Math40003 Linear Algebra and Groups — Problem Sheet 6

1. Let $\mathcal{B} = \{(1,3),(1,2)\} \subseteq \mathbb{R}^2$.
   (a) Show that $\mathcal{B}$ is a basis of $\mathbb{R}^2$.
   (b) Compute the basis change matrix form $\mathcal{B}$ to the canonical basis of $\mathbb{R}^2$ $(\{(1,0)^t,(0,1)^t\})$.
   (c) Compute the basis change matrix from the canonical basis to $\mathcal{B}$.

2. Let
$$\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$
   (a) Show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.
   (b) Compute the basis change matrix from the canonical basis of $\mathbb{R}^3$ to $\mathcal{B}$.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by
$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y \\ x+z \\ x+z \end{pmatrix}$$
   Write the matrix ${}_{\mathcal{B}}[T]_{\mathcal{B}}$ where $\mathcal{B}$ is the basis
$$\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \subseteq \mathbb{R}^3.$$

4. Let V be a vector space of dimension 2 (over an arbitrary field containing $\mathbb{Q}$) and let
$$\mathcal{B} = \{v_1, v_2\} \qquad \mathcal{B}' = \{v'_1, v'_2\}$$
   be two bases of V such that
$$v_1 = 6v'_1 - 2v'_2 \qquad v_2 = 9v'_1 - 4v'_2.$$
   (a) Compute the basis change matrix from $\mathcal{B}$ to $\mathcal{B}'$.
   (b) Compute the coordinates of $-3v_1 + 3v_2$ with respect to $\mathcal{B}'$.
   (c) Why did we assume that the base field contained $\mathbb{Q}$? What happens if we try and answer the same questions with base field $\mathbb{Z}/2\mathbb{Z}$?)

5. Let A be a square matrix of dimension n over an arbitrary field. Assume that there exists $m \in \mathbb{N}$ such that A^m is the zero matrix. Show that $\mathrm{I}_n + A$ is invertible.