Math40003 Linear Algebra and Groups Problem Sheet 2

1. (a) Let M_θ be the reflection in the line $L_\theta = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = x_1 \tan\theta\}$. Using any school geometry or trigonometry you like, show that the matrix representing M_θ is
$$\begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$
   (b) Let R_α be a rotation about the origin, and let M_β be the reflection in a line through the origin. Prove that $M_\beta R_\alpha$ is a reflection.
   (c) Let M_α and M_β be reflections in straight lines through the origin. Prove that $M_\alpha M_\beta$ is a rotation.

2. * Let $\mathbb{R}[x]$ be the set of all polynomials with variable x and real coefficients, with standard addition and scalar multiplication. Show that this is a vector space over $\mathbb{R}$.

3. Decide for each of the following sets, whether it is a vector space with the indicated operations of addition and scalar multiplication:
   (a) The set $\mathbb{R}^2$, with the usual addition but with scalar multiplication defined by
$$r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ry \\ rx \end{pmatrix}.$$
   (b) The set $\mathbb{R}^2$, with the usual scalar multiplication but with addition defined by
$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} y+s \\ x+r \end{pmatrix}.$$
   (c) The set $\mathbb{R}^2$, with addition and scalar multiplication defined by
$$\begin{pmatrix} x \\ y \end{pmatrix} \oplus \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x+a+1 \\ y+b \end{pmatrix} \quad \text{and} \quad r \odot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rx+r-1 \\ ry \end{pmatrix}.$$

4. Let F be a field. Show every F-vector space V with additive identity 0_V has the following properties:
   (a) The vector 0_V is the unique vector satisfying the equation $0_V \oplus v = v$ for all vectors v in V.
   (b) Let 0 be the additive identity in F. Then $0 \odot v = 0_V$ for all vectors v in V.

5. Describe all subspaces of $\mathbb{R}^3$.

6. Let U, W be subspaces of a vector space V over F. Show that $U \cup W$ is a subspace of V iff either $U \subseteq W$ or $W \subseteq U$.