
Let A be $n \times n$ matrix with entries in $\mathbb{F}$, then the following statements are equivalent:

1. rank(A)=n
2. The rows of A form a basis for $\mathbb{F}^n$
3. The columns of A form a basis for $\mathbb{F}^n$
4. A is invertible

4 Linear Transformation

Definition 4.0.1.

Suppose V, W are vector spaces over a field $\mathbb{F}$. Let $T : V \to W$ be a function from $VtoW$. we say:

- T preserves addition if for all $\nu_1, \nu_2 \in V$ we have $T(\nu_1 + \nu_2) = T(\nu_1) + T(\nu_2)$
- T preserves *scalar multiplicaion* if for all $\nu \in V$, $\lambda \in \mathbb{F}, T(\lambda\nu) = (\nu)$
- T is a *linear tranformation (or linear map)* if it:
    1. preserves addition
    2. preserves scalar multiplicaion

Proposition 4.1.3.
Let A be an $m \times n$ matrix over $\mathbb{F}$. Define $T : \mathbb{F}^n \to \mathbb{F}^m$, by $T(\nu) = A\nu$

Proposition 4.1.4
Basic Properties of linear transformations

Let $T : V \to W$ be a linear map. Write $0_V, 0_W$ for the zero vectors in V and W respectively, We have:

1. $T(0_V) = 0_W$
2. Suppose $\nu = \epsilon_1\nu_1 + \ldots + \epsilon_k\nu_k$ for $\epsilon_i \in \mathbb{F}, \nu_i \in V$. Then $T(\nu) = \lambda_1 T(\nu_1) + \ldots +_k T(\nu_k)$

Proposition 4.1.6
Let V and W be vector spaces over $\mathbb{F}$. Let $\nu_1, \ldots, \nu_n$ be a basis for V. Let $w_1, \ldots, w_n$ be any n vectors from W (not necessarily distinct). Then this is a unique linear transformation $T : V \to W$ such that $T(\nu_i) = w_i$ for all i.
Remark 4.1.7
This shows that once we know what a linear transformations does to a basis we know what the transformation is.

4.1 Image and Kernel

Definition 4.1.1.

Let $T : V \to W$ be a linear transformation:

- The *Image of* T is the set $ImT = T(\nu) \in W : \nu \in V \subseteq W$
- The *Kernel of* T is the set $KerT = \nu \in V : T(\nu) = 0_W \subseteq V$

Proposition 4.2.3
Let $T : V \to W$ be a linear transformation. Then:

1. ImT is a subspace of W
2. $KerT$ is a subspace of W

Proposition 4.2.5
Let $T : V \to W$ to be a linear transformation and let $\nu_1, \nu_2 \in V$. Then

$$T(\nu_1) = T(\nu_2) \iff \nu_1 - \nu_2 \in KerT$$

Proposition 4.2.6
Let $T : V \to W$ be a linear transformation. Suppose that $\nu_1, \ldots, \nu_n$ is a basis for V. Then $ImT = SpanT(\nu_1), \ldots, T(\nu_n)$
Proposition 4.2.7
Let A be an $m \times no$ matrix. Let $\mathbb{F}^n \to \mathbb{F}^m$ be given by $T(\nu) = A\nu$. Then:

1. *KerT* is the solution space to $A\nu = 0$
2. *ImT* is the column space of A
3. *dim(ImT)* $=$ *rankA*

Theorem 4.1.

The Rank Nullity theorem Let $T : V \to W$ be a linear tranformation. Then

$$dim(ImT) + dim(KerT) = dim(V)$$

Corollary 4.2.10
A system of linear equations in n unknowns with co-efficients in $\mathbb{F}$:

4.2 Representing vectors and transformations with respect to a basis

Definition 4.3.1
For $\nu \in V$ with $\nu =_1 \nu_1 + \ldots + \lambda_n \nu_n$ the vector of V wrt B is

$$[\nu]_B = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$$

Proposition 4.3.3 Let V be an n-dimensional vector space over$\mathbb{F}$ with a basis B. Then the map:

$$T : V \to \mathbb{F}^n$$
$$\mathrm{T}(\nu) = [\nu]_B$$

is a bijective linear transformation

Definition 4.2.1.

The Matrix A constructed to map $\mathbb{F}^n \to \mathbb{F}^m$ is the matrix of T with respect to B and Cm we wrute this ${}_cT_B[\nu]_B = [T\nu]_C$. If $V = W$ and $B = C$ we sometimes write this simple as $[T]_B$

Proposition 4.3.8 Let V be a vector space. Let $B = \nu_1, \ldots, \nu_n$ and $C = w_1, \ldots, w_n$ be bases for V. Then for $j \in 1, \ldots, n$ we can write $\nu_j = \lambda_{ij} + \ldots + \lambda_{nj} w_n$

Let P be the matrix $(\lambda_{ij}) = \begin{bmatrix} \lambda_{11} & \ldots & \lambda_{1n} \\ \vdots & & \vdots \\ \lambda_{n1} & \ldots & \lambda_{nn} \end{bmatrix}$

So the j^{th} column is $[\nu_j]_C$.

1. $P = [X]_C$ where $X : V \to V$ is the unique linear transformation such that $X(w_j) = \nu_j$ for all j
2. For all $\nu \in V, P[\nu]_B = [\nu_j]_C$
3. $P =_C [Id]_B$ where Id is the identity transformation of V

Definition 4.2.2.

P is the change of basis matrix from B to C.

WARNING. **THIS IS CONFUSING BECAUSE OF 1 IN PROPOSITION 4.3.8 maps basis elements to C to those of B- sometimes described the other way around**

Proposition 4.3.10
Let V, B, C, P as above. Then:

1. P is invertible, and its inverse is the change of the basis matrix from C to B
2. $T : V \to V$ be a linear transformation, Then $[T]_C = P[T]_B P^{-1}$

Remark. 4.3.12
It is a fact that if P is the change of basis matrix $_C[Id]_B$ from B to C amd Q is the change of basis matrix $_D[Id]_C$(where B,C,D and all basis for $\mathbb{F}^n$ then $QP =_D [Id]_{CC}[Id]_B =_D [Id]_B$, the change of basis matrix from B t D.
This gives us a quick method of calculating change of basis matrices for $\mathbb{F}^n$