IMPERIAL COLLEGE LONDON
DEPARTMENT OF MATHEMATICS

Question Sheet 4

MATH40003 Linear Algebra and Groups — Term 2, 2020/21

This is the problem sheet for the problem classes in week 5. Solutions will be released on Friday of week 5, after the classes.

Question 1 The Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by $F_0 = 0$, $F_1 = 1$ and the recurrence relation $F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Find a matrix $A \in M_2(\mathbb{R})$ with the property that

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}.$$

Compute the eigenvalues and eigenvectors of A and express $(1, 0)^T$ as a linear combination of eigenvectors. Hence, or otherwise, find a general expression for F_n (in terms of n).

Question 2 Suppose $A \in M_n(F)$ has characteristic polynomial

$$\chi_A(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n.$$

The *Cayley - Hamilton Theorem* states that

$$\chi_A(A) = a_0 I_n + a_1 A + \ldots + a_{n-1}A^{n-1} + A^n = 0.$$

(A special case of this was given on Question 7 of sheet 3.) Here is the start of a proof of the result. Finish the proof:

Let $B = B(x)$ denote the adjugate matrix of $(xI_n - A)$. So each entry in B is a cofactor of $xI_n - A$ and is therefore a polynomial of degree at most $n - 1$ in x. Thus we can write $B(x) = B_{n-1}x^{n-1} + \ldots + B_1 x + B_0$ for some matrices $B_{n-1}, \ldots, B_1, B_0 \in M_n(F)$ (so these do not involve the variable x). By 5.3.2 in the lectures:

$$(a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + x^n)I_n = \det(xI_n - A)I_n = B(x)(xI_n - A) = (B_{n-1}x^{n-1} + \ldots + B_1 x + B_0)(xI_n - A). \quad (1)$$

Multiplying out the right-hand side and equating coefficients of the various powers of x we obtain:

$$a_0 I_n = -B_0 A, \ldots$$

Question 3 Suppose $S, T : V \to V$ are linear and $S \circ T = T \circ S$. For $\lambda \in F$ let $E_\lambda(S) = \{v \in V : Sv = \lambda v\}$. Show that if $v \in E_\lambda(S)$, then $T(v) \in E_\lambda(S)$.

Question 4 (i) Suppose $v_1, \ldots, v_r \in \mathbb{R}^n$ is an orthogonal set of non-zero vectors. Show that $v_1, \ldots, v_r$ are linearly independent.

(ii) Suppose $A \in M_n(\mathbb{R})$. Prove that A is an orthogonal matrix if and only if for all $u \in \mathbb{R}^n$ we have $\|Au\| = \|u\|$.

(iii) Suppose $A \in M_2(\mathbb{R})$ is an orthogonal matrix and $\det(A) = 1$. Show that there is $\theta \in \mathbb{R}$ such that $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ (so A is a rotation matrix).

(iv) Suppose $C \in M_3(\mathbb{R})$ is an orthogonal matrix and $\det(C) = 1$. Show that 1 is an eigenvalue of C. Is this true for 4×4 matrices?

Question 5 Find an orthogonal matrix $A \in M_4(\mathbb{R})$ whose first column is $\frac{1}{2}(1,1,1,1)^T$.

Question 6 Suppose U is a subspace of $\mathbb{R}^n$ and $T : U \to U$ is linear. Suppose B is an orthonormal basis of U. Prove that $[T]_B$ is symmetric if and only if for all $u, v \in U$ we have $T(u) \cdot v = u \cdot T(v)$.

Question 7 Suppose U is a subspace of $\mathbb{R}^n$. Let $U^\perp = \{v \in \mathbb{R}^n : v \cdot u = 0 \text{ for all } u \in U\}$.

i) Show that $U^\perp$ is a subspace of $\mathbb{R}^n$.

ii) Show that $U \cap U^\perp = \{0\}$.

iii) Show that $\dim(U^\perp) = n - \dim(U)$.

iv) In the case where $n = 4$ and U is the subspace spanned by $u_1 = (1,1,0,1)^T$ and $u_2 = (1,1,1,0)^T$, compute $U^\perp$.