Imperial College London
Department of Mathematics

Question Sheet 6

MATH40003 Linear Algebra and Groups — Term 2, 2020/21

Problem sheet released on Friday of week 7. All questions can be attempted before the tutorials at the end of week 8. Solutions will be released following the tutorials.

Question 1 Suppose $(G, \cdot)$ is a group and H is a subgroup of G. Prove that each of the following is an equivalence relation on G (where g, h are elements of G):

(i) $g \sim_1 h$ if and only if there is $k \in G$ with $h = kgk^{-1}$;

(ii) $g \sim_2 h$ if and only if $h^{-1}g \in H$.

In the case where $(G, .)$ is the group $(\mathbb{R}^2, +)$ and H is the subgroup $\{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$, describe geometrically the $\sim_2$-equivalence classes. What are the $\sim_1$-equivalence classes?

Question 2 Suppose $(G, \cdot)$ is a group and H, K are subgroups of G.

(i) Show that $H \cap K$ is a subgroup of G.

(ii) Show that if $H \cup K$ is a subgroup of G then either $H \subseteq K$ or $K \subseteq H$.

Question 3 Which of the following groups are cyclic?

(a) S_2.

(b) $\mathrm{GL}(2, \mathbb{R})$.

(c) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \{1, -1\} \right\}$ under matrix multiplication.

(d) $(\mathbb{Q}, +)$.

Question 4 Let G and H be finite groups. Let $G \times H$ be the set $\{(g, h) \mid g \in G, h \in H\}$ with the binary operation $(g_1, h_1) * (g_2, h_2) = (g_1 g_2, h_1 h_2)$.

(a) Show that $(G \times H, *)$ is a group.

(b) Show that if $g \in G$ and $h \in H$ have orders a, b respectively, then the order of (g, h) in $G \times H$ is the lowest common multiple of a and b.

(c) Show that if G and H are both cyclic, and $\gcd(|G|, |H|) = 1$, then $G \times H$ is cyclic. Is the converse true?

Question 5 Find an example of each of the following:

(a) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{C})$.

(b) an element of order 3 in the group $\mathrm{GL}(2, \mathbb{R})$.

(c) an element of infinite order in the group $\mathrm{GL}(2, \mathbb{R})$.

(d) an element of order 12 in the group S_7.

Question 6 Prove that if $\{x_1, \ldots, x_n\}$ is any finite subset of $(\mathbb{Q}, +)$, then the subgroup $\langle x_1, \ldots, x_n \rangle$ is cyclic.