IMPERIAL COLLEGE LONDON
DEPARTMENT OF MATHEMATICS

Question Sheet 1

MATH40003 Linear Algebra and Groups — Term 2, 2020/21

All questions can be attempted before the end of week 2. We can discuss some of them in the Q and A sessions in week 2. Some of the questions are meant to be easier than others. If you can do the easy ones without much effort, skip to the harder ones. Solutions will be released on Friday of week 2.

Question 1 Compute the determinants of the following matrices, assuming that the entries are from the field $\mathbb{R}$. Which matrices are invertible (and for which a)?

$$\begin{pmatrix} 6 & 7 \\ 8 & 9 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & -1 \\ 2 & -3 & 2 \\ -1 & 12 & -7 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 4 \\ 3 & 2 & 5 \\ 0 & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ a & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Question 2 (i) Factorise $\det \begin{pmatrix} x & y & x \\ y & x & x \\ x & x & y \end{pmatrix}$. (ii) Solve $\det \begin{pmatrix} t-1 & 3 & -3 \\ -3 & t+5 & -3 \\ -6 & 6 & t-4 \end{pmatrix} = 0$ for $t \in \mathbb{R}$.

Question 3 Calculate the determinant of the matrix $\begin{pmatrix} 6 & 2 & 1 & 0 & 5 \\ 2 & 1 & 1 & -2 & 1 \\ 1 & 1 & 2 & -2 & 3 \\ 3 & 0 & 2 & 3 & -1 \\ -1 & -1 & -3 & 4 & 2 \end{pmatrix}$.

How would you check your answer?

Question 4 For a real number α define

$$A(\alpha) = \begin{pmatrix} 1 & \alpha & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & \alpha & 1 & -1 \\ -1 & \alpha & 1 & 1 \end{pmatrix} \in M_4(\mathbb{R}).$$

(a) Find the determinant of $A(\alpha)$.
(b) Find a value α_0 of α such that the system $A(\alpha_0)x = 0$ has a nonzero solution for $x \in \mathbb{R}^4$.
(c) Prove that when $\alpha < \alpha_0$, there is no real 4×4 matrix B such that $B^2 = A(\alpha)$.

Question 5 Let $A, B \in M_n(F)$ (where F is an arbitrary field). Prove that $|A| = 0$ or $|B| = 0$ if and only if $|AB| = 0$.
Note: you may NOT assume the result $|AB| = |A|\,|B|$ from lectures as this question is supposed to be part of the proof of that result. You MAY assume the result that says a matrix is invertible iff it has nonzero determinant.

Question 6 Let B_n be the $n \times n$ matrix

$$\begin{pmatrix} -2 & 4 & 0 & 0 & \dots & 0 & 0 & 0 \\ 1 & -2 & 4 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & -2 & 4 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & 1 & -2 & 4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 \end{pmatrix}$$

(a) Prove that if $n \geq 4$, then $|B_n| = 8|B_{n-3}|$.
(b) Prove that $|B_n| = 0$ if $n = 3k - 1$ (where k is a positive integer).
(c) Find $|B_n|$ if $n = 3k$ or $3k + 1$.

Question 7 The *trace* $\mathrm{tr}(A)$ of an $n \times n$ matrix $A = (a_{ij})$ over a field F is defined to be the sum of the diagonal entries of A: $\mathrm{tr}(A) = \sum_{i=1}^{n} a_{ii}$. Prove that, for $n \times n$ matrices A, B (over F) we have: (a) $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$; (b) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

Suppose V is an n-dimensional F-vector-space with a basis C and $T : V \to V$ is a linear transformation. We define the trace of T to be $\mathrm{tr}([T]_C)$. Prove that this does not depend on the choice of basis C.

Question 8 If A is an $n \times n$ matrix over a field F, the *characteristic polynomial* of A is $\det(xI_n - A)$. Prove that this is a polynomial of degree n over F and: (i) the coefficient of x^n is 1; (ii) the constant term is $(-1)^n \det(A)$; (iii) the coefficient of x^{n-1} is $-\mathrm{tr}(A)$.

Question 9 Let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

where the a_i are in the field F. Prove that the characteristic polynomial of A is
$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. (*Hint:* Try induction.)