Analysis 2 - Concise Notes

MATH50001

Year 2 Content

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Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

Content from MATH40002 assumed to be known.

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Part I

Term 1

1 Differentiation in Higher Dimensions

1.1 Euclidean Spaces

1.1.1 Preliminaries

Definition - Modulus Function

$$|x| := \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Having the following properties:

(i) $\forall x \in \mathbb{R}, |x| \geq 0, |x| = 0 \iff x = 0$

(ii) $\forall x, y \in \mathbb{R}, |xy| = |x||y|$

(iii) $\forall x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$ *(Triangle inequality)*

1.1.2 Euclidean space of dim. n

Define - Euclidean Space of dim. $n, \mathbb{R}^n$
Defined as the set of ordered n-tuples $(x^1, \ldots, x^n)$, s.t each $x^i \in \mathbb{R} \forall i$
$\mathbb{R}^n$ a vector space.

Define - Inner Product, $< \cdot, \cdot >, : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$

$$< (x^1, x^2, \ldots, x^n), (y^1, y^2, \ldots, y^n) >= \sum_{i=1}^{n} x^i y^i$$

Define - Norm/Lengths, $|| \cdot || : \mathbb{R}^n \to \mathbb{R}$

$$||x|| = \sqrt{< x, x >}$$

Having the following properties:

(i) $\forall x \in \mathbb{R}^n, ||x|| \geq 0, ||x|| = 0 \iff x = \vec{0}$

(ii) $\forall \in \mathbb{R}, x \in \mathbb{R}^n ||\lambda x|| = |\lambda|||x||$

(iii) $\forall x, y \in \mathbb{R}^n, ||x + y|| \leq ||x|| + ||y||$ *(Triangle inequality)*

Definition - Cauchy-Schwartz Inequality

$$|\langle x, y \rangle| \leq ||x||||y||$$

1.1.3 Convergence of Sequences in Euclidean Spaces

Definition - Sequence in $\mathbb{R}^n$
An infinite ordered list, $x_0, x_1, \ldots$, s.t $x_i \in \mathbb{R}^n \ \forall i$. Denoted $(x_i)_{i \geq 1}$ or $(x_i)_{i \in \mathbb{N}}$

Definition 1.1 - **Convergence**
A seq. $(x_i) \in \mathbb{R}^n$ **converges to** $x \in \mathbb{R}^n$ if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ **s.t** $\forall i \geq \mathbb{N}, ||x_i - x|| < \epsilon$
Corollary
seq. $(x_i) \in \mathbb{R}^n$ converges to $x \in \mathbb{R}^n \iff$

For $x_i = (x_i^1, \ldots, x_i^n)$ and $x = (x^1, \ldots, x^n)$

$$x_i \to x \iff \forall k \ x_i^k \to x^k \text{ as } i \to \infty$$

1.2 Continuity

1.2.1 Open sets in Euclidean Spaces

Definition - Open Ball
Open ball of radius r is

$$B_r(x) = \{y \in \mathbb{R}^n : ||x - y|| < r\}$$

Definition 1.2 - Open sets
A set $U \subseteq \mathbb{R}^n$ is called **open**, if

$$\forall x \in U, \exists r > 0 \textbf{ such that} B_r(x) \subseteq U$$

1.2.2 Continuity at a point/on an open set

Definition 1.3 - Continuity at a point
Let $A \subset \mathbb{R}^n$ an open set, with $f : A \to R^n$
f **continuous at** $p \in A$ if

$$\forall \epsilon > 0, \exists \delta > 0 \textbf{ s.t } ||x - p|| < \delta \implies ||f(x) - f(p)|| < \epsilon$$

f is (pointwise) continuous on $A \subseteq \mathbb{R}^n \iff$ continuous $\forall p \in A$, we write f is continuous.
For small enough δ, we have $f(B_\delta(p)) \subseteq B_\epsilon(f(p))$

Theorem 1.2 - Composition of continuous functions
Let $A \subseteq \mathbb{R}^n$ open, $B \subseteq \mathbb{R}^m$ open and suppose $f : A \to B$ continuous at $p \in A$, and $g : B \to \mathbb{R}^l$ continuous at $f(p)$

Then $g \circ f : A \to \mathbb{R}^l$ continuous at p

Definition 1.4 - Limit of a function at a point
$A \subseteq \mathbb{R}^n$ an open set. f a function $f : A \to \mathbb{R}^m$, with $p \in A$ and $q \in \mathbb{R}^m$
Say $\lim_{x\to p} f(x) = q$ if $\forall \epsilon > 0, \exists \delta > 0$ **s.t** $\forall x \in A$ **with** $0 < ||x - p|| < \delta$ **we have** $||f(x) - p|| < \epsilon$

$$f \text{ continuous at } p \iff \lim_{x\to p} f(x) = q$$

Theorem 1.3 - Algebra of Limits
Suppose $A \subseteq \mathbb{R}^n$ open, with $p \in A$ and $f, g : A \to \mathbb{R}^n$

$$\lim_{x\to p} f(x) = F \textbf{ and } \lim_{x\to p} g(x) = G$$

Then:

(i) $\lim_{x\to p}(f(x) + g(x)) = F + G$

(ii) $\lim_{x\to p}(f(x)g(x)) = FG$

(iii) **If,** $G \neq 0$ **then** $\lim_{x\to p} \frac{f(x)}{g(x)} = \frac{F}{G}$

1.3 Derivative of a map of Euclidean Spaces

1.3.1 Derivative of a linear map

Lemma 1.5
The map $f : (a, b) \to \mathbb{R}$ differentiable at $p \in (a, b) \iff \exists$ map of the form $A_\lambda(x) = \lambda(x - p) + f(p)$ for some $\lambda \in \mathbb{R}$ s.t

$$\lim_{x \to p} \frac{|f(x) - A_\lambda(x)|}{|x - p|} = 0$$

Notation

$$h[v] \text{ for h a linear map, v a vector}$$

$$h(v) \text{ h a map, v a point in domain of h}$$

$$L(\mathbb{R}^n; \mathbb{R}^m) - \textbf{Set of linear maps from } \mathbb{R}^n \to \mathbb{R}^m$$

Definition 1.5 - Derivative in higher dimension
Suppose $\Omega \subset \mathbb{R}^n$ open. **The map $f : \Omega \to \mathbb{R}^m$ differentiable** at $p \in \Omega$ if **$\exists$ a linear map $\Lambda \in L(\mathbb{R}^n; R^m)$** such that

$$\lim_{x \to p} \frac{||f(x) - (\Lambda[x - p] + f(p))}{||x - p||} = 0$$

We write

$$Df(p) := \Lambda$$

Calling $Df(p)$ the derivative of f at p
Λ a $m \times n$ matrix called the **Jacobian**

Lemma 1.6 - Differentiable then continuous
$\Omega \subset \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}^m$ differentiable at $p \in \Omega \implies f$ continuous at p

Theorem 1.7 - Uniqueness of Derivative
*The derivative, **if it exists**, **is unique***

1.3.2 Chain Rule

Chain rule in $\mathbb{R}$
$f, g : \mathbb{R} \to \mathbb{R}, g$ differentiable at p, f differentiable at $g(p)$ Then $f \circ g$ differentiable at p with

$$(f \circ g)'(p) = f'(g(p))g'(p)$$

Theorem 1.8 - Chain rule in higher dim.
$\Omega \subset \mathbb{R}^n$ open, $\Omega' \subset \mathbb{R}^m$ open
With $g : \Omega \to \Omega'$ differentiable at $p \in \Omega$, $f : \Omega' \to \mathbb{R}^l$ differentiable at $g(p) \in \Omega'$
Then $h = f \circ g : \Omega \to \mathbb{R}^l$, differentiable at p, s.t

$$Dh(p) = D(f(g(p)) \circ Dg(p)$$

1.4 Directional Derivatives

1.4.1 Rates of change and Partial Derivatives

Definition - Directional Derivative
The **directional derivative** of f at p in the direction v is

$$\frac{\partial f}{\partial v}(p) := \lim_{t \to 0} \frac{1}{t}[f(p + vt) - f(p)] = Df(p)[v]$$

Definition - Partial derivatives
We can find any directional derivative at p, given we know the partial derivatives of f

$$D_i f(p) = \frac{\partial f}{\partial e_i}(p)$$

In $\mathbb{R}^3$ we have,

$$Df(p)[v] = \begin{pmatrix} D_1 f(p) & D_2 f(p) & D_3 f(p) \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix}$$

Definition - Gradient
Gradient of f at p

$$\nabla f(p) = \begin{pmatrix} D_1 f(p) \\ D_2 f(p) \\ D_3 f(p) \end{pmatrix} \qquad Df(p) = (\nabla f(p))^t$$

Theorem 1.9 - Jacobian
Suppose $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}^m$ of the form

$$f(x) = \big(f^1(x), f^2(x), \dots, f^m(x)\big)$$

If f differentiable for some $p \in \Omega$ Then **Jacobian of f at p is:**

$$Df(p) = \begin{pmatrix} D_1 f^1(p) & \dots & D_n f^1(p) \\ \vdots & \ddots & \vdots \\ D_1 f^m(p) & \dots & D_n f^m(p) \end{pmatrix}$$

1.4.2 Relation between partial derivatives and differentiability

Theorem 1.12
Let $\Omega \subset \mathbb{R}^n$ open, $f : \Omega \to \mathbb{R}$. **Suppose the partial derivatives:**

$$D_i f(x) := \lim_{t \to 0} \frac{f(x + te_i - f(x)}{t}$$

exist $\forall x \in \Omega$, with each map $x \mapsto D_i f(x)$ continuous at $p, \forall i$ $\implies$ **f is differentiable at p**

1.5 Higher Derivatives

1.5.1 Higher derivatives as linear maps

Can think of the differential of f, $Df(p)$ as a map

$$Df : \Omega \to L(R^n; R^m) = \Omega \to \mathbb{R}^{mn}$$

$$p \mapsto Df(p)$$

if map Df is continuous $\implies$ $f : \Omega \to \mathbb{R}$ is **continuously differentiable**

Definition - Higher derivative
If $Df : \Omega \to \mathbb{R}^{mn}$ differentiable at p, denote derivative of Df as $DDf(p) : \mathbb{R}^n \to \mathbb{R}^{nm}$

$$DDf(p) \in L(\mathbb{R}^n; \mathbb{R}^{nm}) = L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$$

Where $DDf(p)$ is a linear map $\mathcal{L} \in L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$, satisfying:

$$\lim_{x \to p} \frac{||Df(x) - Df(p) - \mathcal{L}[x-p]||}{||x-p||} = 0$$

$DDf(p)$ takes an n-vector to a $m \times n$ matrix

Definition - Continuously differentiable
$f : \Omega \to \mathbb{R}^m$ is k-times dfferentiable with all continuous derivatives $\implies$ **f is k-times continuously differentiable**
Testing for k-times differentiability
For $f = \big(f^1(x), f^2(x), \dots, f^m(x)\big)$
If f differentiable at $p \in \Omega \implies$ we have partial derivatives $D_i f^j : \Omega \to \mathbb{R}$.
If Df differentiable, then 2^{nd} partial derivatives exist

$$D_k D_i f^j(p) := \lim_{t \to 0} \frac{D_i f^j(p + te_k) - D_i f^j(p)}{t}$$

Easy to check these exist and are continuous $\implies$ k-times differentiability at p

1.5.2 Symmetry of mixed partial derivatives

Theorem 1.13 - Schwartz' Theorem
Suppose $\Omega \subset \mathbb{R}^n$ open and $f : \Omega \to \mathbb{R}$ differentiable $\forall p \in \Omega$
Suppose also, for $i, j \in \{1, \ldots, n\}$, 2^{nd} partial derivatives $D_i D_j f$ and $D_j D_i f$ exist and are continuous $\forall p \in \Omega$

$$\forall p \in \Omega, D_i D_j f(p) = D_j D_i f(p)$$

Definition - Hessian
The matrix of 2^{nd} partial derivatives at the point p

$$\text{Hess } f(p) = [D_i D_j f(p)]_{i,j=1,\ldots,n}$$

Schwartz' Theorem says Hess $f(p$ is a symmetric matrix

1.5.3 Taylor's Theorem

Definition - Multi-inidices
Multi-index $\alpha \in (\mathbb{N})^n, \alpha = \big(\alpha_1, \ldots, \alpha_n\big)$
We define $|a| = \sum_{i=1}^n \alpha_i$ and

$$D^\alpha f := (D_1)^{\alpha_1} (D_2)^{\alpha_2} \ldots (D_n)^{\alpha_n} f,$$

And for a vector $h = \big(h_1, \ldots, h_n\big)$

$$h^\alpha := (h^1)^{\alpha_1} (h^2)^{\alpha_2} \ldots (h^n)^{\alpha_n}$$

Also

$$\alpha! := \alpha_1! \alpha_2! \ldots \alpha_n!$$

helpful examples

$$D^{(0,3,0)} f(p) = D_2^3 f(p)$$
$$D^{(1,0,1)} f(p) = D_1 D_3 f(p)$$
$$(x, y, z)^{(2,1,5)} = x^2 y^1 z^5$$

Theorem 1.14 - Taylor's Theorem in higher dim.
Suppose $p \in \mathbb{R}^n$ and $f : B_r(p) \to \mathbb{R}$ a k-times continuously differentiable $\forall q \in B_r(p)$, for some $k \geq 1 \in \mathbb{N}$
Then $\forall h \in \mathbb{R}^n$ with $||h|| < r$ We have

$$f(p+h) = \sum_{|\alpha| \leq k-1} \frac{h^\alpha}{\alpha!} D^\alpha f(p) + R_k(p, h)$$

Sum over all $\alpha = \big(\alpha_1, \ldots, \alpha_n\big)$
with $|\alpha| \leq k - 1$ and remainder term

$$R_k(p, h) = \sum_{|\alpha| = k} \frac{h^\alpha}{\alpha!} D^a f(x)$$

for some x s.t $0 < ||x - p|| < ||h||$
Evidently

$$\lim_{h \to 0} \frac{|R_k(p, h)|}{||h||^{k-1}} = 0$$

1.6 Inverse & Implicit Function Theorem

1.6.1 Inverse Function Theorem

Theorem 1.15 - *(Inverse Function Theorem)*
Let Ω an open set in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ continuously differentiable on Ω, $\exists q \in \Omega$ s.t $Df(q)$ invertible
Then $\exists$ open sets $U \subset \Omega$ and $V \subset \mathbb{R}^n, q \in U, f(q) \in V$ s.t

(i) $f : U \to V$, a bijection

(ii) $f^{-1} : V \to U$, continuously differentiable

(iii) $\forall y \in V$,

$$Df^{-1}(y) = [Df(f^{-1}(y))]^{-1}$$

1.6.2 Implicit Function Theorem

Theorem 1.16 - ***(Implicit Function Theorem - Simple version)***
$\Omega \subset \mathbb{R}^2$ open
$F : \Omega \to \mathbb{R}$ continuously differentiable and $\exists (x', y') \in \Omega$ s.t

(i) $F(x', y') = 0$, and

(ii) $D_2 F(x', y') \neq 0$

$\implies \exists$ open sets $A, B \subset \mathbb{R}$ with $x' \in A, y' \in B$ with a map $f : A \to B$ s.t

$$(x, y) \in A \times B \text{ satisfies } F(x, y) = 0 \iff y = f(x) \text{for some } x \in A$$

with $f : A \to B$ continuously differentiable.

Definition - C^1**−diffeomorphism**
$\Omega, \Omega' \subset \mathbb{R}^n$ open.
Say $f : \Omega \to \Omega'$ a C^1-diffeormorphism, if $f : \Omega \to \Omega'$ a bijection, continuously differentiable, and $\forall x \in \Omega, Df(x)$ **invertible**
$\mathcal{D}$ the set of all C^1-diffeomorphisms from $\Omega \to \Omega$, a group under group law; composition.

1.6.4 Implicit Function Theorem - General Form

Theorem 1.17 - ***(Implicit Function Theorem)***
$\Omega \subset \mathbb{R}^n, \Omega' \subset \mathbb{R}^m$ open sets
$F : \Omega \times \Omega' \to \mathbb{R}^m$ continuously differentiable on $\Omega \times \Omega'$ and sps $\exists (a, b) \in \Omega \times \Omega'$ s.t

(i) $f(p) = 0$ and,

(ii) $m \times n$ matrix
$$(D_{n+j} f^i(p)), \qquad 1 \leq i, j \leq m$$
invertible

$\implies \exists$ open sets $A \subset \Omega, B \subset \Omega'$ with $a \in A, b \in B$ with a map $g : A \to B$ s.t

$$g(x, y) = 0 \text{ for some } (x, y) \in A \times B \iff y = g(x) \text{for some } x \in A$$

with $g : A \to B$ continuously differentiable.

2 Metric and Topological Spaces

2.1 Metric Spaces

2.1.1 Motivation + Definition

Definition 2.1 - Metric
X an arbitrary set
Metric a function $d : X \times X \to \mathbb{R}$ satisfying:

(M1) $\forall x, y \in X;\ d(x,y) \geq 0, d(x,y) = 0 \iff x = y$ ***(positivity)***

(M2) $\forall x, y \in X;\ d(x,y) = d(y,x)$ ***(symmetry)***

(M3) $\forall x, y, z \in X d(x,y) \leq d(x,z) + d(z,y)$ ***(triangle inequality)***

Definition 2.2 - Metric space
Pair of a set and metric; $M = (X, d)$
Call elements of X points,with $d(x,y)$ distance between x, y w.r.t d

Definition

$$C([a,b]) = \{f : [a,b] \to \mathbb{R} | f : [a,b] \to \mathbb{R}\text{continuous}\}$$

2.1.2 Examples of metrics

Examples

- $d_2(x,y) = ||x - y||$; Euclidean metric on $\mathbb{R}^n$
- $d_{\text{disc}}(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$
- $d_\infty(x,y) = \sup_{k \geq 1} |x^k - y^k|$
- $d_\infty(f,g) = \max_{a \leq t \leq b} |f(t) - g(t)$ where $f, g \in C([a,b])$ ***(supremum/uniform metric)***
- $d_1()$

Definition 2.3. Induced metrics

(X, d) a metric space
$Y \subseteq X$, define $d|_Y : Y \times Y \to \mathbb{R}$ as $d|_Y(x,y) = d(x,y)\ \forall x, y \in Y$

Definition 2.3. Metric Subspace

Say $(Y, d|_Y)$ a metric subspace of (X, d)

Definition 2.4. Product metric space

(X_1, d_1) and (X_2, d_2) metric spaces.
define metric using d_1, d_2 $d : (X_1 \times X_2) \times (X_1 \times X_2) \to \mathbb{R}$.
$(X_1 \times X_2, d)$ a product metric space.

2.1.3 Normed Vector Spaces

Definition 2.5. Norm in Metric Spaces

V a vector space on $\mathbb{R}$. Say $||\cdot|| : V \to \mathbb{R}$ a **norm** on V if

(N1) $\forall v \in V,\ ||v|| \geq 0$ and $||v|| = 0 \iff v = 0$

(N2) $\forall v \in V, \forall \lambda \in \mathbb{R},\ ||\lambda v|| = |\lambda| \cdot ||v||$

(N3) $\forall u, v \in V,\ ||u + v|| \leq ||u|| + ||v||$

Definition - Normed vector space
A pair of a vector space $(V, ||\cdot||)$
note $||\cdot||$ is a metric on $V \implies$ normed vector space a metric space.

2.1.4 Open sets in metric spaces

Definition 2.6. **Open ball in metric spaces**

(X, d), with $x \in X, \epsilon \in \mathbb{R}; \epsilon > 0$

$$\text{Ball radius } \epsilon;\ B_\epsilon(x) = \{x' \in X | d(x, x') < \epsilon\}$$

notation; $B_\epsilon(x, X, d)$

Definition 2.7. **Open set in metric space**

(X, d) a metric space. $U \subseteq X$ open in (X, d) if:

$$\forall u \in U,\ \exists \delta > 0 \in \mathbb{R} \text{ s.t } B_\delta(u) \subset U$$

Definition 2.8. **Topologically equivalent**

d_1, d_2 metrics on a set X **topologically equivalent** if:

$$\forall\ U \subseteq X,\ U \text{ open in } (X, d_1) \iff U \text{ open in } (X, d_2)$$

2.1.5 Convergence in Metric Spaces

Definition 2.9. **Convergence in Metric Spaces**

(X, d) a metric space. $(x_n)_{n\geq 1}$ a sequence in X.
Say $(x_n) \to x \in (X, d)$ if

$$\forall\ \epsilon > 0, \exists N \in \mathbb{N} \text{ s.t } \forall\ n \geq N, d(x, x_n) < \epsilon$$

Lemma 2.7. - if (x_n) converges in $(X, d) \implies$ limit is unique
Corollary - d_1, d_2 topologically equivalent $\iff (x_n)$ converges in (X, d_1) and (X, d_2)

2.1.6 Closed sets in metric spaces

Definition 2.10. **Closed set in Metric Spaces**

(X, d) a metric space. $V \subseteq X$ a set.
V **closed** in (X, d) if $\forall\ (x_n) \in V$ s.t $(x_n) \to x$ convergent in $(X, d) \implies x \in V$

Theorem 2.9.

(X, d) a metric space. $V \subseteq X$

$$V \text{ closed in } (X, d) \iff X \backslash V \text{ open in } (X, d)$$

Lemma 2.10

(i) Intersection of closed sets in (X, d) is a closed set in (X, d)

(ii) Finite union of closed sets in (X, d) a closed set in (X, d)

2.1.7 Interior, isolated, limit, and boundary points in metric spaces

Definition 2.11. - 2.12.

(X, d) a metric space, $V \subset X,\ x \in X$

(i) x an **interior/inner point** of V if
$$\exists \delta > 0, \quad \text{s.t } B_\delta(x) \subset V$$

(a) **Interior of V; V°** - $\{v \in V : v \text{ an interior point of } V\}$

(ii) x a **limit/accumulation point** of V if
$$\forall \delta > 0, (B_\delta(x) \cap V) \backslash \{x\} \neq \emptyset$$

Note: *not all limit points of V are in V*

(b) **Closure of V; $\bar{V}$** - $V \cup \{v \text{ a limit point of } V\}$

(iii) x a **boundary point of V** if
$$\forall \delta > 0, B_\delta \cap V \neq \emptyset \text{ and } B_\delta(x) \backslash V \neq \emptyset$$

(c) **Boundary of V; ∂V -** $\{v \in X : v \text{ a boundary point of } V\}$

(iv) x an **isolated point** of V if
$$\exists \delta > 0,\ \text{s.t } V \cap B_\delta(x) = \{x\}$$

Lemma 2.11 (X, d) a metric space, $V \subseteq X$
$x \in X$ a limit point of $V \iff \exists$ sequence in $V \backslash \{x\}$ converging to x.

Definition 2.13. **Dense and Seperable subsets**

(X, d) a metric space

- $V \subseteq X$ **dense** in X if $\bar{V} = X$
- (X, d) seperable if, $\exists$ dense countable subset of X

2.1.8 Continuous maps of metric spaces

Definition 2.14. **Continuity in metric spaces**

$(X, d_X), (Y, d_Y)$ metric spaces.
$f : X \to Y$ a map

(i) f **continuous** at $x \in X$ if
$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t } \forall x' \in X \text{ s.t } d_X(x', x) < \delta, d_Y(f(x), f(x')) < \epsilon$$

(ii) $f : X \to Y$ continuous if f continuous $\forall x \in X$

(iii) $f : X \to Y$ uniformly continuous if f continuous $\forall x \in X$ with $\delta = \delta(\epsilon)$ not depending on x

Theorem 2.12.

$(A_1, d_1), (A_2, d_2)$ metric spaces
$f : A_1 \to A_2$ continuous $\iff$ pre-image of any open set in A_2 is an open set in A_1
$f : A_1 \to A_2$ continuous $\iff$ pre-image of any closed set in A_2 is a closed set in A_1

Theorem 2.13.

$(X, d_X), (Y, d_Y)$ metric spaces
$f : X \to Y$ a map;

$$f \text{ continuous at } x \in X \iff \text{for any sequence } (x_n) \to x;\ f(x_n) \to f(x) \text{ in } (Y, d_Y)$$

Definition 2.15. **Homeomorphism**

$(X_1, d_1), (X_2, d_2)$ metric spaces.

(i) $f : X_1 \to X_2$ a **homeomorphism** if

- $f : X_1 \to X_2$ a bijection
- $f : X_1 \to X_2$ and $f^{-1} : X_2 \to X_1$ continuous

(ii) Say $(X_1, d_1), (X_2, d_2)$ **homeomorphic** if $\exists$ homeomorphism from X_1 to X_2

Definition 2.16.

$(X, d_X), (Y, d_Y)$ metric spaces with $f : X \to Y$

(i) f is **Lipschitz** if $\exists$ constant $M > 0$ s.t $\forall x_1, x_2 \in X, d_Y(f(x_1), f(x_2)) \leq M \cdot d_X(x_1, x_2)$

(ii) f is **bi-Lipschitz** if $\exists$ constants $M_1, M_2 > 0$ s.t $\forall x_1, x_2 \in X$

$$M_2 \cdot d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq M_1 \cdot d_X(x_1, x_2)$$

Corollary; *any bi-Lipschitz map is injective*

(iii) f an **isometry/distance preserving** if $\forall x_1, x_2 \in X$;

$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$$

2.2 Topological Spaces

2.2.2 Topology on a set

Definition 2.17. **Topology**

A an arbitrary set. τ a collection of subsets of A
τ a **topology** on A if:

(T1) $\emptyset \in \tau$ and $A \in \tau$

(T2) $G_\alpha \in \tau$ for α in a (finite) set $I \implies \bigcup_{\alpha \in I} G_\alpha \in \tau$

(T3) $G_1, G_2, \ldots, G_m \in \tau \implies \bigcap_{i=1}^m G_i \in \tau$

A **topological space**; (A, τ) a pair of a set A and topology τ on A. Each element in τ an open set in (A, τ)
U a neighbourhood of a if $U \in \tau$ and $a \in U$

Example 2.25. Some Topologies

1. **Coarse topology -** A arbitrary set, $\tau = \{\emptyset, A\}$
2. **Induced topology -** (X, d) a metric space, with τ the collection of all open sets in (X, d)
3. **Order Topology -** $A = \mathbb{R}$ with τ collection of subsets of $\mathbb{R}$ of form $(a, +\infty)$, $a \in \mathbb{R} \cup \{-\infty, +\infty\}, (infty, +\infty) := \emptyset$
4. **Discrete Topology** - A arbitrary, $\tau = \mathcal{P}(A)$
5. **Product topology -**

Definition. **Metrisable topological space**
Say topological space (X, τ) **metrisable** if $\exists$ metric on X which induces a topology τ.
Definition. **Induced and Subspace topology**
(X, τ) a topological space. $Y \subset X$

$$\tau_Y = \{U \cap Y | U \in \tau\}$$

τ_Y the **induced topology** on Y from (X, τ)
(Y, τ_Y) has the **subspace topology** induced from (X, τ)

Definition 2.18. **Stronger topology**

A a set, with τ_1, τ_2
Say τ_1 stronger (or finer) than τ_2 if $\tau_2 \subset \tau_1$
Lemma 2.14.
(A, τ)
A set $G \subset A$ open $\iff \forall\ x \in G,\ \exists$ neighbourhood of x contained in G

Definition 2.19. **Interior in Topological space**

(A, τ) a topological space. $\Omega \subseteq A$
$z \in \Omega$ an interior point of Ω if

$$\exists U \in \tau \text{ s.t } z \in U \text{ and } U \subset \Omega$$

interior of Ω; $\Omega^\circ = \{z \in \Omega | z \text{ an interior point of } \Omega\}$
Properties of interior

- $S \subset T \implies S^\circ \subset T^\circ$
- S open in $A \iff S = S^\circ$
- S° largest open set contained in S

2.2.3 Convergence, and Hausdorff property

Definition 2.20. **Convergence in Topological Spaces**

(A, τ) a topological space. $(x_n)_{n \geq 1}$ a sequence in A
(x_n) **converges** in (A, τ) if

$$\exists x \in A \text{ s.t } \forall\ G \in \tau \text{ with } x \in G,\ \exists N \in \mathbb{N},\ \text{s.t } \forall n \geq N, x_n \in G$$

Definition 2.21. **Hausdorff**

(A, τ) called **Hausdorff** if:

$$\forall x, y \in A\ x \neq y,\ \exists \text{ open set } U, V \text{ s.t } x \in U, y \in V \text{ and } U \cap V = \emptyset$$

Say U and V seperate x and y

Theorem 2.14.

(A, τ) a Hausdorff topological space. (x_n) a sequence in A.
if (x_n) convergent in $(A, \tau) \implies$ limit is unique.

2.2.4 Closed sets in topological spaces

Definition 2.22. **Closed set in Topological space**

(A, τ) a topological space.
$V \subseteq A$. Say V closed in $(A, \tau) \iff A \backslash V \in \tau$

Lemma 2.17.
(A, τ) a topological space $\implies \emptyset$ and A closed in (A, τ)

(i) intersection of closed sets in (A, τ) is a closed set in (A, τ)

(ii) union of a finite number of closed sets in (A, τ) is a closed set in (A, τ)

Definition 2.23. **Limit/Accumulation point in Topological Spaces**

(A, τ), a topological space, $S \subseteq A$
$x \in A$ a **limit/accumulation point** of S if

$$\forall\ U \text{ a neighbourhood of } x,\ (S \cap U) \backslash \{x\} \neq \emptyset$$

x not necessarily in S
Closure of S, $\bar{S}$ $= S \cup \{x \in A | x \text{ a limit point of } S\}$
Lemma
S closed in $(A, \tau) \iff S = \bar{S}$

2.2.5 Continuous maps on topological spaces

Definition 2.24. **Continuity in topological space**

$(X, \tau_X), (Y, \tau_Y)$ with $f : X \to Y$
f continuous on X if:

$$\forall \text{open sets } U \in Y,\ f^{-1}(U) \text{ open in } X$$

Theorem 2.20.

$(X, \tau_X), (Y, \tau_Y)$ with $f : X \to Y$
f continuous $\iff$ pre-image of closed set in Y is closed in X

Theorem 2.21.

$(X, \tau_X), (Y, \tau_Y), (Z, \tau_Z)$
$f : X \to Y, g : Y \to Z$ continuous $\implies$ $g \circ f : X \to Z$ continuous

Definition 2.25. **Homeomorphisms in Topological space**

$f : X \to Y$ a homeomorphism is $f : X \to Y$ bijective with f and f^{-1} continuous

Definition 2.25. **Topologically equivalent in Topological space**

$(X, \tau_X), (Y, \tau_Y)$ **topologically equivalent/homeomorphic** if $\exists$ homeomorphism from $X \to Y$

2.3 Connectedness

2.3.1 Connected sets

Definition 2.26. **Disconnected sets**

For (X, d) a metric space, consider $T \subseteq X$. T **disconnected**,if $\exists$ open sets $U, V \in X$ s.t:

(i) $U \cap V = \emptyset$

(ii) $T \subseteq U \cup V$

(iii) $T \cap U \neq \emptyset$ and $T \cap V \neq \emptyset$

Set connected if not disconnected. i.e for any 2 of the properties that hold from above the 3rd cannot.
Lemma 2.23.
(X, d) a metric space. $T \subseteq X$

$$T \text{ disconnected} \iff \exists \text{ continuous } f : T \to \mathbb{R} \text{ s.t } f(T) = \{0, 1\}$$

Theorem 2.22.

Consider $(\mathbb{R}, d)$, $S \subseteq \mathbb{R}$

$$S \text{ connected} \iff S \text{ an interval}$$

2.3.2 Continuous maps + Connected sets

Theorem 2.27.

(A, d_1) and (A, d_2) metric spaces. $f : A_1 \to A_2$ continuous map
$S \subset A$ connected $\implies$ $f(S)$ connected
Corollary 2.28.
$f : (X, d_X) \to (Y, d_Y)$ a homeomorphism

$$X \text{ connected} \iff Y \text{ connected}$$

Theorem 2.29.

(X, d) connected metric space, $f : X \to \mathbb{R}$ continuous. Assume $\exists a, b \in X$ s.t $f(a) < 0, f(b) > 0 \implies \exists c \in X$ s.t $f(c) = 0$

2.3.3 Path Connected Sets

Definition 2.28. Path

Under (X, d) given $a, b \in X$
Path from $a \to b$ a continuous map $f : [0, 1] \to X$ s.t $f(0) = a, f(1) = b$

Definition 2.29. Path Connected

(X, d) path connected if $\forall a, b \in X, \exists$ path from $a \to b$ in X

Theorem 2.30.

if (X, d) path connected $\implies$ connected

2.4 Compactness

2.4.1 Compactness by covers

Definition 2.30. Covers

(X, d) a metric space. $Y \subseteq X$

(i) collection R of open subsets of X an **open cover** for Y if

$$Y \subseteq \bigcup_{v \in R} v$$

(ii) Given open cover R for Y
Say C a **sub-cover** of R for Y if $C \subseteq R$ and $Y \subseteq \bigcup_{v \in R} v$

(iii) Open cover R for Y is a **finite cover** if R has finitely many elements.

Definition 2.31. Compact

(X, d) a metric space
$Y \subseteq X$ compact in (X, d) if every open cover for Y has a finite sub-cover.

Proposition 2.32.
$a, b \in \mathbb{R},\ a \leq b$ in (R, d_1) we have $[a, b]$ compact

Proposition 2.33.
(X, d) a metric space, $Y \subseteq X$
X compact, Y closed $\implies Y$ compact.

Theorem 2.34.

(X, d) a metric space $Y \subset X$

$$Y \text{ compact} \implies Y \text{ closed}$$

Theorem 2.35.

$(X, d_X), (Y, d_Y)$ metric spaces. Considering $(X \times Y, d)$
$d((x_1, y_1), (x_2, y_2)) = d_1(x_1, x_2) + d_2(y_1, y_2)$
X, Y compact $\implies (X \times Y, d)$ compact
Corollary.
$[a_1, b_1] \times [a_2, b_2] \cdots \times [a_{n-1}, b_{n-1}] \times [a_n, b_n]$ compact in $\mathbb{R}^n$

Definition 2.32. Bounded

(X, d) non-empty metric space, $Z \subseteq X$
Z **bounded** in (X, d) if $\exists M \in \mathbb{R}$ s.t $\forall x, y \in Z; d(x, y) \leq M$
S arbitrary set. $f : S \to X$ bounded if $f(S)$ bounded in X
Lemma 2.37.
(X, d) compact metric space $\implies X$ bounded

Theorem 2.36. Heine-Borel

Consider $(\mathbb{R}^n, d_2)$, $X \subseteq \mathbb{R}^n$
X compact $\iff X$ closed and bounded

2.4.2 Sequential Compactness

Definition 2.33. Sequentially compact

(X, d) sequentially compact, if for every sequence in X has convergent subsequence in (X, d)

$$\forall (x_n)_{n\geq 1} \in X,\ \exists (x_{n_k})_{k\geq 1},\ x \in X \text{ s.t } x_{n_k} \to x$$

Lemma 2.39.
(X, d) a metric space. with sequence $(x_n)_{n\geq 1}$ s.t $\exists (x_{n_k})_{k\geq 1},\ x \in X$ s.t $x_{n_k} \to x$.

$$\iff \exists x \in X \text{ s.t } \forall \epsilon > 0 \text{ there are infinitely many } i \text{ s.t } x_i \in B_\epsilon(x)$$

Theorem 2.41. Bolzanno-Weierstrass

Any bounded sequence in $\mathbb{R}^n$ has convergent subsequence.

Theorem 2.40. + 2.42.

(X, d) metric space.

$$X \text{ Compact } \iff X \text{ Sequentially Compact}$$

2.4.3 Continuous maps + Compact Sets

Theorem 2.41.

$(X, d_X), (Y, d_Y)$ metric spaces.
$f : X \to Y$ a continuous map if

$$Z \text{ compact in } X \implies f(Z) \text{ compact in } Y$$

Corollary 2.44.
$(X, d_X), (Y, d_Y)$ metric spaces, $f : X \to Y$ a homeomorphism

$$\implies X \text{ compact } \iff Y \text{ compact}$$

Theorem 2.45.

Every continuous map from compact metric space to a metric space is uniformly continuous.
Corollary 2.46. $f : [a, b] \to \mathbb{R}$ continuous $\implies f$ uniformly continuous

Theorem 2.47.

(X, d_X) compact, $f : X \to \mathbb{R}$ continuous $\implies f$ bounded above and below attaining its upper & lower bounds

Theorem 2.48.

$f : \mathbb{R} \to \mathbb{R}$ continuous w.r.t Euclidean metrics on domain and range.
$\forall\ [a, b]$ we have $f([a, b])$ of the form $[m, M]$ for $m, M \in \mathbb{R}$

2.5 Completeness

2.5.1 Complete metric spaces Banach space

Definition 2.34. Cauchy Sequence

(X, d) a metric $(x_n)_{n\geq 1}$ sequence in X
Say $(x_n)_{n\geq 1}$ a **Cauchy sequence** in (X, d) if

$$\forall \epsilon > 0, \exists N_\epsilon \in \mathbb{N} \text{ s.t } \forall n, m \geq N_\epsilon \text{ we have } d(x_n, x_m) < \epsilon$$

Definition 2.35. Complete & Banach

(i) metric space (X, d) **complete** if every Cauchy sequence in X converges to a limit in X

(ii) Normed vector space $(V, ||\cdot||)$ a **Banach space** if V with induced metric space $d_{||||}$ a complete metric space.

Theorem 2.51.

Assume $(f_n : [a, b] \to \mathbb{R})_{n\geq 1}$ sequence of continuous functions converging uniformly to $f : [a, b] \to \mathbb{R} \implies f : [a, b] \to \mathbb{R}$ continuous

Theorem 2.52.

Metric space $(C([a, b]), d_\infty)$ is complete or equivalently $(C([a, b]), ||\cdot||_\infty)$ a Banach space

Theorem 2.53.

(X, d) a compact metric space $\implies (X, d)$ complete

2.5.2 Arzelà-Ascoli

Definition 2.36. **Uniformly bounded & Uniformly equi-continuous**

Let $\mathcal{C}$ a collection of functions $f : [a, b] \to \mathbb{R}$

1. Say collection $\mathcal{C}$ **uniformly bounded** if $\exists M$ s.t $\forall f \in \mathcal{C}$ and $\forall x \in [a, b] \implies |f(x)| < M$
2. Say collection $\mathcal{C}$ **uniformly equi-continuous** if $\forall \epsilon > 0, \exists \delta > 0$ s.t $\forall f \in \mathcal{C}$ and $\forall x_1, x_2 \in [a, b]$ s.t $|x_1 - x_2| < \delta$ we have $|f(x_1) - f(x_2)| < \epsilon$

Theorem 2.54. **Arzelà-Ascoli**

Assume $\mathcal{C}$ collection of continuous functions $f : [a, b] \to \mathbb{R}$ if $\mathcal{C}$ uniformly bounded and uniformly equi-continuous $\implies$ every sequence in $\mathcal{C}$ has convergent subsequence in $(C([a, b], d_\infty)$

2.5.3 Fixed point theorem

Definition 2.37. **Contracting**

(X_1, d_1) and (X_2, d_2), with $f : X_1 \to X_2$
Say f **contracting** if $\exists K \in (0, 1)$ s.t $\forall a, b \in X$ we have

$$d_2(f(a), f(b)) \leq K \cdot d_1(a, b)$$

Every contracting map is continuous.

Definition 2.37. **Fixed point**

$f : X \to X$ say $x \in X$ a **fixed point** of f if $f(x) = x$

Theorem 2.55. **Banach fixed point theorem**

(X, d) a non-empty complete metric space.
$f : X \to X$ a contracting map $\implies$ f has unique fixed point in X

Part II

Term 2 - Complex Analysis

1 Holomorphic Functions

1.1 Complex Numbers

Definition 1.1. i

$$i = \sqrt{-1}, \quad i^2 = -1$$

Root of $x^2 + 1 = 0$

Basic properties

$$z = x + iy, \quad Re(z) = x, \ Im(z) = y$$

The complex conjugate:

$$\bar{z} = x - iy$$

Polar Coordinates
$z = x + iy$

$$r = |z| = \sqrt{x^2 + y^2}$$

$$x = r\cos\theta, \ y = r\sin\theta$$

$$z = r(\cos\theta + i\sin\theta$$

De-Moivre's Formula

$$z^n = r^n(\cos(n\theta) + i\sin(n'\theta)), \ n \in \mathbb{Z}^+$$

Eulers's Formula

$$e^{i\theta} = (\cos\theta + i\sin\theta)$$

1.2 Sets in Complex Plane

Definition 1.2. Discs in $\mathbb{C}$

Open Disc : $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$
Boundary of Disc : $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$
Unit Disc : $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

Definition 1.3. **Interior Point**

$\Omega \in \mathbb{C}, z_0$ an **interior point** of Ω if $\exists r > 0$ s.t $D_r(z_0) \subset \Omega$

Definition 1.4.

Set Ω **open** if $\forall \omega \in \Omega$, ω an interior point

Definition 1.5.

Set Ω **closed** if $\Omega^C = \mathbb{C} \backslash \Omega$ open
Closed $\iff$ contains all its limit points.

Definition 1.6. Closure

Closure of $\Omega = \bar{\Omega} = \{\Omega \cup$ limit points of $\Omega\}$

Definition 1.7. Boundary

Boundary of $\Omega = \underbrace{\bar{\Omega}}_{\text{Closure}} \backslash \underbrace{\partial\Omega}_{\text{interior}}$

Definition 1.8. Bounded

Ω bounded if $\exists M > 0$ s.t $|\omega| < M \ \forall \omega \in \Omega$

Definition 1.9. Diameter

$$diam(\Omega) = \sup_{z,w \in \Omega} |z - w|$$

Definition 1.10. Compact

Ω compact if closed and bounded

Theorem 1.1.

$$\begin{aligned} \Omega \text{ compact } &\iff \text{every sequence} \{z_n\} \subset \Omega \text{ has a subsequence convergent in } \Omega \\ &\iff \text{every open covering of } \Omega \text{ has a finite subcover} \end{aligned}$$

Theorem 1.2.

if $\Omega_1 \supset \Omega_2 \supset \dots \Omega_n \supset \dots$ a sequence of non-empty compact sets
s.t $\lim_{n \to \infty} diam(\Omega_n) \to 0$

$$\implies \exists! w \in \mathbb{C},\ w \in \Omega_n\ \forall n$$

Definition 1.11. Connected

Open set Ω **connected** $\iff$ any 2 points in Ω joined by curve γ entirely contained in Ω

1.3 Complex Functions

Definition 1.12.

$\Omega_1, \Omega_2 \subset \mathbb{C}$

$$f : \Omega_1 \to \Omega_2$$

a **mapping** $\Omega_1 \to \Omega_2$ if

$$\forall z = x + iy \in \Omega_1$$

$$\exists! w = u + iv \in \Omega_2,\ s.t\ w = f(z)$$

We have $w = f(z) = u(x, y) + iv(x, y)$
$u, v : \mathbb{R}^2 \to \mathbb{R}$

Definition 1.13.

f defined on $\Omega_1 \subset \mathbb{C}$ f **continuous** at $z_0 \in \Omega$ if

$$\forall \epsilon > 0 \exists \delta > 0\ s.t\ |z - z_0| < \delta \implies |f(z) - f(z_0)| < \epsilon$$

f continuous if continuous $\forall z \in \Omega$

1.4 Complex Derivative

Definition 1.14. Holomorphic

$\Omega_1, \Omega_2 \subset \mathbb{C}$ open sets
$f : \Omega_1 \to \Omega_2$
Say f **differentiable/holomorphic** at z_0 if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} = f'(z_0) \textbf{ exists}$$

f holomorphic on open set Ω if holomorphic at every point of Ω
Lemma
f holomorphic at $z_0 \in \Omega \iff \exists\ a \in \mathbb{C}$ s.t

$$f(z_0 + h) - f(z_0) - ah = h\Psi(h)$$

For Ψ a function defined for all small h with $\lim_{h \to 0} \Psi(h) = 0$, $a = f'(z_0)$
Corollary
f holomorphic $\implies$ f continuous

Proposition
f, g holomorphic in $\Omega \implies$

(i) $(f+g)' = f' + g'$

(ii) $(fg)' = f'g + fg'$

(iii) $g(z_0) \neq 0 \implies (\frac{f}{g})' = \frac{f'g - fg'}{g^2}$

(iv) $f : \Omega \to V,\ g : \to \mathbb{C}$ holomorphic
$\implies [g \circ f(z)]' = g'(f(z))f'(z)\ \forall z \in \Omega$

1.5 Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$
$$u'_x = v'_y \qquad u'_y = -v'_x$$

Definition 1.15.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

Theorem 1.3.

$f(z) = u(x,y) + iv(x,y) \quad z = x + iy$
f holomorphic at $z_0 \implies$
$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

Theorem 1.4.

$f = u + iv$ complex-valued function on open set Ω
u, v continuously differentiable, satisfying Cauchy-Riemann equations $\implies f$ holomorphic on Ω with $f'(z) = \frac{\partial f}{\partial z}(z)$

1.6 Cauchy-Riemann equations in polar

For $f = u + iv$ we have
$$u'_r = \frac{1}{r}v'_\theta \qquad v'_r = -\frac{1}{r}u'_\theta$$

1.7 Power Series

Definition 1.16. Power Series

Of the form
$$\sum_{n=0}^{\infty} a_n z^n \quad a_n \in \mathbb{C}$$

Series converge at z if $S_N(z) = \sum_{n=0}^{N} a_n z^n$ has limit $S(z) = \lim_{n\to\infty} S_N(z)$

Theorem 1.5.

Given power series $\sum_{n=0}^{\infty} a_n z^n$, $\exists 0 \leq R \leq \infty$ s.t

(i) if $|z| < R \implies$ series converges absolutely

(ii) $|z| > R \implies$ series diverges

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} \qquad \text{(Radius of Convergence)}$$

Theorem 1.6.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Defines holomorphic function on its disc of convergence. With

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

with same radius of convergence as f.
Power series infinitely differentiable in the disc of convergence, achieved through term-wise differentiation.

Definition 1.17. Entire

A function said to be **entire** if holomorphic $\forall z \in \mathbb{C}$

1.8 Elementary functions

1.8.1 Exponential function

$$e^z = e^x \cos y + i e^x \sin y \qquad z = x + iy \in \mathbb{C}$$

Properties

(i) $y = 0 \implies e^z = e^x$

(ii) e^z is entire

(iii) $g(z)$ holomorphic
$\implies \frac{\partial}{\partial z} e^{g(z)} = e^{g(z)} g'(z)$

(iv) $z_1, z_2 \in \mathbb{C} \quad e^{z_1 + z_2} = e^{z_1} e^{z_2}$

(v) $|e^z| = |e^x||e^{iy}| = e^x \sqrt{\cos^2 x + \sin^2(x)} = e^x$

(vi) $(e^{iy})^n = e^{iny}$

(vii) $arg(z) = \arctan(y/x)$
$arg(e^z) = y + 2\pi k, \quad k \in \mathbb{Z}$

1.8.2 Trigonometric functions

Definition 1.18.

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \qquad \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Properties

(i) $\sin z, \cos z$ are entire

(ii) $\frac{\partial}{\partial z} \sin z = \cos z \quad \frac{\partial}{\partial z} \cos z = -\sin z$

(iii) $\sin^2 z + \cos^2 z = 1$

(iv) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$
$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

1.8.3 Logarithmic Functions

Definition 1.19.

$$log(z) = ln|z| + i\arg(z) = log(r) + i(\theta + 2\pi k) \quad z \neq 0,\ k \in \mathbb{Z}$$

$\log(z)$ a multi-valued function

Definition 1.20.

$\text{Log}(z) = \ln|z| + i\,\text{Arg}(z)$ for $\text{Arg}(z)$ principal value $\in [-\pi, \pi]$
Properties

(i) $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

(ii) $\text{Log}(z)$ holomorphic in $\mathbb{C}\backslash\{(\infty, 0]\}$

1.8.4 Powers

Definition 1.21.

$\alpha \in \mathbb{C}$ define $z^\alpha := e^{\alpha \log(z)}$ as a multi-valued function

Definition 1.22.

Principal value of $z^\alpha,\ \alpha \in \mathbb{C}$ as $z^\alpha = e^{\alpha\,\text{Log}(z)}$ ***Properties***

(i) $z^{a_1} z^{a_2} = z^{a_1 + a_2}$

2 Cauchy's Integral Formula

2.1 Parametrised Curve

Definition 2.1.

Parametrised curve a function $z(t) : [a, b] \to \mathbb{C}$

Smooth if $z'(t)$ exists and is continuous on $[a, b]$ with $z'(t) \neq 0 \forall t \in [a, b]$ Taking one-sided limits for $z'(a), z'(b)$.

Piecewise-smooth if z continuous on $[a, b]$ and if $\exists$ finitely many points $a = a_0 < a_1 < \cdots < a_n = b$ s.t $z(t)$ smooth on $[a_k, a_{k+1}]$

$$z : [a, b] \to \mathbb{C} \quad \tilde{z} : [c, d] \to \mathbb{C}$$

equivalent if $\exists$ continuously differentiable bijection $s \to t(s)$ from $[c, d]$ to $[a, b]$ s.t $t'(s) > 0$ and $\tilde{z}(s) = z(t(s)) =$

Definition 2.2. Path Integrals

Path integral given smooth $\gamma \subset \mathbb{C}$ parametrised by $z : [a, b] \to \mathbb{C}$.
f continuous function on γ

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$$

independent of choice of parametrization.
If γ piece-wise smooth

$$\int_\gamma f(z)dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))z'(t)dt$$

Definition 2.3.

Define curve γ^- obtained by reversing orientation of γ
Can take $z^- : [a, b] \to \mathbb{C}$ s.t $z^-(t) = z(b + a - t)$

Definition 2.4. Closed Curve

Smooth/piece-wise smooth curve closed if $z(a) = z(b)$ for any parametrisation.

Definition 2.5. Simple Curve

Smooth/piece-wise smooth curve simple if not **self-intersecting**

$$z(t) \neq z(s) \text{ unless } s = t \in [a, b]$$

2.2 Integration along Curves

Definition 2.6. Length of smooth curve

$$\text{Length}(\gamma) = \int_a^b |z'(t)|dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$$

Theorem 2.1. Properties of Integration

(i) $\int_\gamma af(z) + bg(z)dz = a\int_\gamma f(z)dz + b\int_\gamma g(z)dz$

(ii) γ^- reverse orientation of γ

$$\implies \int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz$$

(iii) **M-L inequality**

$$\left|\int_\gamma f(z)dz\right| \leq \sup_{z\in\gamma} |f(z)| \cdot \text{length}(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}dt$$

2.3 Primitive Functions

Definition 2.7. Primitive

A **Primitive** for f on $\Omega \subset \mathbb{C}$ a function F holomorphic on Ω s.t $F'(z) = f(z)\ \forall z \in \Omega$

Theorem 2.2.

Continuous function f with primitive F in open set Ω and curve γ in Ω from $w_1 \to w_2$

$$\int_\gamma f(z)dz = F(w_2) - F(w_1)$$

Corollary
γ closed curve in open set Ω f continuous and has primitive in $\Omega \implies$

$$\oint_\gamma f(z)dz = 0$$

Corollary
$Omega$ with $f' = 0 \implies f$ constant

2.4 Properties of Holomorphic functions

Theorem 2.3.

Let $\Omega \subset \mathbb{C}$ open set
$T \subset \Omega$ a triangle whose interior contained in Ω

$$\implies \oint_T f(z)dz = 0$$

for f holomorphic in Ω
Corollary
f holomorphic on open set Ω containing rectangle R in its interior

$$\implies \oint_R f(z)dz = 0$$

2.5 Local existence of primitives and Cauch-Goursat theorem in a disc

Theorem 2.4.

Holomorphic functions in open disc have a primitive in that disc
Corollary **- (Cauchy-Goursat Theorem for a disc)**
f holomorphic in disc $\implies \oint_\gamma f(z)dz = 0$
for any closed curve γ in that disc
Corollary
Suppose f holomorphic in open set containing circle C and its interior

$$\implies \oint_C f(z)dz = 0$$

2.6 Homotopies and simply connected domains

Definition 2.8. Homotopic

γ_0, γ_1 **homotopic** in Ω if $\forall s \in [0,1], \exists$ curve $\gamma \subset \Omega$ with $\gamma_s(t)$ s.t

$$\gamma_s(a) = \alpha \qquad \gamma_s(b) = \beta$$

$\forall t \in [a,b] : \gamma_s(t)|_{s=0} = \gamma_0(t) \quad \gamma_s(t)|_{s=1} = \gamma_1(t)$
With $\gamma_s(t)$ jointly continuous in $s \in [0,1]$ and $t \in [a,b]$

Theorem 2.5.

γ_0, γ_1 homotopic, f holomorphic

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Definition 2.9.

Open set $\Omega \subset \mathbb{C}$ **simply connected** if any 2 pair of curves in Ω with shared end-points homotopic.

Theorem 2.6.

Any holomorphic function in simply connected domain has a primitive.

***Corollary* - (Cauchy-Goursat Theorem)**

f holomorphic in simply connected open set Ω

$$\implies \oint_\gamma f(z)dz = 0$$

for any closed piecewise-smooth curve $\gamma \subset \Omega$

Theorem 2.7. (Deformation Theorem)

γ_1 and γ_2, 2 simple closed piecewise-smooth curves with γ_2 lying wholly inside γ_1
f holomorphic in domain containing region between γ_1, γ_2

$$\implies \oint_{\gamma_1} f(z)dz = \oint_{\gamma_2} f(z)dz$$

2.7 Cauchy's Integral Formulae

Theorem 2.8. (Cauchy's Integral Formula)

f holomorphic inside and on simple closed piecewise-smooth curve γ
$\forall z_0$ interior $to \gamma$

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} dz$$

Theorem 2.9. *(Generalised cauchy's integral formula)*

f holomorphic in open set Ω.
γ simple,closed piecewise-smooth Ω
$\forall z$ interior to γ

$$\implies \frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(t)}{(t-z)^{n+1}} dt$$

Corollary

f holomorphic $\implies$ all its derivatives are too.

3 Applications of Cauchy's integral formula

Corollary - (Liouville's theorem)
if an entire function bounded $\implies f$ constant

Theorem 3.1. (Fundamental theorem of algebra)

Every polynomial of degree > 0 with complex coefficients has at least one zero.
Corollary Every polynomial $P(z) = a_n z^n + \cdots + a_0$ of degree $n \geq 1$ has precisely n roots in C

Theorem 3.2. (Morera's theorem)

Suppose f continuous in open disc D s.t $\forall$ triangle $T \subset D$

$$\int_T f(z)dz = 0 \implies f \text{ holomorphic}$$

3.1 Taylor + Maclaurin Series

Theorem 3.3. (Taylor's expansion theorem

f holomorphic in Ω, $z_0 \in \Omega$

$$\implies f(z) = f(z_0) = f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots$$

Valid in all circles $\{z : |z - z_0| < r\} \subset \Omega$

Definition 3.1. (Taylor Series)

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots = \sum_{i=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Definition 3.2. (Maclaurin Series)

Taylor series for $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} z^n$$

3.2 Sequences of holomorphic functions

Theorem 3.4.

if $\{f_n\}_{n=1}^{\infty}$ a sequence of holomorphic functions converging uniformly to f in every compact subset of $\Omega \implies f$ holomorphic in Ω
Corollary

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

f_n holomorphic in $\Omega \subset \mathbb{C}$
Given series converges uniformmly in compact subsets of $\Omega \implies F(z)$ holomorphic

Theorem 3.5.

Sequence $\{f_n\}_{n=1}^{\infty} \xrightarrow[unif]{} f$ in every compact subset of $\Omega \implies$ sequence $\{f_n'\}_{n=1}^{\infty} \xrightarrow[unif]{} f'$ in every compact subset of Ω

3.3 Holomorphic functions defined in terms of integrals

Theorem 3.6.

Let $F(z, s)$ defined for $(z, s) \in \Omega \times [0, 1]$
$\Omega \subset \mathbb{C}$ open set. Given F satisfies

(i) $F(z, s)$ holomorphic in $\Omega \forall s$

(ii) F continuous on $\Omega \times [0, 1]$

$\implies f(z) := \int_0^1 F(z, s)ds$ holomorphic

3.4 Schwarz reflection principle

Definition 3.3.

$\Omega \subset \mathbb{C}$ open and **symmetric** w.r.t real line

$$z \in \Omega \iff \bar{z} \in \Omega$$

Definition 3.4.

$$\Omega^+ = \{z \in \Omega : Im(z) > 0\} \quad \Omega^- = \{z \in \Omega : Im(z) < 0\} \quad I = \{z \in \Omega : Im(z) = 0\}$$

Theorem 3.7. (Symmetry Principle)

f^+, f^- holomorphic in Ω^+, Ω^- respectively.
Extend continuously to I s.t $f^+(x) = f^-(x) \quad \forall x \in I$

$$f(z) := \begin{cases} f^+(z), & z \in \Omega^+ \\ f^+(z) = f^-(z), & z \in I \\ f^-(z), & z \in \Omega^- \end{cases} \qquad \text{holomorphic}$$

Theorem 3.8. (Schwarz relfection principle)

f holomorphic in Ω^+ extend continuously to I s.t f real-valued on I
$\implies \exists F$ holomorphic in Ω s.t $F|_{\Omega^+} = f$

4 Meromorphic Functions

4.1 Complex Logarithm

Theorem 4.1.

Ω simply connected, $1 \in \Omega, 0 \notin \Omega$
$\implies$ in Ω there is a branch of logarithm

$$F(z) = \log_{\Omega}(z)$$

Satisfying

(i) F holomorphic in Ω

(ii) $e^{F(z)} = z \ \forall z \in \Omega$

(iii) $F(r) = \log(r),\ r \in \mathbb{R}$ close to 1

Theorem 4.2.

Holomorphic f has 0 of order m at z_0
$\iff$ can be written in form

$$f(z) = (z - z_0)^m g(z)$$

g holomorphic at z_0, $g(z_0) \neq 0$

Corollary
Os of non-constant holomorphic function are isolated.
Every zero has neighbourhood, inside of which it is the only 0

4.2 Laurent Series

Definition 4.1.

Laurent Series for f at z_0, where series converge

$$f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n = \dots + a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1} + a_0 + a_1(z-z_0)^1 + a_2(z-z_0)^2 + \dots$$

Theorem 4.3. (Laurent Expansion theorem)

f holomorphic in anunulus $D = \{z : r < |z-z_0| < R\}$
$\implies f(z)$ expressed in form $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$

$$a_n = \frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)}{(\eta - z_0)^{n+1}} d\eta$$

γ simple,closed piecewise smooth curve in D with z_0 in its interior.

4.3 Poles of holomorphic functions

Definition 4.2.

z_0 a **singularity** of complex function f
if f not holomorphic at z_0, but every neighbourhood of z_0 has at least 1 holomorphic point.

Definition 4.3.

Singulartiy z_0 is **isolated** if $\exists$ neighbourhood of z_0, where it is the only singularity.

Definition 4.4.

f holomorphic with isolated singularity z_0
Considering Laurent expansion valid in some annulus

$$f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$$

$\implies$

- $a_n = 0 \; \forall n < 0 \implies z_0$ a **removable singularity**
- $a_n = 0 \forall n < -m, m \in \mathbb{Z}^+, a_{-m} \neq 0 \implies z_0$ pole of order m
- $a_n \neq 0$ for infinitely many negative $n \implies z_0$ a **essential singularity**

Theorem 4.4.

f has pole of order m at $z_0 \iff$ written in form

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

g holomorphic at z_0, $g(z_0) \neq 0$

4.4 Residue Theory

Definition 4.5.

Let $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$ for $0 < |z-z_0| < R$ the Laurent series for f at z_0
Residue of f at z_0 is

$$\implies Res[f, z_0] - a_{-1}$$

Theorem 4.5.

$\gamma \subset \{z : 0 < |z-z_0| < R\}$ simple closed piecewise-smooth curve containing z_0

$$\implies Res[f, z_0] = \frac{1}{2\pi i}\oint_\gamma f(z)dz$$

Theorem 4.6.

f holomorphic function inside and on simple closed piecewise-smooth curve γ except at the singularities $z_1, \dots, z_n$ in its interior

$$\implies \oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{n} Res[f, z_j]$$

4.5 The argument principle

Theorem 4.7. (Princple of argument)

f holomorphic in open Ω, except for finitely many poles.
γ simple closed piecewise-smooth curve in Ω not passing through poles or zeroes of f

$$\implies \oint_\gamma \frac{f'(z)}{f(z)} dz = 2\pi i(N - P)$$

$$N = \sum order(zeroes) \quad P = \sum order(poles)$$

Theorem 4.8. (Rouche's Theorem)

f, g holomorphic in open Ω
$\gamma \subset \Omega$ simple closed piecewise-smooth curve with interior containing only points of Ω

if $|g(z)| < |f(z)|, z \in \gamma$

$$\implies \sum_{0s \text{ of } f+g \text{ in } \gamma} order(zeros) = \sum_{0s \text{ of } f \text{ in } \gamma} order(zeros)$$

Definition 4.6.

Mapping **open** if maps open sets $\mapsto$ open sets

Theorem 4.9. (Open mapping theorem)

if f holomorphic and non-constant in open $\Omega \subset \mathbb{C}$

$$\implies f \text{ open}$$

Remark
f open $\implies |f|$ open

Theorem 4.10. (Max modulus principle)

f non-constant holomorphic in open $\Omega \subset \mathbb{C}$
$\implies f$ cannot attain maximum in Ω
Corollary
Ω open with closure $\bar{\Omega}$ compact
f holomorphic on Ω and continuous on $\bar{\Omega}$

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{z \in \bar{Omega} \setminus \Omega} |f(z)|$$

4.6 Evaluation of definite integrals

5 Harmonic Functions

5.1 Harmonic functions

Definition 5.1.

$\varphi = \varphi(x, y) : \mathbb{R}^2 \to \mathbb{R},\ x, y \in \mathbb{R}$
φ **harmonic** in open $\Omega \subset \mathbb{R}^2$ if

$$\begin{aligned} \underbrace{\Delta\varphi(x, y)}_{\text{laplace operator}} &:= \frac{\partial^2 \varphi}{\partial x^2}(x, y) + \frac{\partial^2 \varphi}{\partial y^2}(x, y) \\ &:= \varphi''_{xx}(x, y) + \varphi''_{yy}(x, y) \\ &:= 0 \end{aligned}$$

Theorem 5.1.

$f(z) = u(x, y) + iv(x, y)$ holomorphic in open $\Omega \subset \mathbb{C}$
$\implies u, v$ harmonic

Theorem 5.2. (Harmonic conjugate)

u harmonic in open disc $D \subset \mathbb{C}$
$\implies \exists$ harmonic v s.t $f = u + iv$ holomorphic in D
v the **harmonic conjugate** to u
Remark
In simply connected domain $\Omega \subset \mathbb{R}^2$ every harmonic function u has harmonic conjugate v s.t

$$v(x, y) = \int_\gamma \left(-\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy \right)$$

Integral independent of path, by Green's theorem as u harmonic and Ω simply connected.

5.2 Properties of real + imaginary parts of holomorphic function

Theorem 5.3.

Assume $f = u + iv$ holomorphic on open connected $\Omega \subset \mathbb{C}$

$$u(x, y) = C \tag{1}$$
$$v(x, y) = K \tag{2}$$
$$C, K \in \mathbb{R} \tag{3}$$

If (1) and (2) have same solution (x_0, y_0) and $f'(x_0 + iy_0) \neq 0$
$\implies$ curve defined by (1) orthogonal to curve defined by (2)

5.3 Preservation of angles

Definition 5.2.

Consider smooth curve $\gamma \subset \mathbb{C}$
$z(t) = x(t) + iy(t) \quad t \in [a, b]$
$\forall t_0 \in [a, b]$ we have direction vector

$$\begin{aligned} L_{t_0} &= \{z(t_0) + tz'(t_0) : t \in \mathbb{R}\} \\ &= \{x(t_0) = tx'(t_0 + i(y(t_0) + ty'(t_0)) : t \in \mathbb{R}\} \end{aligned}$$

For γ_1, γ_2 curves parameterised by functions $z_1(t), z_2(t)$, $t \in [0, 1]$ s.t $z_1(0) = z_2(0)$

Define angle between γ_1, γ_2 as angle between tangents

$$\arg z_2'(0) - \arg z_1'(0)$$

Theorem 5.4. (Angle preservation theorem)

f holomorphic in open $\Omega \subset \mathbb{C}$
Given γ_1, γ_2 inside Ω, parameterised by $z_1(t), z_2(t)$

Take $z_0 = z_1(0) = z_2(0)$ with $z_1'(0), z_2'(0), f'(z_0) \neq 0$

$$\underbrace{\arg z_2'(t) - \arg z_1'(t)}_{\text{angle between } z_1(0), z_2(0)} |_{t=0} = \underbrace{\arg f(z_2'(t)) - \arg f(z_1'(t))}_{\text{angle between } f(z_1(0)), f(z_2(0))} |_{t=0} (\mod 2\pi)$$

Definition 5.3.

Ω open $\subset \mathbb{C}$
$f : \Omega \to \mathbb{C}$ **conformal** if holomorphic in Ω and if $f'(z) \neq 0 \forall z \in \Omega$
Conformal mappings preserve angles.

Definition 5.4.

Holomorphic function a **local injection** on open $\Omega \subset \mathbb{C}$
if

$$\forall z_0 \in \Omega, \exists D = \{z : |z - z_0| < r\} \subset \Omega \text{ s.t } f : D \to f(D) \text{ an injection}$$

Theorem 5.5.

$f : \Omega \to \mathbb{C}$ local injection and holomorphic

$$\implies f'(z) \neq 0 \quad \forall z \in \Omega$$

Inverse of f defined on its range holomorphic
$\implies$ inverse of conformal mapping also holomorphic

5.4 Möbius Transformations

Definition 5.5.

Mobius Transformation/ Bilinear transformation a map

$$f(z) = \frac{az+b}{cz+d} \quad a, b, c, d \in \mathbb{C}, ad - bc \neq 0$$

Remark
Mobius Transformations holomorphic except for simple pole $z = -\frac{d}{c}$ with derivative

$$f'(z) = \frac{ad - bc}{(cz+b)^2}$$

$\implies$ mapping conformal for $\mathbb{C} \backslash \{-\frac{d}{c}\}$

Theorem 5.6.

(i) Inverse of mobius transformation a mobius transformation

(ii) Composition of mobius transformations a mobius transformations

Corresponding to matrix multiplication and inverses

Definition 5.6. (Special/Simple mobius tranformations)

(M1) $f(z) = az$ Scaling and rotation by a

(M2) $f(z) = z + b$ Translation by b

(M3) $f(z) = \frac{1}{z}$ Inverse and reflection w.r.t real axis

Theorem 5.7.

Every mobius transformations a composition of $M1, M2, M3$

Corollary
Mobius transformations:

circles $\mapsto$ circles
interior points $\mapsto$ interior points

Straight lines, considered to be circles of infinite radius

5.5 Cross-ratios Mobius Transformations

Theorem 5.8.

$w = f(z)$ a Mobius Transformation
s.t distinct $(z_1, z_2, z_3) \mapsto (w_1, w_2, w_3)$

$$\implies \left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right) = \left(\frac{w - w_1}{w - z_3}\right)\left(\frac{w_2 - w_3}{w_2 - w_1}\right) \quad \forall z$$

5.6 Conformal mapping of half-plane to unit disc

Theorem 5.9.

$$\mathbb{D} = \{z : |z| < 1\} \quad \mathbb{H} = \{z = x + iy : Im(z) = y > 0\}$$
$$w = f(z) = \tfrac{i-z}{i+z} \quad g(w) = \tfrac{1-w}{1+w}$$

5.7 Riemann mapping theorem

Definition 5.7.

$\Omega \subset \mathbb{C}$ **proper** if non-empty and $\Omega \neq \mathbb{C}$

Theorem 5.10.

Ω proper and simply connected
if $z_0 \in \Omega \implies \exists!$ conformal $f : \Omega \to \mathbb{D}$ s.t $f(z_0) = 0$ and $f'(z_0) > 0$
Corollary
Any 2 simply connected open subsets in $\mathbb{C}$ conformally equivalent.