Topic: Counting

In today's problem class we will be reviewing concepts from combinatorics.

1. (a) 10 people say goodbye to each other and shake hands. Everybody goes home alone. How many handshakes are there in total?

   (b) 10 couples say goodbye to each other and shake hands. Every couple goes home alone. How many handshakes are there in total?

   (c) 10 couples say goodbye to each other: The men shake hands, the women kiss each other on each cheek and a man and a woman also kiss each other on each cheek. How many handshakes and how many kisses are there in total? [For the purpose of this exercise, we shall assume that a couple consists of a man and a woman. You are welcome to modify this assumption and update your counts according to the assumptions you make.]

2. Explain, without direct calculation that, for $n \in \mathbb{N}$,

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

   Use a proof where you only comment on sampling from sets of an appropriate cardinality.

   Remark 0.1 *In many situations, a probability calculation can be reduced to an exercise in counting equally likely sample outcomes using combinatorial techniques. If the sample space comprises* $\mathrm{card}(\Omega)$ *equally likely outcomes, and event* E *represents a collection of* $\mathrm{card}(E)$ *of them, then we can legitimately define* $\mathrm{P}(E)$ *by*

$$\mathrm{P}(E) = \frac{\mathrm{card}(E)}{\mathrm{card}(\Omega)},$$

   and so the probability calculation only requires enumeration of $\mathrm{card}(E)$ *and* $\mathrm{card}(\Omega)$.

3. Outlook: Hypergeometric distribution. Consider an urn filled with N balls, with $K \in \mathbb{N}$ being white balls and $N-K$ being black. Suppose we draw $n \in \mathbb{N}$ balls from the urn *without replacement* and we denote by x the number of observed white balls. We compute the probability of having x white balls when we draw n balls without replacement:

$$\mathrm{P}(X=x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}, \text{ for } x \in \{0,1,\ldots,K\} \text{ and } n-x \in \{0,1,\ldots,N-K\},$$

   and $\mathrm{P}(X=x)=0$ otherwise. We justify the above formula as follows: For the denominator, we report the total number of possibilities of drawing n balls from an urn of N balls, so $\binom{N}{n}$ in total. For the numerator, we have $\binom{K}{x}$ possibilities of choosing x white balls from the total number of K white balls and $\binom{N-K}{n-x}$ possibilities of choosing $n-x$ black balls from the total number of $N-K$ black balls. We claim that

$$\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}} = \frac{\binom{n}{x}\binom{N-n}{K-x}}{\binom{N}{K}}.$$

   (a) Prove the above identity by expanding the binomial coefficients/factorials.

   (b) Describe in words what the left and right hand side represent.

   (c) Suppose your sock drawer is a mess and contains 18 black socks and 10 blue socks that otherwise look alike. What is the probability that you randomly select two black socks if you select exactly 2 socks?

(d) If I deal you a hand of 13 cards at random from a well shuffled pack. What is the probability that your hand contains exactly 10 hearts?

(e) A 12-member jury for a criminal case will be selected from a pool of 14 men and 6 women. What is the probability that at least 3 of the jury will be women?

4. Use counting approaches in the solution of the following problems;

   (a) Each of n sticks is broken into two parts, long and short, and a new set of n sticks formed by pairing and joining the $2n$ parts at random. What is the probability that:

      i. each stick is paired and rejoined into its original form that is, there is a match between the rejoined long and short parts for all n sticks.

      ii. each of the n long parts are rejoined with a short part.

   (b) Six fair dice are rolled. What is the probability that a full set of scores $\{1, 2, 3, 4, 5, 6\}$ is obtained?

   (c) If the letters M,I,I,I,I,S,S,S,S,P,P are arranged at random, what is the probability that:

      i. the arrangement spells the word MISSISSIPPI?

      ii. the arrangement has no adjacent I's?

      iii. the arrangement has at least 2 consecutive S's?