5 Topics: Continuous random variables, transformations of random variables, expectation

5.1 Prerequisites: Lecture 12

Exercise 5- 1: Let $X \sim \mathrm{Exp}(\lambda)$. Show that, if $x, y > 0$ then

$$\mathrm{P}(X > x + y | X > x) = \mathrm{P}(X > y).$$

This is called the *Lack of memory* property (for a continuous random variable).

Exercise 5- 2: (Suggested for personal/peer tutorial) The length of time (in hours) that a student takes to complete a one hour exam is a continuous random variable X with probability density function f_X defined by

$$f_X(x) = cx^2 + x, \quad 0 < x \leq 1,$$

for some constant c, and zero otherwise.

(a) Find the value of c.

(b) By integration, find the cumulative distribution function F_X of X.

(c) Find the probability that a student completes the exam in less than half an hour.

(d) **Given** that a student takes longer than fifteen minutes to complete the exam, find the probability that they require at least half an hour, that is, find the conditional probability

$$\mathrm{P}\left(X > \frac{1}{2} \middle| X > \frac{1}{4}\right)$$

(e) In a class of two hundred students, find the probability that at most three students complete the exam in fewer than ten minutes.

Assume that the exam completion times for the two hundred students are independent random variables having the distribution specified above.

Hint: Consider discrete random variables $Y_1, ...Y_{200}$ where

$$Y_i = \begin{cases} 1 & \textit{student } i \textit{ completes the exam in fewer than ten minutes} \\ 0 & \textit{otherwise} \end{cases}$$

You need to calculate $\mathrm{P}(Y \leq 3)$ *where* $Y = \sum_{i=1}^{200} Y_i$.

Exercise 5- 3: The probability density function of continuous random variable X taking values in the range $\mathrm{Im}X = (0, 2)$ is specified by

$$f_X(x) = \begin{cases} x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \end{cases}$$

and zero otherwise. Find the cumulative distribution function of X, F_X, and hence find $\mathrm{P}(0.8 < X \leq 1.2)$.

Exercise 5- 4: The *median* of a continuous random variable X is that value x such that $F_X(x) = 1/2$. Find the median of X when

(a) X has an *Exponential* distribution with parameter $\lambda > 0$, that is

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0$$

and zero otherwise.

(b) $\log X$ has a normal distribution with parameters μ and σ^2, where $\mu \in \mathbb{R}, \sigma > 0$.

5.2 Prerequisites: Lecture 13

Exercise 5- 5: Suppose that X is a continuous random variable with range $\mathrm{Im}X = [0, 1]$, and probability density function f_X specified by

$$f_X(x) = 2(1 - x), \quad 0 \le x \le 1,$$

and zero otherwise. Find the probability distributions of random variables Y_1, Y_2 and Y_3 defined respectively by

(a) $Y_1 = 2X - 1$,

(b) $Y_2 = 1 - 2X$,

(c) $Y_3 = X^2$,

that is, in each case, find the range and the density function.

Exercise 5- 6: The continuous random variable X has a Uniform distribution on the interval $[-1, 1]$. Find the probability density function of random variables

(a) $Y = |X|$,

(b) $Z = X^2$.

Exercise 5- 7: If X is **any** continuous random variable with distribution function F_X, show that

(a) the random variable $U = F_X(X)$ has a Uniform distribution on $(0, 1)$; (*Hint:* You may first assume that F_X is invertible/strictly monotonically increasing. Then, as an extra challenge, can you find a proof which allows you to drop this additional assumption?)

(b) the random variables $Y_1 = -\log F_X(X)$ and $Y_2 = -\log(1-X)$ have an exponential distribution.

Exercise 5- 8: Suppose that random variable X has a standard normal distribution.

(a) Find the cumulative distribution function (cdf) of $Y = X^2$ in terms of the standard normal c.d.f. Φ.

Hint: For the c.d.f. of Y, we have

$$\mathrm{P}(Y \le y) = \mathrm{P}(X^2 \le y) = \mathrm{P}(|X| \le \sqrt{y}).$$

(b) Find the probability density function of Y, f_Y.

(c) Identify (by name) the probability distribution of Y.

Exercise 5- 9: [From the January test in 2021] Let X be a continuous random variable with probability density given by

$$f_X(x) = \begin{cases} 2x, & \text{for } 0 \le x \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Define the random variable $Y = g(X)$, where

$$Y = g(X) = \begin{cases} X, & \text{for } 0 \le X \le 1/2, \\ \frac{1}{2}, & \text{for } X > \frac{1}{2}. \end{cases}$$

a) Find the image of Y.

b) Find the cumulative distribution function of Y.

c) Sketch the cumulative distribution function. Is Y a continuous or discrete random variable or is it neither discrete nor continuous?

5.3 Prerequisites: Lecture 14

Exercise 5- 10: The annual profit (in millions of pounds) of a manufacturing company is a function of product demand. If X is the continuous random variable corresponding to the demand in a given year, then the annual profit is also a continuous random variable, Y say, where

$$Y = 2(1 - e^{-2X})$$

If X has an Exponential distribution with parameter $\lambda = 6$, find the expected annual profit.

Exercise 5- 11: Consider the random variable Y as defined in Exercise 5- 9. How could you define the expectation for this random variable?