

- 1.* (a) Which of the following functions $T : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ are linear transformations?
- $T(x_1, x_2, x_3) = (x_1 + x_2 - x_3, 2x_1 + x_2)$
 - $T(x_1, x_2, x_3) = (0, \sqrt{2}x_3)$
 - $T(x_1, x_2, x_3) = (x_1x_2, x_3)$
- (b) Let V be the vector space of all 2×2 matrices over \mathbb{R} . Which of the following functions $T : V \rightarrow V$ are linear transformations?
- $T(A) = A^2$ for all $A \in V$
 - $T(A) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} A$ for all $A \in V$
- (c)
 - Find a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ which sends $(1, 0)$ to $(1, 1, 0)$ and $(1, 1)$ to $(1, 0, -1)$.
 - Find two different linear transformations $\mathbb{R}^3 \rightarrow \mathbb{R}^2$ which send $(1, 1, 0)$ to $(1, 1)$ and $(0, 1, 1)$ to $(0, 1)$.
- (d) Let V be the vector space (over \mathbb{R}) of all functions $f : \mathbb{R} \rightarrow \mathbb{R}$. Which of the following are linear transformations (thinking of \mathbb{R} as \mathbb{R}^1 in parts (i) and (iii))?
- $T_1 : V \rightarrow \mathbb{R}$ where $T_1(f) = f(1)$ (for $f \in V$).
 - $T_2 : V \rightarrow V$ where $T_2(f) = f \circ f$ (for $f \in V$).
 - $T_3 : \mathbb{R} \rightarrow V$ where $T_3(\mu)$ is the function $f_\mu \in V$ given by $f_\mu(x) = \mu x$ (for $\mu, x \in \mathbb{R}$).
2. (a) Give an example of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for exactly one vector $v \in \mathbb{R}^2$.
- (b) Give an example of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for no vector $v \in \mathbb{R}^2$.
- (c) Give an example of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for infinitely many vectors $v \in \mathbb{R}^2$.
- (d) Show that there is no linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ such that $T(v) = (1, 0, 0)$ for exactly two vectors $v \in \mathbb{R}^2$.
3. (Harder) (a) Suppose V, W are vector spaces (over a field F) and $S, T : V \rightarrow W$ are linear transformations. Prove that $S + T : V \rightarrow W$ defined by $(S + T)(v) = S(v) + T(v)$ (for $v \in V$) is a linear transformation. If $\lambda \in F$, show that $\lambda S : V \rightarrow W$ defined by $(\lambda S)(v) = \lambda S(v)$ (for $v \in V$) is a linear transformation. Explain why the set U of all linear transformations from V to W is a vector space with these operations.
- (b) In the case where $V = F^2$ and $W = F^3$, what is the dimension of the vector space U ? What is the dimension of U for arbitrary finite dimensional vector spaces V and W ?

4. (a) Define $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ by $T(x_1, x_2, x_3) = (x_1 - x_2, x_2 - x_3, x_3 - x_1)$. Find bases of $\text{Ker } T$ and $\text{Im } T$. For which values of k is the vector $(1, 3, k)$ in $\text{Ker } T$ or $\text{Im } T$?
- (b) Let V be the vector space of polynomials of degree at most 2 over \mathbb{R} . Define $T : V \rightarrow V$ by

$$T(ax^2 + bx + c) = (a + b + c)x^2 + (c - a)x + (a + 3b + 5c).$$

Find bases of $\text{Ker } T$ and $\text{Im } T$.

- (c) Let V be as in 4b, and define $S : V \rightarrow V$ by

$$S(p(x)) = p(1 + x) - p(x) \text{ for } p(x) \in V.$$

(So for example, $S(x^2) = (x + 1)^2 - x^2 = 2x + 1$.) Show that S is a linear transformation, and find bases of $\text{Ker } S$ and $\text{Im } S$.

5. (a) Let V be a finite-dimensional vector space, and $T : V \rightarrow V$ a linear transformation.
- i. Prove that T is injective if and only if $\text{Ker } T = \{0\}$.
 - ii. Prove that T is surjective if and only if $\text{Ker } T = \{0\}$.
- (b) Find an example of a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $\text{Ker } T = \text{Im } T$.
- (c) Prove that there does not exist a linear transformation $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ such that $\text{Ker } T = \text{Im } T$.