6 Topics: Expectation, independence of random variables, probability generating functions

6.1 Prerequisites: Lecture 14

Exercise 6- 1: (Suggested for personal/peer tutorial) Indicator variables and their expectation: Recall that we defined the indicator of an event $A \in \mathcal{F}$ in Definition 7.4.2 as follows: For an event $A \in \mathcal{F}$, we denote by

$$\mathbb{I}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A, \\ 0, & \text{if } \omega \notin A, \end{cases}$$

the *indicator variable of the event* A.

(a) For events $A, B \in \mathcal{F}$, show that

i. $(\mathbb{I}_A)^k = \mathbb{I}_A$ for any $k \in \mathbb{N}$,

ii. $\mathbb{I}_{A^c} = 1 - \mathbb{I}_A$,

iii. $\mathbb{I}_{A \cap B} = \mathbb{I}_A \mathbb{I}_B$,

iv. $\mathbb{I}_{A \cup B} = \mathbb{I}_A + \mathbb{I}_B - \mathbb{I}_A \mathbb{I}_B$.

(b) Prove the fundamental bridge between probability and expectation, i.e. show that there is a one-to-one correspondence between events and indicator random variables and for any $A \in \mathcal{F}$ we have

$$\mathrm{P}(A) = \mathrm{E}(\mathbb{I}_A).$$

Exercise 6- 2: Prove Theorem 10.2.6: Consider a discrete/continuous random variable X with finite expectation.

(a) If X is non-negative, then $\mathrm{E}(X) \geq 0$.

(b) If $a, b \in \mathbb{R}$, then $\mathrm{E}(aX + b) = a\mathrm{E}(X) + b$.

Exercise 6- 3: Prove Theorem 10.3.3: Let X be a discrete random variable with finite variance and consider deterministic constants $a, b \in \mathbb{R}$. Then

$$\mathrm{Var}(aX + b) = a^2 \mathrm{Var}(X).$$

6.2 Prerequisites: Lecture 15

Exercise 6- 4: Consider a sequence of *Bernoulli* random variables $X_1, ..., X_n$ each with parameter θ resulting from independent binary trials, so that

$$\mathrm{P}(X = 0) = 1 - \theta, \quad \mathrm{P}(X = 1) = \theta.$$

Find the probability mass functions of the random variables

(a) $Y = \text{Min}\{X_1, ..., X_n\}$

(b) $Z = \text{Max}\{X_1, ..., X_n\}$

[*Hint: find the ranges of* Y *and* Z*, and consider* $\mathrm{P}(Y = 1)$, $\mathrm{P}(Z = 0)$.]

6.3 Prerequisites: Lecture 16

Exercise 6- 5: Suppose that $F_{X,Y}(x,y)$ is the joint distribution function of (X,Y). Find an expression for $\mathrm{P}(X \leq x, Y > y)$ in terms of $F_{X,Y}$.

Exercise 6- 6: Consider a probability space given by $\Omega = \{-1, 0, 1\}$, $\mathcal{F} = \mathcal{P}(\Omega)$ and P is defined by

$$\mathrm{P}(-1) = \mathrm{P}(0) = \mathrm{P}(1) = \frac{1}{3}.$$

We define two discrete random variables by $X(\omega) = \omega$ and $Y(\omega) = |\omega|$.

(a) Compute $\mathrm{P}(X = 0, Y = 1)$, $\mathrm{P}(X = 0)$ and $\mathrm{P}(Y = 1)$. What can you conclude?

(b) Compute $\mathrm{E}(XY)$ and $\mathrm{E}(X)\mathrm{E}(Y)$. What can you conclude?

Exercise 6- 7: Show that discrete random variables X and Y on $(\Omega, \mathcal{F}, \mathrm{P})$ are independent if and only if

$$\mathrm{E}(g(X)h(Y)) = \mathrm{E}(g(X))\mathrm{E}(h(Y)), \tag{6.1}$$

for all functions $g, h : \mathbb{R} \to \mathbb{R}$ for which the expectations on the right hand side exist.

Hint: For the part, where you assume (6.1) for all g, h and want to show that X and Y are independent, proceed as follows: Write down the definition of independence of discrete random variables using the factorisation of the corresponding joint pmf. Now, choose specific functions g and h (as suitably defined indicator functions), which allow you to derive the factorisation formula for the pmf from (6.1).

Exercise 6- 8: Convolution theorem: Consider jointly discrete/continuous random variables X and Y and define their sum as $Z = X + Y$. Find the probability mass function/density function of Z (leaving your expression as a sum/integral). If you assume that X and Y are independent, can you simplify the p.m.f./p.d.f. of Z?

Hint: Recall that $\mathrm{P}((X,Y) \in A) = \sum\sum_{(x,y)\in A} \mathrm{P}(X = x, Y = y)$ in the discrete case, and $\mathrm{P}((X,Y) \in A) = \int\int_A f_{X,Y}(x,y)dxdy$ in the jointly continuous case for "nice" sets $A \subseteq \mathbb{R}^2$.

Exercise 6- 9: Prove Theorem 12.7.2: For jointly discrete/continuous random variables X, Y with finite expectations, we have

$$\mathrm{Cov}(X,Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y).$$

Exercise 6- 10: Prove Theorem 12.7.7: Let X, Y denote two jointly discrete/continuous random variables with finite variances. Then

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y).$$

6.4 Prerequisites: Lecture 17

Exercise 6- 11: Suppose that $X_1, \ldots, X_n$ are independent $\mathrm{Ber}(p)$ random variables. Define $S_n = \sum_{i=1}^n X_i$. Use probability generating functions to show that $S_n \sim \mathrm{Bin}(n,p)$.

Exercise 6- 12: Suppose that $X_1, \ldots, X_n$ are independent Poisson random variables – not necessarily with the same parameter, i.e. $X_i \sim \mathrm{Poi}(\lambda_i)$. Define $S_n = \sum_{i=1}^n X_i$. Use probability generating functions to show that $S_n \sim \mathrm{Poi}(\sum_{i=1}^n \lambda_i)$.