2 Topics: Counting, axiomatic definition of probability, conditional probability

2.1 Prerequisites: Lecture 4

Exercise 2- 1: (Suggested for personal/peer tutorial)

Explain, without direct calculation that, for $k, N \in \mathbb{N}, k \leq N$,

$$\sum_{n=k}^{N} \binom{n}{k} = \binom{N+1}{k+1}.$$

Use a proof where you only comment on sampling from sets of an appropriate cardinality. You might want to write out the sum on the left hand side as

$$\sum_{n=k}^{N} \binom{n}{k} = \binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{N}{k}.$$

2.2 Prerequisites: Lecture 5

Exercise 2- 2: Given two events $E, F \subseteq \Omega$, prove that the probability of *one and only one* of them occurring is

$$\mathrm{P}(E) + \mathrm{P}(F) - 2\mathrm{P}(E \cap F).$$

Exercise 2- 3: Consider the following statements, which are claimed to be true for events A_1, A_2 in a sample space Ω:

$$\begin{array}{lll} (a) & \mathrm{P}(A_1) = 0 & \implies \mathrm{P}(A_1 \cup A_2) = 0 \\ (b) & \mathrm{P}(A_1) = \mathrm{P}(A_2^c) & \implies A_1^c = A_2 \\ (c) & A_1 \subseteq A_2 \text{ and } \mathrm{P}(A_1) = \mathrm{P}(A_2^c) & \implies \mathrm{P}(A_1) \leq 1/2 \\ (d) & \mathrm{P}(A_1^c) = x_1, \ \mathrm{P}(A_2^c) = x_2 & \implies \mathrm{P}(A_1 \cup A_2) \geq 1 - x_1 - x_2 \end{array}$$

In each case, either prove that the statement is true for all Ω, A_1, A_2, or provide a specific counter-example to show that there exists Ω, A_1, A_2 for which it is false

Exercise 2- 4: Show the so-called *Boole's inequality*: For any events $A_1, \ldots, A_n$ with $n \in \mathbb{N}$, we have

$$\mathrm{P}\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) \leq \mathrm{P}(A_1) + \cdots + \mathrm{P}(A_n).$$

Exercise 2- 5: Show the so-called *inclusion-exclusion principle*: For any events $A_1, \ldots, A_n$ with $n \in \mathbb{N}$, we have

$$\begin{aligned} &\mathrm{P}\left(A_1 \cup A_2 \cup \cdots \cup A_n\right) \\ &= \sum_{1 \leq i \leq n} \mathrm{P}\left(A_i\right) - \sum_{1 \leq i_1 < i_2 \leq n} \mathrm{P}\left(A_{i_1} \cap A_{i_2}\right) + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} \mathrm{P}\left(A_{i_1} \cap A_{i_2} \cap A_{i_3}\right) - \cdots \\ &\quad + (-1)^n \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-1} \leq n} \mathrm{P}\left(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_{n-1}}\right) \\ &\quad + (-1)^{n+1} \mathrm{P}\left(A_1 \cap A_2 \cap \cdots \cap A_n\right). \end{aligned}$$

2.3 Prerequisites: Lecture 6

Exercise 2- 6: Consider a standard 52-card deck which has been shuffled well. You pick two cards at random, one at a time without replacement. We denote by A the event that the first card is a spade and by B the event that the second card is black. Find $\mathrm{P}(A|B)$ and $\mathrm{P}(B|A)$.