Math40003 Linear Algebra and Groups — Problem Sheet 3

1. (a) Which of these sets of vectors are linearly independent? Which span $\mathbb{R}^3$?

   i. $(5,3,0)$, $(2,1,1)$
   ii. $(1,0,1)$, $(-1,1,0)$, $(0,1,1)$
   iii. $(1,3,1)$, $(2,1,1)$, $(-1,7,-5)$
   iv. $(1,-3,2)$, $(2,-1,1)$, $(2,-5,4)$, $(1,2,5)$

   (b) For which a, b, c are the vectors $(1,3,1)$, $(2,1,1)$, (a,b,c) linearly dependent?

2. *Let V be a finite-dimensional vector space. For each of the following statements, say whether it is true or false. If it is true, give a justification; otherwise find a counterexample.

   (a) If $\{v_1, \ldots, v_n\}$ is a basis, for V, and $\{x_1, \ldots, x_r\}$ is a linearly independent subset of V with $r < n$, and if $v_i \notin Span\{x_1, \ldots, x_r\}$ for all $i = 1, \ldots, n$, then $\{x_1, \ldots, x_r, v_{r+1}, \ldots, v_n\}$ is a basis for V.

   (b) If U is a subspace of V, then $U + U = U$.

   (c) If U and W are subspaces of V, and $\dim U + \dim W = \dim V$, then $U \cap W = \{0_V\}$.

   (d) If $\dim V = n$ and $v_1 \in V$, then there exist vectors $v_2, \ldots, v_n$ in V such that $\{v_1, \ldots, v_n\}$ spans V.

   (e) If W is a subspace of V , then $\dim W \leq \dim V$ and $\dim W = \dim V$ if and only if $W = V$.

3. Which of the following sets of vectors in $\mathbb{R}^4$ are linearly independent? Extend those which are linearly independent to a basis of $\mathbb{R}^4$.

   (a) $(1,2,3,0)$, $(-1,2,3,0)$
   (b) $(1,2,3,0)$, $(-1,2,3,0)$, $(0,1,2,3)$
   (c) $(1,1,-1,-1)$, $(1,-1,1,-1)$, $(-1,1,1,-1)$, $(0,1,2,-3)$

4. Let $V = \mathbb{R}^{\mathbb{R}}$ (the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$).

   (a) Show that the functions

   $$f_1(x) = 1, \quad f_2(x) = 1 + x + x^2, \quad f_3(x) = \sin x, \quad f_4(x) = \cos x$$

   are linearly independent.

   (b) Which of the following functions lie in $Span(f_1, f_2, f_3, f_4)$?

   $$5 - 3x - 3x^2, \quad \tan x, \quad 10 - x - x^2 + \sin(x + \pi/3).$$

5. (a) Write down an infinite number of different bases of $\mathbb{R}^2$ (in finite time).

   (b) Find a basis for $W = Span(x^2 - 1,\ x^2 + 1,\ 4,\ 2x - 1,\ 2x + 1) \leq \mathbb{R}[x]$.

   Recall: $\mathbb{R}[x]$ is the set of real polynomials in the variable x

6. Let V be the vector space of all 3×3 matrices over $\mathbb{R}$.

(i) Find a basis of V consisting of invertible matrices.

(ii) Let $W = \{A \in V\colon A^t = A\}$. Show $W \leq V$ and compute $\dim W$.

(iii) Let $W \subset V$ be the set of matrices whose columns, rows, and both diagonals add to 0. Show $W \leq V$ and find a basis for W.