Imperial College London
Department of Mathematics

Question Sheet 3

MATH40003 Linear Algebra and Groups — Term 2, 2020/21

This is the problem sheet for the problem classes in week 4. Solutions will be released after the problem classes at the end of week 4.

Question 1 For each of the following matrices $A \in M_3(\mathbb{R})$, find the eigenvalues and eigenvectors. Then diagonalise A, or prove it cannot be diagonalised.

(i) $\begin{pmatrix} -1 & -2 \\ 4 & 5 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ (iii) $\begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 1 \end{pmatrix}$.

Question 2 For which values of c is the matrix $\begin{pmatrix} 1-2c & 4c & -c \\ -c & 2c+1 & -c \\ 0 & 0 & -1 \end{pmatrix} \in M_3(\mathbb{R})$ diagonalisable?

Question 3 Let $A = \begin{pmatrix} -10 & -18 \\ 9 & 17 \end{pmatrix} \in M_2(\mathbb{R})$.

(a) Find an invertible 2×2 matrix P such that $P^{-1}AP$ is diagonal.
(b) Find A^n, where n is an arbitrary positive integer.
(c) Find a matrix $B \in M_2(\mathbb{R})$ such that $B^3 = A$.
(d) Find a matrix $C \in M_2(\mathbb{C})$ such that $C^2 = A$.
(e) Prove that there is no $C \in M_2(\mathbb{R})$ such that $C^2 = A$.

Question 4 Suppose V is a vector space over a field F and $T : V \to V$ is linear. If $\lambda \in F$, let $E_\lambda = \{v \in V : T(v) = \lambda v\}$. Prove that this is a subspace of V and λ is an eigenvalue of T if and only if $E_\lambda \neq \{0\}$.

Question 5 For each of the linear maps θ_i below, write down the matrix representing θ_i with respect to the standard basis. Hence find the eigenvalues of θ_i and for each eigenvalue λ, find the eigenspace E_λ. Determine whether θ_i is diagonalizable.

i) $\theta_1 : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$\theta_1 : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} c-b \\ a-c \\ c \end{pmatrix}.$$

ii) $\theta_2 : \mathbb{C}^3 \to \mathbb{C}^3$ given by

$$\theta_2 : \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} c-b \\ a-c \\ c \end{pmatrix}.$$

Question 6 For each of the linear maps T in Question 2 of Sheet 2, compute the eigenvalues and eigenvectors of T and determine whether or not T is diagonalisable.

Question 7 As in Question 9 of Sheet 1, let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

where the a_i are in the field F. Let $e_1, \dots, e_n$ be the standard basis of F^n.

i) Prove that F^n is spanned by the vectors $e_1, Ae_1, \dots, A^{n-1}e_1$. What is $A^n e_1$ as a linear combination of these?

ii) Show that for every $v \in F^n$ there is a polynomial $q(x)$ (over F) of degree at most $n-1$ such that $v = q(A)e_1$ (where $q(A)$ is the result of substituting A for x into the polynomial q).

iii) Deduce that $\chi_A(A)$ is the zero matrix (this is a special case of the Cayley - Hamilton Theorem).

Question 8 In this question you can use Q7. Unless stated otherwise, you can choose which field to use.

(a) Find a 3×3 matrix which has characteristic polynomial $x^3 - 7x^2 + 2x - 3$.
(b) Find a 3×3 matrix A such that $A^3 - 2A^2 = I_3$.
(c) Find a 4×4 invertible matrix B such that $B^{-1} = B^3 + I_4$.
(d) Find a 5×5 invertible matrix B such that $B^{-1} = B^3 + I_5$.
(e) Find a real 4×4 matrix C such that $C^2 + C + I_4 = 0$.
(f) For each $n \geq 2$ find an $n \times n$ matrix D such that $C^n = I_n$ but $C \neq I_n$.