Math40003 Linear Algebra and Groups Problem Sheet 1

1. Find all solutions of the following systems of linear equations.

(a)
$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 8 \\ 3x_1 - 6x_2 + 2x_3 &= 18 \\ x_3 - 2x_4 &= 5 \\ 2x_1 - 2x_2 + 3x_4 &= 4 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 3x_2 + x_3 &= 2 \\ 3x_1 - 8x_2 + 2x_3 &= 5 \\ 2x_1 - 5x_2 + x_3 &= 1 \end{aligned}$$

(c)
$$\begin{aligned} x_1 - 2x_3 + x_4 &= 0 \\ 2x_1 - x_2 + x_3 - 3x_4 &= 0 \\ 4x_1 - 3x_2 - x_3 - 7x_4 &= 4 \end{aligned}$$

(d)
$$\begin{aligned} -x_2 + x_3 - 3x_4 &= 0 \\ x_1 + 3x_2 + x_3 - x_4 &= 0 \\ 2x_1 + 5x_2 + 3x_3 - 5x_4 &= 0 \end{aligned}$$

In each case write down all solutions with $x_2 = 5$.

2. Which of these matrices A is invertible (and for which a)?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Calculate A^{-1} when it exists.

3. * How can you use Gaussian elimination to solve $\begin{pmatrix} 2 & 4 & 1 \\ 2 & 6 & 1 \\ 3 & 9 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$?

Find *all* solutions (x_1, x_2, x_3).

4. Let A be an $n \times m$ matrix, and $b \in \mathbb{R}^n$. Suppose $Ax = b$ has at least one solution $x_0 \in \mathbb{R}^m$. Show all solutions are of the form $x = x_0 + h$, where h solves $Ah = 0$.

5. Let A and B be square $n \times n$ matrices with real entries. For each of the following statements, either give a **proof**, or find a **counterexample with** $n = 2$.

(i) If $AB = 0$ then A and B cannot both be invertible.

(ii) If A and B are invertible then $A + B$ is invertible.

(iii) If A and B are invertible then AB is invertible.

(iv) If A and B are invertible and $(AB)^2 = A^2B^2$, then $AB = BA$.

(v) If $ABA = 0$ and B is invertible then $A^2 = 0$.

(vi) If $ABA = I$ then A is invertible and $B = (A^{-1})^2$.

(vii) If A has a left inverse B and a right inverse C then $B = C$.

6. Let $n \geq 2$ and let $A_n = (a_{ij})$ be the $n \times n$ matrix such that

$$a_{i-1,i} = 1 \text{ for } i = 2, \ldots n,$$
$$a_{i+1,i} = 1 \text{ for } i = 1, \ldots n-1,$$

and $a_{ij} = 0$ for all other i, j. Write down A_2, A_3 and A_4. Prove that A_n is invertible for all even values of n, and is not invertible for all odd values of n. Find A_2^{-1} and A_4^{-1}.

7.* For which $a, b \in \mathbb{R}$ does the system of equations

$$\begin{aligned} x_1 + x_2 + x_3 &= -1 \\ 2x_1 + x_2 + ax_3 &= 1 \\ 3x_1 + x_2 + x_3 &= b \end{aligned}$$

have (i) no solutions, (ii) exactly one solution, (iii) infinitely many solutions?

What about the system

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 - x_2 + ax_3 + x_4 &= 1 \\ 2x_1 + ax_2 + x_3 + 2x_4 &= b\ ? \end{aligned}$$

8. Which of the following are possible, find examples if possible:

(a) Two simultaneous equations in two unknowns which defines a line in $\mathbb{R}^2$.

(b) Two simultaneous equations in two unknowns which defines the empty set in $\mathbb{R}^2$.

(c) One equation in no unknowns which defines the empty set.

(d) Two simultaneous equations in three unknowns which defines a point in $\mathbb{R}^3$.