3 Topics: Bayes' rule, law of total probability

3.1 Prerequisites: Lecture 7

Exercise 3- 1: (Suggested for personal/peer tutorial) *The Prisoner's Dilemma*: Three prisoners A, B, C are in solitary confinement under sentence of death, but each knows that one of them, chosen at random with equal probability, is to be pardoned. Prisoner A begs the governor to tell him whether he, A, is to be pardoned or executed. The governor refuses to answer this, but he does say that B is to be executed. The governor thinks that he isn't giving useful information, as A knows that at least one of B and C must die.

(a) A suddenly feels much happier, as he believes his chances of being pardoned have *risen* from 1/3 to 1/2. The governor, who, if A were actually to be pardoned, would be equally likely to give C's name rather than B's, is mystified by A's euphoria. Who is correct?

[Hint: Let A, B, C be the events that A, B or C respectively are pardoned. Then A, B, C partition Ω. Now let G_{AB} be the event that the governor tells A that B is to be executed. You want $\mathrm{P}(A|G_{AB})$, so consider the three conditional probabilities of G_{AB} given A, B and C respectively, and then use Bayes Theorem.]

(b) What should C feel if he overhears the governor's reply, but assumes that the question was asked by one of the warders? (consider the event G_{WB} that the governor tells a warder that B is to be executed).

Exercise 3- 2: A diagnostic test has a probability 0.95 of giving a positive result when applied to a person suffering from a certain disease, and a probability 0.10 of giving a (false) positive when applied to a non-sufferer. It is estimated that 0.5 % of the population are sufferers. Suppose that the test is now administered to a person about whom we have no relevant information relating to the disease (apart from the fact that he/she comes from this population). Calculate the following probabilities:

(a) that the test result will be positive;

(b) that, given a positive result, the person is a sufferer;

(c) that, given a negative result, the person is a non-sufferer;

(d) that the person will be misclassified.

Exercise 3- 3: Show that the Bayes' rule and the law of total probability also hold with extra conditioning:

(a) Bayes' rule with extra conditioning: For events A, B, E with $\mathrm{P}(A \cap E) > 0, \mathrm{P}(B \cap E) > 0$, we have

$$\mathrm{P}(A|B \cap E) = \frac{\mathrm{P}(B|A \cap E)\mathrm{P}(A|E)}{\mathrm{P}(B|E)}$$

(b) Consider events A, E with $\mathrm{P}(E) > 0$ and let $\{B_i : i \in \mathcal{I}\}$ denote a partition of Ω, with $\mathrm{P}(B_i \cap E) > 0$ for all $i \in \mathcal{I}$. Then,

$$\mathrm{P}(A|E) = \sum_{i \in \mathcal{I}} \frac{\mathrm{P}(A \cap B_i \cap E)}{\mathrm{P}(E)} = \sum_{i \in \mathcal{I}} \mathrm{P}(A|B_i \cap E)\mathrm{P}(B_i|E).$$

3.2 Prerequisites: Lecture 8

Exercise 3- 4: Athletes are routinely tested for the use of performance-enhancing drugs. When a test is to be carried out, the athlete provides two blood samples, the first of which is then tested. The test used is quite accurate, in that it correctly indicates the *presence* of drugs in 99.5% of tests, and correctly indicates the *absence* of drugs in 98% of tests. If this test is positive, indicating that drugs are present, the second sample is tested, using the same test, and if the second test is also positive, then the athlete is deemed to be using drugs.

Suppose that an athlete is selected at random, and two blood samples (regarded as identical) are obtained. Let events T_1 and T_2 correspond respectively to the events that first and second samples test positive, and let C be the event that drugs are actually present in the samples.

It is estimated that only 1 athlete in 1000 gives samples in which drugs are actually present

If it is assumed the results of the two tests are *conditionally independent* given the presence or absence of drugs in the samples, give expressions for, and evaluate

(a) the probability that the first test is positive

(b) the conditional probability that drugs are actually present in the sample, given that the first test is positive.

(c) the probability that both tests are positive, so that the athlete fails the test

(d) the conditional probability that drugs are actually present in the sample, given that both tests are positive.

Exercise 3- 5: Prove Lemma 6.1.11: Any countable union can be written as a countable union of disjoint sets. I.e. let $A_1, A_2, \dots \in \mathcal{F}$ and define $D_1 = A_1, D_2 = A_2 \setminus A_1, D_3 = A_3 \setminus (A_1 \cup A_2), \dots$. Then $\{D_i\}$ is a collection of disjoint sets and $\cup_{i=1}^n A_i = \cup_{i=1}^n D_i$ for n being any positive integer or ∞.

3.3 Prerequisites: Lecture 9

Exercise 3- 6: Consider a function with domain $\mathcal{X}$ and co-domain $\mathcal{Y}$, i.e. $f : \mathcal{X} \to \mathcal{Y}$. For any collection of subsets $B_i \subseteq \mathcal{Y}$, $i \in \mathcal{I}$ where $\mathcal{I}$ denotes an (arbitrary) index set, show that

$$f^{-1}\left(\bigcap_{i\in\mathcal{I}} B_i\right) = \bigcap_{i\in\mathcal{I}} f^{-1}(B_i).$$

Exercise 3- 7: Let X denote a discrete random variable on $(\Omega, \mathcal{F}, \mathrm{P})$ and let $g : \mathbb{R} \to \mathbb{R}$ be a deterministic function. Show that $Y = g(X)$ is also a discrete random variable and find the probability mass function of Y.