7 Topics: Moment generating functions, conditional distribution and conditional expectation

7.1 Prerequisites: Lecture 18

Exercise 7- 1: Use moment generating functions to find the mean and variance of

(a) $X \sim \mathrm{Poi}(\lambda)$,

(b) $X \sim \mathrm{Bin}(n, p)$,

(c) $X \sim \mathrm{Exp}(\lambda)$,

(d) $X \sim \mathrm{N}(\mu, \sigma^2)$.

Exercise 7- 2: (Suggested for personal/peer tutorial) Use moment generating functions to prove that for independent random variables $X \sim \mathrm{N}(\mu_X, \sigma_X^2), Y \sim \mathrm{N}(\mu_Y, \sigma_Y^2)$, we have that $X + Y \sim \mathrm{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$.

Exercise 7- 3: Suppose that X_1 and X_2 are independent and identically distributed random variables, each having a standard normal distribution. Let random variable V be defined by

$$V = X_1^2 + X_2^2.$$

Find the pdf of V.

7.2 Prerequisites: Lecture 19

Exercise 7- 4: Consider tossing a coin repeatedly, where the probability of heads appearing in one toss is given by $p \in (0, 1)$. Let X denote the length of the initial run (i.e. if you toss heads first, how many heads do you toss before tossing tail and vice versa if you toss tail first). By conditioning on the outcome of the first coin toss and by using the law of total expectation, find $\mathrm{E}(X)$.

Exercise 7- 5: Consider two discrete random variables X and Y with joint probability mass function given by

$$\mathrm{P}(X = x, Y = y) = \begin{cases} c(2x + y), & \text{if } x \in \{0, 1, 2\} \text{ and } y \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise,} \end{cases}$$

where c is an appropriately-chosen constant.

(a) Find the value of c.

(b) Find $\mathrm{P}(X = 2, Y = 1)$

(c) Find $\mathrm{P}(X \geq 1, Y = 1)$

(d) Find $\mathrm{P}(X \geq 1, Y \leq 1)$

(e) Find the marginal probability mass function of X.

(f) Find the marginal probability mass function of Y.

(g) Are X and Y independent random variables?

(h) Find the probability mass function of Y given $X = 2$.

(i) Compute $\mathrm{P}(Y = 1|X = 2)$.

(j) Compute $\mathrm{E}(Y|X = 2)$.

Exercise 7- 6: Consider two jointly continuous random variables (X, Y) with joint density function given by

$$f_{X,Y}(x, y) = cxy, \quad \text{for} \quad 0 \leq x \leq 1, 0 \leq y \leq 1,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

(a) Determine the constant c such that $f_{X,Y}$ is a valid density function.

(b) Find the marginal density of X.

(c) Find the conditional density of $Y|X = x$.

(d) Find the conditional distribution of $Y|X = x$.

(e) Are X and Y independent?

Exercise 7- 7: Consider two jointly continuous random variables (X, Y) with joint density function given by

$$f_{X,Y}(x, y) = c(x + y), \quad \text{for} \quad 0 \leq y \leq x \leq 1,$$

and $f_{X,Y}(x, y) = 0$ otherwise.

(a) Determine the constant c such that $f_{X,Y}$ is a valid density function.

(b) Find the marginal density of X.

(c) Find the marginal density of Y.

(d) Are X and Y independent?

(e) Find the conditional density of $Y|X = x$.

(f) Find the conditional distribution of $Y|X = x$.