MATH50001 COMPLEX ANALYSIS 2021
LECTURES

Lecture 1

Section: Syllabus & Historical Remarks

• Holomorphic Functions: Definition using derivative, Cauchy-Riemann equations, Polynomials, Power series.

• Cauchys Integral Formula: Complex integration along curves, Goursats theorem, Local existence of primitives and Cauchys theorem in a disc, Evaluation of some integrals, Homotopies and simply connected domains, Cauchys integral formulas.

• Applications of Cauchys integral formula: Moreras theorem, Sequences of holomorphic functions, Holomorphic functions defined in terms of integrals, Schwarz reflection principle.

• Meromorphic Functions: Zeros and poles. Laurent series. The residue formula, Singularities and meromorphic functions, The argument principle and applications, The complex logarithm.

• Harmonic functions: Definition, and basic properties, Maximum modulus principle. Conformal Mappings: Definitions, Preservation of Angles, Statement of the Riemann mapping theorem, Rational functions, Möbius transformations.

Course website: http://www2.imperial.ac.uk/~alaptev/CA21
see also Blackboard

Section: Complex numbers

The complex number $i = \sqrt{-1}$ ie associated with solutions of the equation

$$x^2 + 1 = 0$$

that does not have real solutions. However, historically complex numbers came through the cubic equation

$$x^3 - ax - b = 0.$$

In 1515 Scipione del Ferro (1465-1526, Italian) found but not published the solution

$$x = \sqrt{3}\frac{b}{2} + \sqrt{\frac{b^2}{4} - \frac{a^3}{27}} + \sqrt{3}\frac{b}{2} - \sqrt{\frac{b^2}{4} - \frac{a^3}{27}}$$

It was interesting that even if $\frac{b^2}{4} - \frac{a^3}{27} < 0$ the equation has real solutions for a, b real. This formula was published by Girolamo Cardano (1501-15-76, Italian) in 1545.

In 1572, Rafael Bombelli (1526-1572, Italian) published a book which spelled out rules of arithmetic for complex numbers and used them in Cardanos formula for finding real solutions of cubics.

Key later work is by John Wallis (1616 - 1703, English) and Leonhard Euler (1707-1783, Swiss). In particular, Euler clarified complex roots of unity and found the multiple roots. He used complex numbers extensively. He introduced i as the symbol for $\sqrt{-1}$ and linked the exponential and trigonometric functions in the famous formula

$$e^{it} = \cos t + i \sin t.$$

Carl Friedrich Gauss (1777-1855, German), who gave a proof of the Fundamental Theorem of Algebra in 1799.

It took almost another century before mathematicians as a community fully accepted complex numbers.

The founding fathers of complex analysis are Augustin-Louis Cauchy, Karl Weierstrass and Bernhard Riemann.

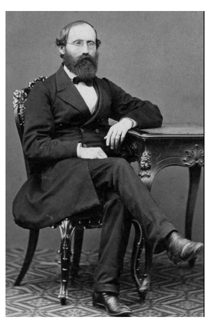
• To A.-L. Cauchy - the central aspect is the differential and integral calculus of complex-valued functions of a complex variable. Here the fundamentals are the Cauchy integral theorem and Cauchy integral formula

Augustin-Louis Cauchy (1789 -1857) - French

• To K. Weierstrass - sums and products and especially power series are the central object.

Karl Weierstrass (1815-1897) - German

• To B. Riemann - conformal maps and associated geometry.



Bernhard Riemann (1826-1866) - German

Modern state of art:

- The Mandelbrot set, Complex dynamics:

Benoit Mandelbrot, (1924, Warszawa, - 2010, Cambridge)

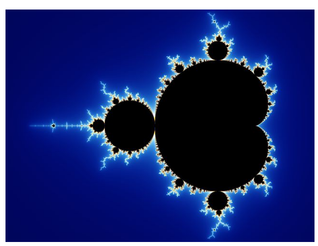
The Mandelbrot set is the set of complex numbers η for which the function

$$f_\eta(z) = z^2 + \eta$$

does not diverge when iterating from $z = 0$ so that the sequence

$$f_\eta(0), f_\eta(f_\eta(0)), f_\eta(f_\eta(f_\eta(0))), \ldots$$

remains bounded



- Riemann Hypothesis is still open (1859)

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{\text{p prime}} \frac{1}{1 - p^{-z}}.$$

This series converges if $\operatorname{Re} z > 1$.

If z is a complex number then in the above sum there some cancellation. In particular the Riemann Hypothesis states

$$\zeta(z) = 0 \quad \Longrightarrow \quad \operatorname{Re} z = 1/2.$$

Section: Basic properties

A complex number takes the form $z = x + iy$, where x and y are real, $x, y \in \mathbb{R}$, and i is an imaginary number that satisfies $i^2 = -1$. We call x and y the real part and the imaginary part of z, respectively, and we write

$$x = \mathrm{Re}\,(z) \quad \text{and} \quad y = \mathrm{Im}\,(z).$$

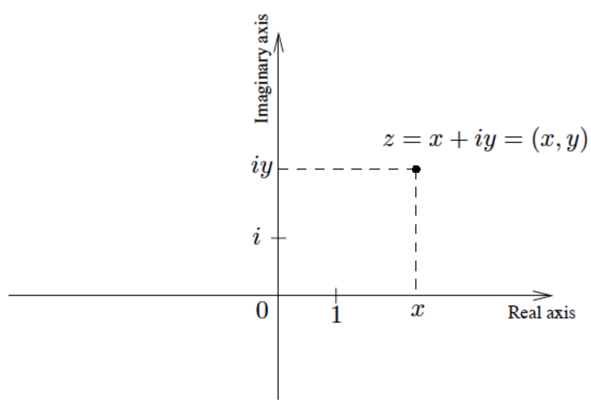
The real numbers are complex numbers with zero imaginary parts. A complex number with zero real part is said to be purely imaginary.

The complex conjugate of $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

The complex numbers can be visualised as the usual Euclidean plane:
$z = x + iy \in \mathbb{C}$ is identified with the point $(x, y) \in \mathbb{R}^2$.

- in this case 0 corresponds to the origin,
- i corresponds to $(0, 1)$.
- the x and y axis of $\mathbb{R}^2$ are called the real axis and imaginary axis respectively.



- **Polar coordinates.**

$$z = x + iy, \quad r = |z| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \bar{z}},$$

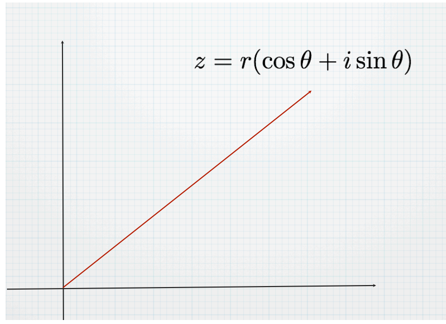
$$x = r\cos\theta, \qquad y = r\sin\theta,$$

where

$$\cos\theta = \frac{x}{r} \qquad \sin\theta = \frac{y}{r}.$$

and thus

$$z = r(\cos\theta + i\sin\theta).$$



Example. Let $z = 1 - i$. Then $r = \sqrt{2}$ and $\sin\theta = -1/\sqrt{2}$. Then

$$\theta = -\frac{\pi}{4} + 2\pi k, \quad k = 0, \pm 1, \pm 2, \ldots.$$

So $\arg z = -\pi/4 + 2\pi k$.

Definition. $\operatorname{Arg} z = \theta$ such that $-\pi < \theta \leq \pi$ is called the Principal value of the argument of z.

Example.

$$\operatorname{Arg}(1 - i) = -\frac{\pi}{4}.$$

Theorem. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$. Then

$$z_1 \cdot z_2 = r_1 r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

Proof. Use elementary trigonometric formulae.

Corollary. (De Moivres formula)

$$z^n = r^n(\cos n\theta + i\sin n\theta), \quad n = 1, 2, 3, \ldots.$$

Abraham De Moivres (French, 1667-1754)

Remark. Theorem implies

$$\arg z_1 + \arg z_2 = \arg (z_1 \cdot z_2),$$

however,

$$\operatorname{Arg} z_1 + \operatorname{Arg} z_2 \neq \operatorname{Arg} (z_1 \cdot z_2).$$

WHY ???

Section: Sets in the complex plane

Definition. Let $z_0 \in \mathbb{C}$ and $r > 0$. Define the open disc $D_r(z_0)$

$$D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

The boundary of the open or closed disc is the circle

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The unit disc is the disc centred at the origin and of radius one

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}.$$

Given a set $\Omega \subset \mathbb{C}$, a point z_0 is an interior point of Ω if there exists $r > 0$ such that $D_r(z_0) \subset \Omega$. The interior of Ω consists of all its interior points.

Definition. A set Ω is *open* if every point in that set is an interior point of Ω. This definition coincides precisely with the definition of an open set in $\mathbb{R}^2$.

Definition. A set Ω is *closed* if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

A set is closed if and only if it contains all its limit points. The closure of any set Ω is the union of Ω and its limit points, and is often denoted by $\overline{\Omega}$.

Definition. The *boundary* of a set Ω is equal to its closure minus its interior, and is often denoted by $\partial\Omega$.

Definition. A set Ω is *bounded* if there exists $M > 0$ such that $|z| < M$ whenever $z \in \Omega$.

Definition. If Ω is bounded, we define its *diameter* by

$$\operatorname{diam}(\Omega) = \sup_{z,w \in \Omega} |z - w|.$$

Definition. A set Ω is said to be *compact* if it is closed and bounded. Arguing as in the case of real variables, one can prove the following.

Theorem. The set $\Omega \subset \mathbb{C}$ is compact if and only if every sequence $\{z_n\} \subset \Omega$ has a subsequence that converges to a point in Ω.

An open covering of Ω is a family of open sets $\{U_\alpha\}$ (not necessarily countable) such that

$$\Omega \subset \cup_\alpha U_\alpha.$$

In analogy with the situation in $\mathbb{R}^2$, we have the following equivalent formulation of compactness.

Theorem. A set Ω is compact if and only if every open covering of Ω has a finite subcovering.

Another property of compactness is that of "nested sets".

Theorem. If $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_n \ldots$ is a sequence of non-empty compact sets in $\mathbb{C}$ with the property that $\text{diam}\,(\Omega_n) \to 0$ as $n \to \infty$, then there exists a unique point $w \in \mathbb{C}$ such that $w \in \Omega_n$ for all n.

Proof. Choose a point z_n in each Ω_n. The condition $\text{diam}\,(\Omega_n) \to 0$ says that $\{z_n\}$ is a Cauchy sequence, therefore this sequence converges to a limit that we call w. Since each set Ω_n is compact we must have $w \in \Omega_n$ for all n. Finally, w is the unique point satisfying this property, for otherwise, if w' satisfied the same property with $w' \neq w$ we would have $|w' - w| > 0$ and the condition $\text{diam}\,(\Omega_n) \to 0$ would be violated.

Definition. An open set Ω is *connected* if and only if any two points in Ω can be joined by a curve γ entirely contained in Ω.

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Lecture 2

Section: Complex functions

Definition. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$.

$$f: \quad \Omega_1 \to \Omega_2$$

is said to be a mapping from Ω_1 to Ω_2 if for any $z = x + iy \in \Omega_1$ there exists only one complex number $w = u + iv \in \Omega_2$ such that

$$w = f(z).$$

We use notations:

$$w = f(z) = u(x, y) + iv(x, y),$$

where u and v are two real functions of two real variables.

Example. Let $w = f(z) = z^2 = x^2 - y^2 + i2xy$, $z \in \mathbb{C}$. Then

$$u(x, y) = x^2 - y^2 \quad \text{and} \quad v(x, y) = 2xy.$$

Example. Let $w = f(z) = 1/z = \bar{z}/|z|^2$, $z \in \mathbb{C} \setminus \{0\}$. Then

$$u(x, y) = \frac{x}{x^2 + y^2} \quad \text{and} \quad v(x, y) = -\frac{y}{x^2 + y^2}.$$

Example. Möbius transformation

$$w = f(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \quad cz + d \neq 0.$$

Definition. Let f be a function defined on a set $\Omega \subset \mathbb{C}$. We say that f is continuous at the point $z_0 \in \Omega$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $z \in \Omega$ and $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$.

Definition. The function f is said to be continuous on Ω if it is continuous at every point of Ω.

Section: Complex derivative

Definition. Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be open sets and let $f : \Omega_1 \to \Omega_2$. We say that f is *differentiable (holomorphic)* at $z_0 \in \Omega_1$ if the quotient

$$\frac{f(z_0 + h) - f(z_0)}{h}$$

converges to a limit when $h \to 0$. Here $h \in \mathbb{C}$, $h \neq 0$ and $z_0 + h \in \Omega_1$. The limit of this quotient, when it exists, is denoted by $f'(z_0)$, and is called the derivative of f at z_0:

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

This means that for any $\varepsilon > 0$ there is $\delta > 0$ such that as soon $|h| < \delta$ we have

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - f'(z_0) \right| < \varepsilon.$$

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

It should be emphasised that in the above limit that $h = h_1 + ih_2 \in \mathbb{C}$ is a complex number that may approach 0 from any direction.

Remark. The word "holomorphic" was introduced by two of Cauchy's students, Briot (1817-1882) and Bouquet (1819-1895), and derives from the Greek (holos) meaning "entire", and (morphe) meaning "form" or "appearance".

Definition. The function f is said to be holomorphic on open set Ω if f is holomorphic at every point of Ω .

If C is a closed subset of $\mathbb{C}$, we say that f is holomorphic on C if f is holomorphic in some open set containing C. Finally, if f is holomorphic in all of $\mathbb{C}$ we say that f is entire.

Example. The function $f(z) = z$ is holomorphic on any open set in $\mathbb{C}$ and $f'(z) = 1$.

Example. If $f(z) = z^n$ then $f'(z) = nz^{n-1}$. Indeed we use induction to find:

- If $n = 1$ then $(z)' = 1$.
- Assuming $(z^n)' = nz^{n-1}$ we obtain

  $$(z^{n+1})' = (z \cdot z^n)' = z' \cdot z^n + z \cdot (z^n)' = z^n + z \cdot nz^{n-1} = (n+1)z^n.$$

Example. Any polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

is holomorphic in the entire complex plane and

$$p'(z) = a_1 + \cdots + n a_n z^{n-1}.$$

Example. The function $1/z$ is holomorphic on any open set in $\mathbb{C}$ that does not contain the origin, and $f'(z) = -1/z^2$.

Proof it.

Example. The function $f(z) = \bar{z}$ is not holomorphic. Indeed, we have

$$\frac{f(z_0 + h) - f(z_0)}{h} = \frac{\bar{h}}{h}.$$

which has no limit as $h \to 0$, as one can see by first taking h real and then h purely imaginary.

Proposition. A function f is holomorphic at $z_0 \in \Omega$ if and only if there exists a complex number a such that

$$f(z_0 + h) - f(z_0) - ah = h\,\psi(h),$$

where ψ is a function defined for all small h and

$$\lim_{h \to 0} \psi(h) = 0.$$

In this case

$$a = f'(z_0).$$

Proof. The proof follow directly from the Definition. Indeed, dividing by h we have

$$\frac{f(z_0 + h) - f(z_0)}{h} - a = \psi(h) \to 0 \quad \text{as} \quad h \to 0.$$

Corollary. If a function f is holomorphic then it is continuous.

Proposition. If f and g are holomorphic in Ω then:

(i) $f + g$ is holomorphic in Ω and $(f + g)' = f' + g'$.
(ii) fg is holomorphic in Ω and $(fg)' = f'g + fg'$.
(iii) If $g(z_0) \neq 0$, then f/g is holomorphic at z_0 and

$$(f/g)' = \frac{f'g - fg'}{g^2}.$$

(iv) Moreover, if $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic, the chain rule holds

$$(g \circ f)(z) = g'(f(z))f'(z), \qquad \forall z \in \Omega.$$

Proof. Arguing as in the case of one real variable, use the expression

$$f(z_0 + h) - f(z_0) - ah = h\,\psi(h).$$

Section: Cauchy-Riemann equations

Consider first

$$f'(z_0) = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}, \qquad h = h_1 + ih_2,$$

assuming that $h = h_1$ (namely that $h_2 = 0$). Then if

$$f(z_0) = f(x_0 + iy_0) = u(x_0, y_0) + iv(x_0, y_0),$$

we have

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h_1 \to 0} \frac{u(x_0 + h_1, y_0) + iv(x_0 + h_1, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h_1} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\,\frac{\partial v}{\partial x}(x_0, y_0) = u'_x(x_0, y_0) + i\,v'_x(x_0, y_0). \end{aligned}$$

Let now $h = ih_2$ (namely that $h_1 = 0$). Then

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h_2 \to 0} \frac{u(x_0, y_0 + h_2) + iv(x_0, y_0 + h2) - u(x_0, y_0) - iv(x_0, y_0)}{ih_2} \\ &= \frac{1}{i}\,\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) = \frac{1}{i}\,u'_y(x_0, y_0) + v'_y(x_0, y_0) \\ &= -i\,u'_y(x_0, y_0) + v'_y(x_0, y_0). \end{aligned}$$

Thus the function u and v satisfy the following

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

- *Cauchy-Riemann equations.*

Example. Let $f(z) = z^2$. Then $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$. Then

$$u'_x = 2x = v'_y \quad \text{and} \quad u'_y = -2y = -v'_x, \qquad -\mathsf{O'K}.$$

Example. Let $f(z) = \bar{z}$. Then $u(x,y) = x$ and $v(x,y) = -y$.

$$u'_x = 1 \neq -1 = v'_y.$$

This means that $f(z) = \bar{z}$ is not differentiable.

The Cauchy-Riemann equations link real and complex analysis.

Definition.

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right)$$

and

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right).$$

Theorem. Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$. If f is holomorphic at z_0, then

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \quad \text{and} \quad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0).$$

Proof. Using the Cauchy-Riemann equations $u'_x = v'_y$ and $u'_y = -v'_x$ we obtain

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(u'_x - \frac{1}{i}u'_y\right) + \frac{i}{2}\left(v'_x - \frac{1}{i}v'_y\right) = \frac{1}{2}\left(u'_x + iu'_y + iv'_x - v'_y\right) = 0.$$

and

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(u'_x + \frac{1}{i}u'_y\right) + \frac{i}{2}\left(v'_x + \frac{1}{i}v'_y\right) = \frac{1}{2}\left(u'_x - iu'_y + iv'_x + v'_y\right)$$
$$= \frac{1}{2}\left(2u'_x - i2u'_y\right) = u'_x + \frac{1}{i}u'_y = 2\frac{\partial u}{\partial z}.$$

The fact that $f'(z_0) = \frac{\partial f}{\partial z}(z_0)$ follows from our computations before. Indeed, we have seen that

$$f'(z_0) = u'_x(x_0,y_0) + iv'_x(x_0,y_0) = u'_x(x_0,y_0) - iu'_y(x_0,y_0) = 2\frac{\partial u}{\partial z}(x_0,y_0).$$

The proof is complete.

The next theorem contains an important converse.

Theorem. Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the

Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $\mathsf{f}'(z) = \partial\mathsf{f}(z)/\partial z$.

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Lecture 3

The next theorem contains an important converse.

Theorem. Suppose $f = u + iv$ is a complex-valued function defined on an open set Ω. If u and v are continuously differentiable and satisfy the Cauchy-Riemann equations on Ω, then f is holomorphic on Ω and $f'(z) = \partial f(z)/\partial z$.

Proof. Assuming $h = h_1 + ih_2$ we have

$$u(x + h_1, y + h_2) - u(x, y) = u'_x(x, y)\, h_1 + u'_y(x, y)\, h_2 + |h|\psi_1(h),$$

where $\psi_1(h) \to 0$ as $h \to 0$. Indeed,

$$\begin{aligned} u(x{+}h_1, y{+}h_2){-}u(x, y) &= u(x{+}h_1, y{+}h_2){-}u(x, y{+}h_2){+}u(x, y{+}h_2){-}u(x, y) \\ &= u'_x(x, y + h_2)h_1 + h_1\varphi_1(h) + u'_y(x, y)h_2 + h_2\varphi_2(h). \end{aligned}$$

Since $u'_x(x, y + h_2)$ is continuous we have

$$u'_x(x, y + h_2) - u'_x(x, y) = \varphi_3(h) \to 0 \quad \text{as} \quad h_2 \to 0$$

and thus

$$\begin{aligned} u(x + h_1, y + h_2) - u(x, y) &= u'_x(x, y)\, h_1 + u'_y(x, y)\, h_2 \\ &\quad + h_1(\varphi_3(h) + \varphi_1(h)) + h_2\varphi_2(h) = |h|\psi_1(h), \end{aligned}$$

where $\psi_1(h) = |h|^{-1}(h_1(\varphi_3(h) + \varphi_1(h)) + h_2\varphi_2(h)) \to 0$, $h \to 0$.

Similarly

$$v(x + h_1, y + h_2) - v(x, y) = v'_x(x, y)\, h_1 + v'_y(x, y)\, h_2 + |h|\psi_2(h),$$

where $\psi_2(h) \to 0$ as $h \to 0$.

Using the Cauchy-Riemann equations $v'_x = -u'_y$ and $v'_y = u'_x$, we find

$$f(z+h) - f(z) = u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y)$$
$$= u'_x(x,y)\, h_1 + u'_y(x,y)\, h_2 + i(v'_x(x,y)\, h_1 + v'_y(x,y)\, h_2) + |h|\psi(h)$$
$$= u'_x(x,y)\, h_1 + u'_y(x,y)\, h_2 - iu'_y(x,y) h_1 + iu'_x h_2 + |h|\psi(h)$$
$$= (u'_x - iu'_y)(h_1 + ih_2) + |h|\psi(h),$$

where $\psi(h) = \psi_1(h) + i\psi_2(h) \to 0$, as $h \to 0$. Therefore f is holomorphic and

$$f'(z) = 2\,\frac{\partial u}{\partial z} = \frac{\partial f}{\partial z}.$$

Section: Cauchy-Riemann equations in polar coordinates

Usual Cauchy-Riemann equations for a holomorphic function $f = u + iv$ as they were defined before are:

$$u'_x = v'_y \qquad u'_y = -v'_x$$

Introduce polar coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad r = \sqrt{x^2+y^2}, \quad \theta = \arctan y/x.$$

Then

$$\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \sin\theta,$$
$$\frac{\partial \theta}{\partial x} = \frac{1}{1+(y/x)^2}\,(-1)\,\frac{y}{x^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial \theta}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{\cos\theta}{r}.$$

Therefore

$$u'_x = u'_r\,\cos\theta + u'_\theta\,\frac{-\sin\theta}{r}, \qquad v'_y = v'_r\,\sin\theta + v'_\theta\,\frac{\cos\theta}{r},$$
$$u'_y = u'_r\,\sin\theta + u'_\theta\,\frac{\cos\theta}{r}, \qquad v'_x = v'_r\,\cos\theta + v'_\theta\,\frac{-\sin\theta}{r}.$$

Multiplying u'_x by $\cos\theta$ and u'_y by $\sin\theta$ and adding the results we find

$$u'_r = u'_r\,\cos^2\theta + u'_r\,\sin^2\theta = u'_x\,\cos\theta + u'_y\,\sin\theta.$$

Using $u'_x = v'_y$ and $u'_y = -v'_x$ we conclude

$$u'_x\,\cos\theta + u'_y\,\sin\theta = v'_y\,\cos\theta - v'_x\,\sin\theta$$
$$= \left(v'_r\,\sin\theta + v'_\theta\,\frac{\cos\theta}{r}\right)\,\cos\theta - \left(v'_r\,\cos\theta - v'_\theta\,\frac{\sin\theta}{r}\right)\,\sin\theta = v'_\theta\,\frac{1}{r}.$$

Then

$$u'_r = \frac{1}{r} v'_\theta \quad \text{and similarly} \quad v'_r = -\frac{1}{r} u'_\theta.$$

Example. Let

$$f(z) = u(x,y) + iv(x,y) = \ln(x^2 + y^2) + 2i \arctan \frac{y}{x}$$
$$= \ln |z|^2 + 2i\text{Arg}\,(z) = 2(\ln r + i\theta),$$

where $z = r(\cos\theta + i \sin\theta)$. Then

$$u'_r = \frac{2}{r} = \frac{1}{r} \cdot 2 = \frac{1}{r} v'_\theta \quad \text{and} \quad 0 = v'_r = -\frac{1}{r} u'_\theta = 0.$$

Section: Power series

Definition. A power series is an expansion of the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where $a_n \in \mathbb{C}$.

The series is convergent at z if the partial sum $S_N(z) = \sum_{n=0}^{N} a_n z^n$ has a limit

$$S(z) = \lim_{N\to\infty} S_N(z).$$

In this case we write $S(z) = \sum_{n=0}^{\infty} a_n z^n$.

For its absolute convergence we consider

$$\sum_{n=0}^{\infty} |a_n||z|^n.$$

Proposition. If $S(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\lim_{N\to\infty}(S(z) - S_N(z)) = 0$.

Theorem. Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \le R \le \infty$ such that:

(i) If $|z| < R$ the series converges absolutely.

(ii) If $|z| > R$ the series diverges.

Moreover, R is given by the formula

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}$$

The number R is called the *radius of convergence* of the power series, and the domain $|z| < R$ the *disc of convergence.*

Example. The complex exponential function, which is defined by

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

converges absolutely for any $z \in \mathbb{C}$ and $R = \infty$.

Example. The geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

converges absolutely $|z| < 1$ and its radius of convergence $R = 1$.

Proof. Let $L = 1/R$ and suppose that $L \neq 0, \infty$. If $|z| < R$, choose $\varepsilon > 0$ so that

$$(L + \varepsilon)|z| = r < 1.$$

By the definition L, we have $|a_n|^{1/n} \leq L + \varepsilon$ for all large n, therefore

$$|a_n||z|^n \leq ((L + \varepsilon)|z|)^n = r^n$$

Comparison with the geometric series $\sum_{n=0}^{\infty} r^n$ shows that $\sum_{n=0}^{\infty} a_n z^n$ converges.

If $|z| > R$, then a similar argument proves that there exists a sequence of terms in the series whose absolute value goes to infinity, hence the series diverges.

Remark. Prove the above result for $R = 0$ and $R = \infty$ ($L = \infty$ and $L = 0$ respectively).

Remark. On the boundary of the disc of convergence, $|z| = R$, one can have either convergence or divergence.

Power series provide an important class of holomorphic functions.

Theorem. The power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

defines a holomorphic function in its disc of convergence. The derivative of f is also a power series obtained by differentiating term by term the series for f, that is,

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Moreover, f has the same radius of convergence as f.

Proof. Indeed, note that

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} e^{\frac{1}{n}\ln n} = e^0 = 1.$$

Therefore

$$\sum_{n=1}^{\infty} a_n z^{n-1} \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n z^n$$

have the same radius of convergence and thus this is also true for $\sum_{n=1}^{\infty} a_n z^{n-1}$ and $\sum_{n=1}^{\infty} n a_n z^{n-1}$.
It remains to show that $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ coincides with $f'(z)$.

Let R be the radius of convergence of f, $|z_0| < r < R$ and let

$$S_N(z) = \sum_{n=0}^{N} a_n z^n, \qquad E_N(z) = \sum_{n=N+1}^{\infty} a_n z^n.$$

Then if h is chosen so that $|z_0 + h| < r$ we have

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right).$$

We find that

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \le \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \le \sum_{n=N+1}^{\infty} |a_n|\, n\, r^{n-1} \to 0, \quad \text{as} \quad N \to \infty.$$

Given $\varepsilon > 0$ there is N_1 such that for any $N > N_1$ we have

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| < \varepsilon.$$

Since $\lim_{N\to\infty} S_N'(z_0) \to g(z_0)$ there is N_2 such that for any $N > N_2$ we have

$$|S_N'(z_0) - g(z_0)| < \varepsilon$$

Finally for any fixed $N > \max(N_1, N_2)$ we choose $\delta > 0$ such that if $|h| < \delta$

$$\left|\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right| < \varepsilon.$$

We now conclude

$$\left|\frac{f(z_0+h) - f(z_0)}{h} - g_(z_0)\right| < 3\varepsilon, \quad |h| < \delta.$$

The proof is complete.

Corollary. A power series is infinitely complex differentiable in its disc of convergence, and the higher derivatives are also power series obtained by termwize differentiation.

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LECTURES

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Lecture 4

Section: Elementary functions.

1. Exponential function.

Definition. We define exponential e^z ($z = x + iy \in \mathbb{C}$) as:

$$e^z = e^x \cos y + ie^x \sin y.$$

Properties:

a) If $y = 0$ then $e^z = e^x$.

b) e^z is entire (holomorphic for any $z \in \mathbb{C}$)

Indeed, for that we check the C-R equations. Since $u = \operatorname{Re} f = e^x \cos y$ and $v = \operatorname{Im} f = e^x \sin y$, we have

$$u'_x = e^x \cos y = v'_y \quad \text{and} \quad u'_y = e^x(-\sin y) = -v'_x.$$

c)

$$\frac{\partial}{\partial z} e^z = \frac{\partial}{\partial x} e^x \cos y + i \frac{\partial}{\partial x} e^x \sin y = e^z.$$

d) Let $g(z)$ be holomorphic. Then

$$\frac{\partial}{\partial z} e^{g(z)} = e^{g(z)} g'(z).$$

e) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} e^{z_1+z_2} &= e^{x_1+x_2}\big(\cos(y_1+y_2) + i\sin(y_1+y_2)\big) \\ &= e^{x_1+x_2}\big(\cos y_1 \cos y_2 - \sin y_1 \sin y_2 + i(\sin y_1 \cos y_2 + \cos y_1 \sin y_2)\big) \\ &= e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i \sin y_2) = e^{z_1} e^{z_2}. \end{aligned}$$

f) $|e^z| = |e^x|\,|e^{iy}| = e^x \sqrt{\cos^2 y + \sin^2 y} = e^x$.

The function e^z is 2π-periodic with respect to y.

g) Applying the De Moivres formula

$$(\cos y + i\,\sin y)^n = \cos ny + i\,\sin ny$$

we obtain

$$\left(e^{iy}\right)^n = e^{iny}.$$

h) Since $\arg z = \arctan y/x$

$$\arg e^z = \arctan\frac{e^x\,\sin y}{e^x\,\cos y} = \arctan(\tan y) = y + 2\pi k, \quad k = 0, \pm 1, \pm 2, \dots.$$

Definition. If f is holomorphic for all $z \in \mathbb{C}$ then it calls *entire*.

Clearly the exponential function e^z is entire.

2. Trigonometric functions.

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \quad \Rightarrow \quad \begin{cases} \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right). \end{cases}$$

Definition. For any $z \in \mathbb{C}$ we define

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right), \qquad \cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right).$$

Properties:

a) $\sin z$ and $\cos z$ are entire functions

b) $\frac{\partial}{\partial z}\sin z = \cos z$ and $\frac{\partial}{\partial z}\cos z = -\sin z$.

c) $\sin^2 z + \cos^2 z = 1$.
Indeed:

$$-\frac{1}{4}\left(e^{iz} - e^{-iz}\right)^2 + \frac{1}{4}\left(e^{iz} + e^{-iz}\right)^2 = \dots = 1.$$

d)

$$\sin(z_1 \pm z_2) = \sin z_1\,\cos z_2 \pm \cos z_1\,\sin z_2,$$
$$\cos(z_1 \pm z_2) = \cos z_1\,\cos z_2 \mp \sin z_1\,\sin z_2.$$

3. Logarithmic functions.

Let $z = r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$.

Definition. $\log z = \ln|z| + i\arg z = \log r + i(\theta + 2\pi k), \quad z \neq 0,$

where $\quad k = 0, \pm 1, \pm 2, \dots.$

Clearly:

$$e^{\log z} = e^{\ln r + i\,(\theta + 2\pi k)} = r\, e^{i\,(\theta+2\pi k)} = r\,(\cos\theta + i\,\sin\theta) = x + i\,y = z.$$

Remark. The function log is a multi-valued function.

Definition. We define $\operatorname{Log} z$ as the singe-valued function:

$$\operatorname{Log} z = \ln|z| + i\operatorname{Arg} z,$$

where $\operatorname{Arg} z$ is the principal value of the argument, namely, $-\pi < \operatorname{Arg} z \le \pi$.

Remark. The function Log is a single-valued function.

Examples.

$\operatorname{Log}(-1) = i\pi$,
$\operatorname{Log}(2i) = \ln 2 + i\pi/2$,
$\operatorname{Log}(1-i) = \ln\sqrt{2} - i\pi/4$.

Properties:

a) $\log(z_1 \cdot z_2) = \log(z_1) + \log(z_2)$. Indeed

$$\begin{aligned}\log(z_1 \cdot z_2) &= \ln|z_1 z_2| + i\arg(z_1 \cdot z_2)\\ &= \ln|z_1| + \ln|z_2| + i\arg z_1 + i\arg z_2 = \log z_1 + \log z_2.\end{aligned}$$

Remark. $\operatorname{Log}(z_1 \cdot z_2) \ne \operatorname{Log} z_1 + \operatorname{Log} z_2$, because $\operatorname{Arg}(z_1 \cdot z_2) \ne \operatorname{Arg} z_1 + \operatorname{Arg} z_2$.

b) The function $\operatorname{Log} z$ is holomorphic in $\mathbb{C} \setminus \{(-\infty, 0]\}$.
Indeed, we have already checked that the C-R equations are satisfied:

$$\operatorname{Log} z = \ln r + i\,\theta = u + iv, \qquad -\pi < \theta \le \pi.$$

Therefore we have

$$\frac{\partial u}{\partial r} = \frac{1}{r}\cdot 1 = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = 0 = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Exercise. Compute $(\operatorname{Log} z)'$.

4. Powers.

Definition. For any $\alpha \in \mathbb{C}$, we define $z^\alpha = e^{\alpha \log z}$ as a multi-valued function.

Example. $i^i = e^{i\,\log i} = e^{i\,(i\pi/2 + i2\pi k)} = e^{-\pi/2}\, e^{-2\pi k}$, $k = 0, \pm1, \pm2, \ldots$.

Definition. We define the principal value of z^α, $\alpha \in \mathbb{C}$, as

$$z^\alpha = e^{\alpha \operatorname{Log} z}.$$

Property:

a) $z^{\alpha_1} \cdot z^{\alpha_2} = e^{\alpha_1 \operatorname{Log} z} e^{\alpha_2 \operatorname{Log} z} = e^{(\alpha_1+\alpha_2)\operatorname{Log} z} = z^{\alpha_1+\alpha_2}$.

Section: Parametrised curve.

Definition. A *parametrised curve* is a function $z(t)$ which maps a closed interval $[a,b] \subset \mathbb{R}$ to the complex plane. We say that the parametrised curve is smooth if $z'(t)$ exists and is continuous on $[a,b]$, and $z'(t) \neq 0$ for $t \in [a,b]$. At the points $t = a$ and $t = b$, the quantities $z'(a)$ and $z'(b)$ are interpreted as the one-sided limits

$$z'(a) = \lim_{h \to 0,\, h>0} \frac{z(a+h) - z(a)}{h}, \quad z'(b) = \lim_{h \to 0,\, h<0} \frac{z(b+h) - z(b)}{h}.$$

Similarly we say that the parametrised curve is piecewise - smooth if z is continuous on $[a,b]$ and if there exist a finite number of points $a = a_0 < a_1 < \cdots < a_n = b$, where $z(t)$ is smooth in the intervals $[a_k, a_{k+1}]$. In particular, the righthand derivative at a_k may differ from the left-hand derivative at a_k for $k = 1, 2, \ldots, n-1$.

Two parametrisations,

$$z : [a,b] \to \mathbb{C} \quad \text{and} \quad \tilde{z} : [c,d] \to \mathbb{C},$$

are equivalent if there exists a continuously differentiable bijection $s \to t(s)$ from $[c,d]$ to $[a,b]$ so that $t'(s) > 0$ and

$$\tilde{z}(s) = z(t(s)).$$

The condition $t'(s) > 0$ says precisely that the orientation is preserved: as s travels from c to d, then $t(s)$ travels from a to b.

Given a smooth curve γ in $\mathbb{C}$ parametrised by $z : [a,b] \to \mathbb{C}$, and f a continuous function on γ we define the integral of f along γ by

$$\int_\gamma f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt.$$

In order for this definition to be meaningful, we must show that the right-hand integral is independent of the parametrisation chosen for γ. Say that $\tilde{z}$ is an equivalent parametrisation as above. Then the change of variables formula and the chain rule imply that

$$\int_a^b f(z(t))\, z'(t)\, dt = \int_c^d f(z(t(s)))\, z'(t(s))\, t'(s)\, ds$$

$$= \int_c^d f(\tilde{z}(s))\, \tilde{z}'(s)\, ds.$$

This proves that the integral of f over γ is well defined.

If γ is piecewise smooth, then the integral of f over γ is the sum of the integrals of f over the smooth parts of γ, so if $z(t)$ is a piecewise-smooth parametrisation as before, then

$$\int_\gamma f(z)\, dz = \sum_{k=0}^{n-1} \int_{a_k}^{a_{k+1}} f(z(t))\, z'(t)\, dt.$$

We can define a curve γ^- obtained from the curve γ by reversing the orientation (so that γ and γ^- consist of the same points in the plane). As a particular parametrisation for γ^- we can take $z^- : [a, b] \to \mathbb{C}$ defined by

$$z^-(t) = z(b + a - t).$$

A smooth or piecewise-smooth curve is closed if $z(a) = z(b)$ for any of its parametrisations. A smooth or piecewise-smooth curve is simple if it is not self-intersecting, that is, $z(t) \neq z(s)$ unless $s = t$, $s, t \in (a, b)$.

A basic example consists of a circle. Consider the circle $C_r(z_0)$ centred at z_0 and of radius r, which by definition is the set

$$C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}.$$

The positive orientation (counterclockwise) is the one that is given by the standard parametrisation

$$z(t) = z_0 + r\, e^{it}, \quad \text{where} \quad t \in [0, 2\pi],$$

while the negative orientation (clockwise) is given by

$$z(t) = z_0 + r\, e^{-it}, \quad \text{where} \quad t \in [0, 2\pi].$$

Section: Integration along curves.

By definition, the length of the smooth curve γ is

$$\text{length}\,(\gamma) = \int_a^b |z'(t)|\, dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt.$$

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Lecture 5

Section: Integration along curves.

By definition, the length of the smooth curve γ is

$$\text{length}\,(\gamma) = \int_a^b |z'(t)|\,dt = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

Theorem. Integration of continuous functions over curves satisfies the following properties:

- $$\int_\gamma (\alpha\, f(z) + \beta\, g(z))\,dz = \alpha \int_\gamma f(z)\,dz + \beta \int_\gamma g(z)\,dz.$$
- If γ^- is γ with the reverse orientation, then
$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$
- (ML-inequality)
$$\left|\int_\gamma f(z)\,dz\right| \leq \sup_{z\in\gamma}|f(z)| \cdot \text{length}\,(\gamma).$$

Proof. The first property follows from the definition and the linearity of the Riemann integral. The second property is left as an exercise. For the third one, we note that

$$\left|\int_\gamma f(z)\,dz\right| \leq \sup_{t\in[a,b]}|f(z(t))| \int_a^b |z'(t)|\,dt = \sup_{z\in\gamma}|f(z)| \cdot \text{length}\,(\gamma).$$

Section: Primitive functions.

Definition. A primitive for f on $\Omega \subset \mathbb{C}$ is a function F that is holomorphic on Ω and such that $F'(z) = f(z)$ for all $z \in \Omega$.

Theorem. If a continuous function f has a primitive F in an open set Ω, and γ is a curve in Ω that begins at w_1 and ends at w_2, then

$$\int_\gamma f(z)\,dz = F(w_2) - F(w_1).$$

Proof. If γ is smooth, the proof is a simple application of the chain rule and the fundamental theorem of calculus. Indeed, if $z(t) : [a,b] \to \mathbb{C}$ is a parametrization for γ, then $z(a) = w_1$ and $z(b) = w_2$, and we have

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt = \int_a^b F'(z(t))z'(t)\,dt$$
$$= \int_a^b \frac{d}{dt}F(z(t))\,dt = F(z(b)) - F(z(a)).$$

If γ is only piecewise-smooth then arguing the same as we did we have

$$\int_\gamma f(z)\,dz = \sum_{k=0}^{n-1}(F(z(a_{k+1}) - F(z(a_k))$$
$$= F(z(a_n)) - F(z(a_0)) = F(z(b)) - F(z(a)).$$

Corollary. If γ is a closed curve in an open set Ω, f is continuous and has a primitive in Ω, then

$$\oint_\gamma f(z)\,dz = 0.$$

Proof. This is immediate since the end-points of a closed curve coincide.

For example, the function $f(z) = 1/z$ does not have a primitive in the open set $\mathbb{C}\setminus\{0\}$, since if C is the unit circle parametrized by $z(t) = e^{it}$, $0 \le t \le 2\pi$, we have

$$\oint_C f(z)\,dz = \int_0^{2\pi} \frac{i\,e^{it}}{e^{it}}\,dt = 2\pi i \neq 0.$$

Corollary. If f is holomorphic in an open connected set Ω and $f' = 0$, then f is constant.

Proof. Fix a point $w_0 \in \Omega$. It suffices to show that $f(w) = f(w_0)$ for all $w \in \Omega$. Since Ω is connected, for any $w \in \Omega$, there exists a curve γ which joins w_0 to w. Since f is clearly a primitive for f', we have

$$\int_\gamma f'(z)\,dz = f(w) - f(w_0),$$

By assumption, $f' = 0$ so the integral on the left is 0, and we conclude that $f(w) = f(w_0)$ as desired.

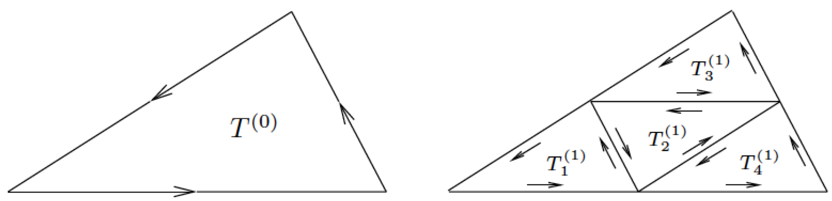
Section: Properties of holomorphic functions.

Theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle whose interior is also contained in Ω, then

$$\oint_T f(z)\,dz = 0,$$

whenever f is holomorphic in Ω.

Proof. Let $T^{(0)}$ be our original triangle (with a fixed orientation which we choose to be positive), and let $d^{(0)}$ and $p^{(0)}$ denote the diameter and perimeter of $T^{(0)}$, respectively. At the first step we find middle point of each side of $T^{(0)}$ and introduce four triangles $T_1^{(1)}$, $T_2^{(1)}$, $T_3^{(1)}$, $T_4^{(1)}$ that are similar to the original triangle as follows:



Then

$$\oint_{T^{(0)}} f(z)\,dz = \oint_{T_1^{(1)}} f(z)\,dz + \oint_{T_2^{(1)}} f(z)\,dz + \oint_{T_3^{(1)}} f(z)\,dz + \oint_{T_4^{(1)}} f(z)\,dz.$$

There is some $j \in \{1, 2, 3, 4\}$ such that (WHY?)

$$\left| \oint_{T^{(0)}} f(z)\,dz \right| \leq 4 \left| \oint_{T_j^{(1)}} f(z)\,dz \right|.$$

We choose a triangle that satisfies this inequality, and rename it $T^{(1)}$. Observe that if $d^{(1)}$ and $p^{(1)}$ denote the diameter and perimeter of $T^{(1)}$, respectively. Then

$$d^{(1)} = \frac{1}{2} d^{(0)} \quad \text{and} \quad p^{(1)} = \frac{1}{2} p^{(0)}.$$

We now repeat this process for the triangle $T^{(1)}$. Continuing this process, we obtain a sequence of triangles

$$T^{(1)}, T^{(1)}, T^{(2)}, \dots, T^{(n)}, \dots$$

with the properties that

$$\left| \oint_{T^{(0)}} f(z)\, dz \right| \le 4^n \left| \oint_{T_j^{(n)}} f(z)\, dz \right|$$

and

$$d^{(n)} = 2^{-n}\, d^{(0)} \quad \text{and} \quad p^{(n)} = 2^{-n}\, p^{(0)},$$

where $d^{(n)}$ and $p^{(n)}$ denote the diameter and perimeter of $T^{(n)}$.
Let $\Omega^{(n)}$ be the closed triangle such that $\partial\Omega^{(n)} = T^{(n)}$. Clearly we have a sequence of compact nested sets

$$\Omega^{(0)} \supset \Omega^{(1)} \supset \cdots \supset \Omega^{(n)} \supset \dots,$$

whose diameter goes to 0. Then there exists a unique point z_0 that belongs to all triangles $\Omega^{(n)}$. Since f is holomorphic then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + (z - z_0)\,\psi(z),$$

where $\psi(z) \to 0$ as $z \to z_0$.

Since the constant $f(z_0)$ and the linear function $f(z_0)(z-z_0)$ have primitives, we can integrate the above equality over $T^{(n)}$ and obtain

$$\oint_{T^{(n)}} f(z)\, dz = \oint_{T^{(n)}} \psi(z)(z - z_0)\, dz.$$

Since z_0 belongs to all triangles we have $|z - z_0| \le d^{(n)}$ and using the ML-inequality we arrive at

$$\left| \oint_{T^{(n)}} f(z)\, dz \right| \le \varepsilon_n\, d^{(n)}\, p^{(n)},$$

where $\varepsilon_n = \sup_{z \in T^{(n)}} |\psi(z)| \to 0$ as $n \to \infty$. Therefore

$$\left| \oint_{T^{(n)}} f(z)\, dz \right| \le \varepsilon_n\, 4^{-n} d^{(0)}\, p^{(0)},$$

and thus finally we obtain

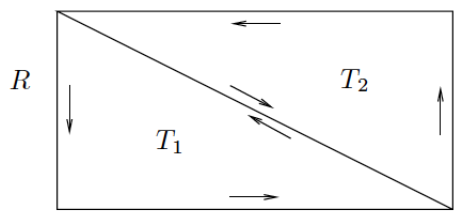
$$\left| \oint_{T^{(0)}} f(z)\, dz \right| \le 4^n \left| \oint_{T_j^{(n)}} f(z)\, dz \right| \le \varepsilon_n\, d^{(0)}\, p^{(0)} \to 0, \quad \text{as} \quad n \to \infty.$$

Corollary. If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then

$$\oint_R f(z)\, dz = 0.$$

Proof. This immediately follows from the equality

$$\oint_R f(z)\, dz = \oint_{T_1} f(z)\, dz + \oint_{T_2} f(z)\, dz.$$



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Lecture 6

Section: Properties of holomorphic functions.

In Lecture 5 we have proved the following

Theorem. Let $\Omega \subset \mathbb{C}$ be an open set and $T \subset \Omega$ be a triangle whose interior is also contained in Ω, then

$$\oint_T f(z)\, dz = 0,$$

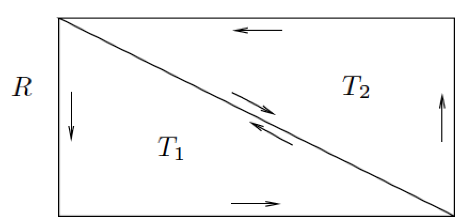
whenever f is holomorphic in Ω.

Corollary. If f is holomorphic in an open set Ω that contains a rectangle R and its interior, then

$$\oint_R f(z)\, dz = 0.$$

Proof. This immediately follows from the equality

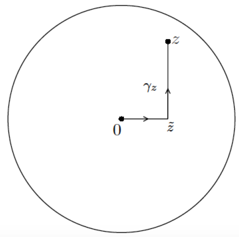
$$\oint_R f(z)\, dz = \oint_{T_1} f(z)\, dz + \oint_{T_2} f(z)\, dz.$$



Section: Local existence of primitives and Cauchy-Goursat theorem in a disc.

Theorem. A holomorphic function in an open disc has a primitive in that disc.

Proof. We may assume that the disc D is centered at the origin. For any $z \in D$ we consider γ_z given by



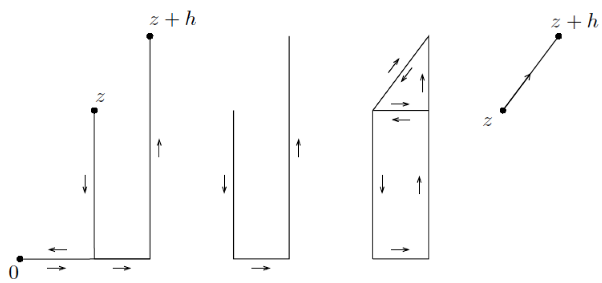
Define

$$F(z) = \int_{\gamma_z} f(w)\, dw.$$

Consider the difference

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)\, dw - \int_{\gamma_z} f(w)\, dw$$

The function f is first integrated along γ_{z+h} with the original orientation, and then along γ_z with the reverse orientation.



(a) (b) (c) (d)

Using the fact that the integration over the triangle and the rectangle equal zero we obtain

$$F(z+h) - F(z) = \int_{\eta} f(w)\, dw,$$

where η is the straight line segment from z to $z+h$. Since f is continuous at z we can write

$$f(w) = f(z) + \psi(w),$$

where $\psi(w) \longrightarrow 0$ as $w \longrightarrow z$. Then

$$F(z+h) - F(z) = \int_{\eta} f(z)\, dw + \int_{\eta} \psi(w)\, dw = f(z)\, h + \int_{\eta} \psi(w)\, dw.$$

Finally we note that using the LM-inequality

$$\left| \int_{\eta} \psi(w)\, dw \right| \leq |h| \sup_{w \in \eta} |\psi(w)|$$

Since $\psi(w) \to 0$ as $w \to z$ we obtain

$$\lim_{h\to 0} \frac{F(z+h)-F(z)}{h} = f(z).$$

Corollary. (Cauchy-Goursat theorem for a disc)
If f is holomorphic in a disc, then

$$\oint_\gamma f(z)\, dz = 0$$

for any closed curve γ in that disc.

Corollary. Suppose f is holomorphic in an open set containing the circle C and its interior. Then

$$\oint_C f(z)\, dz = 0.$$

Proof. Let D be the disc with boundary circle C. Then there exists a slightly larger disc $\tilde{D} \supset D$ and so that f is holomorphic on $\tilde{D}$. We may now apply Cauchy-Goursat theorem in $\tilde{D}$ to conclude that $\oint_C f(z)\, dz = 0$.

Section: Homotopies and simply connected domains.

Let γ_0 and γ_1 be two curves in an open set Ω with common end-points. That is if γ_0 and γ_1 are two parametrizations defined on $[a,b]$, we have

$$\gamma_0(a) = \gamma_1(a) = \alpha \quad \text{and} \quad \gamma_0(b) = \gamma_1(b) = \beta.$$

Definition. The curves γ_0 and γ_1 are said to be *homotopic* in Ω if for each $0 \le s \le 1$ there exists a curve $\gamma_s \subset \Omega$, parametrized by $\gamma_s(t)$ defined on $[a,b]$, such that for every s

$$\gamma_s(a) = \alpha \quad \text{and} \quad \gamma_s(b) = \beta,$$

and for all $t \in [a,b]$

$$\gamma_s(t)|_{s=0} = \gamma_0(t) \quad \text{and} \quad \gamma_s(t)|_{s=1} = \gamma_1(t).$$

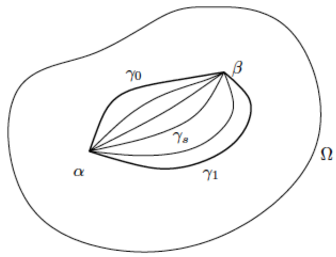
Moreover, $\gamma_s(t)$ should be jointly continuous in $s \in [0,1]$ and $t \in [a,b]$.

Theorem. If f is holomorphic in Ω, then

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

Proof. We first show that if two curves are close to each other and have the same end-points, then the integrals over them are equal.

Due to definition, the function $F(s,t) = \gamma_s(t)$ is continuous on $[0,1] \times [a,b]$. Then the image of F denoted by K is compact.



Then there is $\varepsilon > 0$ such that every disc of radius $3\varepsilon > 0$ centred at a point in the image of F is completely contained in Ω.

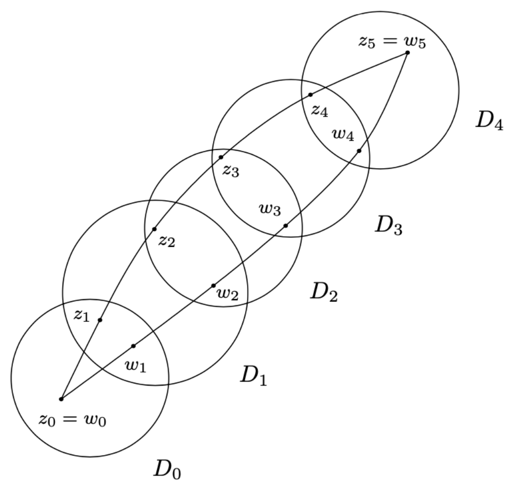
WHY ??? Show it.

Since F is uniformly continuous we choose δ such that

$$\sup_{t\in[a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \varepsilon \quad \text{whenever} \quad |s_1 - s_2| < \delta.$$

We now choose discs $\{D_0, \dots, D_n\}$ of radius 2ε, and points $\{z_0, \dots, z_{n+1}\}$ on γ_{s_1} and $\{w_0, \dots, w_{n+1}\}$ on γ_{s_2} such that the union of these discs covers both curves, and

$$z_i, z_{i+1}, w_i, w_{i+1} \in D_i.$$



Here $z_0 = w_0 = \gamma_{s_1}(a) = \gamma_{s_2}(a)$ and
$z_{n+1} = w_{n+1} = \gamma_{s_1}(b) = \gamma_{s_2}(b)$.
On each D_i, let F_i be a primitive of f.
In $D_i \cap D_{i+1}$ the primitives F_i and F_{i+1}
are two primitives of the same function, so they must

differ by a constant.
Therefore

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1}),$$

or

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1}).$$

Finally we have

$$\int_{\gamma_{s_1}} f(z)\,dz - \int_{\gamma_{s_2}} f(z)\,dz$$

$$= \sum_{i=0}^{n+1} (F_i(z_{i+1}) - F_i(z_i)) - \sum_{i=0}^{n+1} (F_i(w_{i+1}) - F_i(w_i))$$

$$\sum_{i=0}^{n+1} (F_i(z_{i+1}) - F_i(w_{i+1}) - (F_i(z_i) - F_i(w_i)))$$

$$= F_n(z_{n+1}) - F_n(w_{n+1}) - (F_0(z_0) - F_0(w_0)) = 0.$$

By subdividing the interval $[0, 1]$ into subintervals $[s_k, s_{k+1}]$, $k = 0, \ldots m$, of length less than δ and using the above arguments for each pair γ_{s_k} and $\gamma_{s_{k+1}}$ with $\gamma_{s_0} = \gamma_0$ and $\gamma_{s_{m+1}} = \gamma_1$ we complete the proof.

Definition. An open set $\Omega \subset \mathbb{C}$ is *simply connected* if any two pair of curves in Ω with the same end-points are homotopic.

Example. A disc D is simply connected. Indeed, let $\gamma_0(t)$ and $\gamma_1(t)$ be two curves lying in D. We can define $\gamma_s(t)$ by $\gamma_s(t) = (1 - s)\gamma_0(t) + s\gamma_1(t)$. Note that if $0 \leq s \leq 1$, then for each t, the point $\gamma_s(t)$ is on the segment joining $\gamma_0(t)$ and $\gamma_1(t)$, and so is in D.

The same argument works if D is replaced any open convex set.
WHY ??? - show it

Example. The set $\mathbb{C} \setminus \{(-\infty, 0]\}$ is simply connected.
WHY ??? - show it

Example. The punctured plane $\mathbb{C} \setminus \{0\}$ is not simply connected.

Theorem. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define

$$F(z) = \int_\gamma f(w)\, dw,$$

where the integral is taken over any curve in Ω joining z_0 to z. This definition is independent of the curve chosen, since Ω is simply connected. Consider

$$F(z+h) - F(z) = \int_\eta f(w)\, dw,$$

where η is the line segment joining z and $z+h$. Arguing as in the proof of the Theorem where we constructed a primitive to a holomorphic function in a disc, we obtain

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

The proof is complete.

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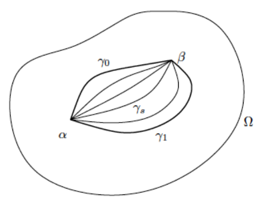
Lecture 7

To remind:

In the previous lecture we introduced homotopic curves:



and proved

Theorem. If γ_0 and γ_1 are homotopic in Ω and if f is holomorphic in Ω, then

$$\int_{\gamma_0} f(z)\, dz = \int_{\gamma_1} f(z)\, dz.$$

Besides, we had

Definition. An open set $\Omega \subset \mathbb{C}$ is *simply connected* if any two pair of curves in Ω with the same end-points are homotopic.

The next theorem is about holomorphic function in a simply connected domains:

Theorem. Any holomorphic function in a simply connected domain has a primitive.

Proof. Fix a point z_0 in Ω and define

$$F(z) = \int_{\gamma} f(w)\, dw,$$

where the integral is taken over any curve in Ω joining z_0 to z. This definition is independent of the curve chosen, since Ω is simply connected. Consider

$$F(z+h) - F(z) = \int_{\eta} f(w)\, dw,$$

where η is the line segment joining z and $z+h$. Arguing as in the proof of the Theorem where we constructed a primitive to a holomorphic function in a disc, we obtain

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z).$$

The proof is complete.

Corollary. (Cauchy-Goursat theorem)
If f is holomorphic in the simply connected open set Ω, then

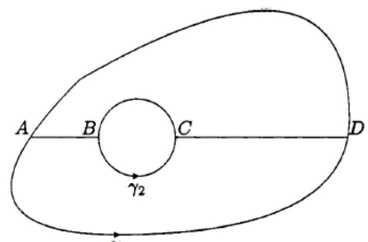
$$\oint_{\gamma} f(z)\, dz = 0,$$

for any closed, piecewise-smooth, curve $\gamma \subset \Omega$.

Theorem. (Deformation Theorem)
Let γ_1 and γ_2 be two simple, closed, piecewise-smooth curves with γ_2 lying wholly inside γ_1 and suppose f is holomorphic in a domain containing the region between γ_1 and γ_2. Then

$$\oint_{\gamma_1} f(z)\, dz = \oint_{\gamma_2} f(z)\, dz.$$

Proof.



Example. Let $\gamma = \{z \in \mathbb{C} : \ |z-1| = 2\}$. Then

$$\oint_\gamma \frac{1}{z^2-4}\,dz = \oint_\gamma \frac{1}{(z-2)(z+2)}\,dz = \frac{1}{4}\oint_\gamma \left(\frac{1}{z-2} - \frac{1}{z+2}\right)dz.$$

Since $1/(z+2)$ is holomorphic inside and on γ, then

$$\oint_\gamma \frac{1}{z+2}\,dz = 0.$$

On the other hand

$$\oint_\gamma \frac{1}{z-2}\,dz = \oint_{\{z:\,|z-2|=1\}} \frac{1}{z-2}\,dz = 2\pi i.$$

Therefore

$$\oint_\gamma \frac{1}{z^2-4}\,dz = i\,\frac{\pi}{2}.$$

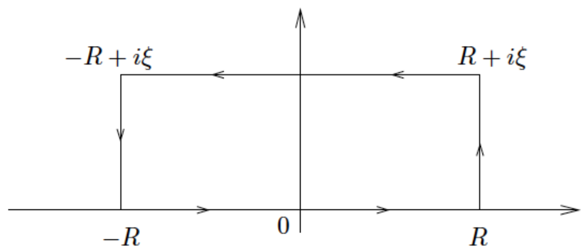
Example. We show that if $\xi \in \mathbb{R}$ then

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2}\, e^{-2\pi i x\xi}\,dx.$$

This gives a proof of the fact that $e^{-\pi x^2}$ is its own Fourier transform. If $\xi = 0$, the formula is precisely the known integral

$$1 = \int_{-\infty}^{\infty} e^{-\pi x^2}\,dx.$$

Now suppose that $\xi > 0$, and consider the function $f(z) = e^{-\pi z^2}$, which is entire, and in particular holomorphic in the interior of the contour γ_R



The contour γ_R consists of a rectangle with vertices R, $R + i\xi$, $-R + i\xi$, $-R$ and the positive counterclockwise orientation.
By the Cauchy-Goursat theorem theorem

$$\oint_{\gamma_R} f(z)dz = 0 \qquad (*)$$

The integral over the real segment is simply

$$\int_{-R}^{R} e^{-\pi x^2}\, dx$$

which converges to 1 as $R \to \infty$. The integral on the vertical side on the right is

$$|I(R)| = \left|\int_0^{\xi} f(R + iy)\, i\, dy\right| = \left|\int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}\, dy\right|$$
$$\leq e^{-\pi R^2} \int_0^{\xi} |e^{-\pi(2iRy-y^2)}|\, dy \leq e^{-\pi R^2}\, \xi\, e^{\pi \xi^2} \to 0,$$

as $R \to \infty$.

Similarly, the integral over the vertical segment on the left also goes to 0 as $R \to \infty$ for the same reasons.

Finally, the integral over the horizontal segment on top is

$$\int_R^{-R} e^{-\pi(x+i\xi)^2}\,dx = -\int_{-R}^{R} e^{-\pi(x+i\xi)^2}\,dx$$

$$= -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2}e^{-2\pi i x\xi}\,dx.$$

Therefore, in the limit as $R \to \infty$ we obtain that $(*)$ gives

$$0 = 1 - e^{\pi\xi^2}\int_{-\infty}^{\infty} e^{-\pi x^2}e^{-2\pi i x\xi}\,dx.$$

Section: Cauchy's integral formulae.

Theorem. Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ. Then for any point z_0 interior to γ we have

$$f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,dz.$$

Proof. If z_0 is interior to γ then for any $r > 0$ such that $\gamma_r = \{z : |z - z_0| = r\}$ lying wholly inside γ, using the deformation theorem we obtain

$$\oint_\gamma \frac{f(z)}{z - z_0}\,dz = \oint_{\gamma_r} \frac{f(z)}{z - z_0}\,dz.$$

Then

$$\frac{1}{2\pi i}\oint_{\gamma_r} \frac{f(z)}{z - z_0}\,dz$$

$$= \frac{1}{2\pi i} f(z_0)\oint_{\gamma_r} \frac{1}{z - z_0}\,dz + \frac{1}{2\pi i}\oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\,dz$$

$$= f(z_0) + \frac{1}{2\pi i}\oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\,dz.$$

Since f is holomorphic it is continuous at z_0. Therefore for a given $\varepsilon > 0$ there is $\delta > r > 0$ such that as soon $|z - z_0| < \delta$ we have

$$|f(z) - f(z_0)| < \varepsilon.$$

Then, by using the ML-inequality we have

$$\Big|\frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0}\, dz\Big| \leq \frac{1}{2\pi} \frac{\varepsilon}{r} 2\pi r = \varepsilon.$$

So we have proved that for any $\varepsilon > 0$

$$\Big|\oint_{\gamma} \frac{f(z)}{z - z_0}\, dz - f(z_0)\Big| < \varepsilon$$

and hence

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{z - z_0}\, dz = f(z_0).$$

The proof is complete.

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Lecture 8

Section: Cauchy's integral formulae.

Theorem. Let f be holomorphic inside and on a simple, closed, piecewise-smooth curve γ. Then for any point z_0 interior to γ we have

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Example.

$$\frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{(z-i)(z+i)}\, dz$$
$$= \frac{1}{2\pi i}\frac{1}{2i} \oint_{|z|=2} \left(\frac{e^z}{z-i} - \frac{e^z}{z+i}\right) dz$$
$$= \frac{1}{2i}\left(e^i - e^{-i}\right) = \sin 1.$$

Theorem. (Generalised Cauchy's integral formula)

Let f be holomorphic in an open set Ω, then f has infinitely many complex derivatives in Ω. Moreover, for simple, closed, piecewise-smooth curve $\gamma \subset \Omega$ and any z lying inside γ we have

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}}\, d\eta.$$

Proof. The proof is by induction on n. The case $n = 0$ is simply the Cauchy integral formula. Suppose that f has up to $n-1$ complex derivatives and that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^n}\, d\eta.$$

Let $h \in \mathbb{C}$ be small enough, so that $z + h$ is lying inside γ. Then

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} = \frac{(n-1)!}{2\pi i} \oint_\gamma f(\eta)\, \frac{1}{h} \Big(\frac{1}{(\eta - z - h)^n} - \frac{1}{(\eta - z)^n} \Big)\, d\eta.$$

Recall

$$A^n - B^n = (A - B)(A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1})$$

and apply it with $A = 1/(\eta - z - h)$ and $B = 1/(\eta - z)$. Then we obtain as $h \to 0$

$$\begin{aligned} &\frac{1}{h} \Big(\frac{1}{(\eta - z - h)^n} - \frac{1}{(\eta - z)^n} \Big) \\ &= \frac{1}{h}\, \frac{h}{(\eta - z - h)(\eta - z)}\, (A^{n-1} + A^{n-2}B + \cdots + AB^{n-2} + B^{n-1}) \\ &\qquad \to \frac{1}{(\eta - z)^2}\, \frac{n}{(\eta - z)^{n-1}}. \end{aligned}$$

This implies

$$\begin{aligned} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} &\to \frac{(n-1)!}{2\pi i} \oint_\gamma f(\eta)\, \frac{1}{(\eta - z)^2}\, \frac{n}{(\eta - z)^{n-1}}\, d\eta \\ &= \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}}\, d\eta. \end{aligned}$$

The proof is complete.

Corollary. If f is holomorphic in Ω, then all its defivatives $f', f'', \ldots,$ are holomorphic.

Exercise:

Let f be continuous on a piecewise-smooth curve γ. At each point $z \notin \gamma$ define the value of a function F by

$$F(z) = \int_\gamma \frac{f(\eta)}{\eta - z}\, d\eta.$$

Show that F is holomorphic at $z \notin \gamma$ and

$$F'(z) = \int_\gamma \frac{f(\eta)}{(\eta - z)^2}\, d\eta.$$

Section: Applications of Cauchy's integral formulae.

Corollary. (Liouville's theorem)
If an entire function is bounded, then it is constant.

Proof. Suppose that f is entire and bounded. Then there is a constant M such that

$$|f(z)| \leq M, \qquad \forall z \in \mathbb{C}.$$

Let $z_0 \in \mathbb{C}$ and let $\gamma_r = \{z : |z - z_0| = r\}$. Then

$$|f'(z_o)| = \left| \frac{1!}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{(z - z_0)^2}\, dz \right| \leq \frac{M}{r} \to 0 \quad \text{as} \quad r \to \infty.$$

Therefore for any $z_0 \in \mathbb{C}$ we have $f'(z_0) = 0$ and thus f is constant.

Theorem. (Fundamental theorem of Algebra) Every polynomial of degree greater than zero with complex coefficients has at least one zero.

Proof. Assume that

$$p(z) = a_n z^n + a_{n-1} z^{n-1} \cdots + a_0 = 0.$$

has no zeros. Then $1/p(z)$ is entire. Clearly $|1/p(z)| \to 0$ as $|z| \to \infty$. Indeed, given $\varepsilon > 0$ there is R such that

$$\left|\frac{1}{p(z)}\right| < \varepsilon, \qquad \forall z: |z| > R.$$

Since $1/p(z)$ is entire it is also continuous and therefore there is a constant $M > 0$ such that

$$\left|\frac{1}{p(z)}\right| \leq M, \qquad z: |z| \leq R$$

and thus $|1/p(z)|$ is bounded in $\mathbb{C}$. This implies $1/p$ is constant and this contradicts the fact that $p(z)$ is a polynomial of degree greater than zero.

Corollary.
Every polynomial

$$P(z) = a_n z^n + \cdots + a_0$$

of degree $n \geq 1$ has precisely n roots in $\mathbb{C}$. If these roots are denoted by $w_1, \ldots w_n$, then P can be factored as

$$P(z) = a_n(z - w_1)(z - w_2)\ldots(z - w_n).$$

Proof. We now know that P has at least one root, say w_1. Then writing $z = (z - w_1) + w_1$. Substituting this in $P(z) = a_n z^n + \ldots a_0$ and using the binomial formula we get

$$P(z) = b_n(z - w_1)^n + \cdots + b_1(z - w_1) + b_0,$$

where $b_n = a_n$. Since $P(w_1) = 0$ we have $b_0 = 0$ and thus

$$P(z) = (z - w_1)Q(z).$$

Repeating this we find

$$P(z) = a_n(z - w_1)(z - w_2)\ldots(z - w_n).$$

Theorem. (Moreras theorem)

Suppose f is a continuous function in the open disc D such that for any triangle T contained in D

$$\int_T f(z)\,dz = 0,$$

then f is holomorphic.

Proof. We have proved before that f has a primitive F in D that satisfies $F' = f$. Then F is indefinitely complex differentiable, and therefore f is holomorphic.

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

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Lecture 9

Section: Taylor and Maclaurin series.

Theorem. (Taylor Expansion theorem)
Let f be holomorphic in an open set Ω and let $z_0 \in \Omega$. Then

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 \dots,$$

valid in all circles $\{z : |z - z_0| < r\} \subset \Omega$.

Proof. Let $\gamma = \{\eta : |\eta - z_0| = r\} \subset \Omega$ and let $z : |z - z_0| < r$.

$$\begin{aligned} f(z) = \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{\eta - z}\, d\eta &= \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{(\eta - z_0) - (z - z_0)}\, d\eta \\ &= \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{\eta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\eta - z_0}}\, d\eta \\ &= \frac{1}{2i\pi} \oint_\gamma \frac{f(\eta)}{\eta - z_0} \cdot \Bigg\{ 1 + \frac{z - z_0}{\eta - z_0} + \left(\frac{z - z_0}{\eta - z_0}\right)^2 + \dots \\ &\qquad + \left(\frac{z - z_0}{\eta - z_0}\right)^{n-1} + \frac{\left(\frac{z - z_0}{\eta - z_0}\right)^n}{1 - \frac{z - z_0}{\eta - z_0}} \Bigg\}\, d\eta \end{aligned}$$

Using Cauchy's generalised integral formula applied to the first n terms we obtain

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \dots + \frac{f^{(n-1)}(z_0)}{(n - 1)!}(z - z_0)^{n-1} + R_n,$$

where

$$R_n = \frac{(z - z_0)^n}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)(\eta - z_0)^n}\, d\eta.$$

Let $M = \max_{\eta\in\gamma} |f(\eta)|$ and let $|z - z_0| = \rho$. Then by using the ML-inequality we obtain

$$|R_n| \leq \frac{\rho^n}{2\pi} \frac{M}{(r-\rho)\, r^n} (2\pi r) = \frac{rM}{r-\rho} \left(\frac{\rho}{r}\right)^n.$$

Since $\rho < r$ we conclude that $R_n \to 0$ as $n \to \infty$.

Definition. The expansion

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \frac{f''(z_0)}{2!} (z - z_0)^2 \ldots,$$

is called the Taylor series of f about z_0. The special case in which $z_0 = 0$

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n,$$

is called the Maclaurin series for f.

Example.

$f(z) = e^z$, $f^{(n)}\Big|_{z=0} = 1$. Therefore

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n, \qquad R = \lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty.$$

Example.

$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$, $|z| < 1$ $(R = 1)$.

Example.

$\text{Log}\,(1 - z)$. Note that

$$(\text{Log}\,(1 - z))' = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Integrating both sides we arrive at

$$\text{Log}\,(1 - z) = -\sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} + C = -\sum_{n=1}^{\infty} \frac{1}{n} z^n + C,$$

where $C = \text{Log}\,(1 - 0) = 0$.

Example.

$f(z) = \frac{1}{1+z}$ about $z_0 = i$.

$$\frac{1}{1+z} = \frac{1}{1+i+z-i} = \frac{1}{1+i} \cdot \frac{1}{1 - \left(-\frac{z-i}{1+i}\right)}$$
$$= \frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^n} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(1+i)^{n+1}} (z-i)^n.$$

where R is defined by the inequality

$$\frac{|z-i|}{|1+i|} < 1 \quad \text{or} \quad |z-i| < \sqrt{2}.$$

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Proof. Let D be any disc whose closure is contained in Ω and T any triangle in that disc. Then, since each f_n is holomorphic, Goursats theorem implies

$$\oint_T f_n(z)\, dz = 0, \qquad \text{for all } n.$$

By assumption$f_n \to f$ uniformly in the closure of D, so f is continuous and

$$\oint_T f_n(z)\, dz = \oint_T f(z)\, dz.$$

Therefore

$$\oint_T f(z)\, dz = 0.$$

Using Morera's theorem we find that f is holomorphic in D. Since this conclusion is true for every D whose closure is contained in Ω, we find that f is holomorphic in all of Ω.

Remark. This is not true in the case of real variables: the uniform limit of continuously differentiable functions need not be differentiable. WHY??

Remark. Consider

$$F(z) = \sum_{n=1}^{\infty} f_n(z)$$

where f_n are holomorphic in $\Omega \subset \mathbb{C}$. Assume that the series converges uniformly in compact subsets of Ω, then the theorem guarantees that F is also holomorphic in Ω.

Theorem. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω. Then the sequence of derivatives $\{f'_n\}_{n=1}^{\infty}$ converges uniformly to f' on every compact subset of Ω.

Proof. For any $\widetilde{\Omega} \subset \Omega$ such that $\overline{\widetilde{\Omega}} \subset \Omega$ and given $\delta > 0$ we define $\widetilde{\Omega}_\delta \subset \widetilde{\Omega}$ by

$$\widetilde{\Omega}_\delta = \{z \in \widetilde{\Omega} : \overline{D_\delta}(z) \subset \widetilde{\Omega}\}.$$

By the previous theorem it is enough to show that $\{f'_n\}_{n=1}^{\infty}$ converges uniformly to f' on $\widetilde{\Omega}_\delta$. For any holomorphic function F in Ω_δ we have

$$|F'(z)| = \left| \frac{1}{2\pi i} \oint_{|\eta - z| = \delta} \frac{F(z)}{(\eta - z)^2}\, d\eta \right|$$

$$\leq \frac{1}{2\pi} \max_{\eta \in \overline{\widetilde{\Omega}}} |F(\eta)| \frac{1}{\delta^2} 2\pi\delta \leq \frac{1}{\delta} \max_{\eta \in \overline{\widetilde{\Omega}}} |F(\eta)|.$$

Applying this inequality to $F(z) = f_n - f$ we conclude the proof.

Corollary.

Let each f_n be holomorphic in a given open set $\Omega \subset \mathbb{C}$ and the series

$$F(z) := \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly in compact subsets of Ω. Then F is holomorphic in Ω.

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LECTURES

MATH50001 Complex Analysis 2021

Lecture 10

Last time:

Section: Sequences of holomorphic functions.

Theorem. If $\{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions that converges uniformly to a function f in every compact subset of Ω, then f is holomorphic in Ω.

Corollary.

Let each f_n be holomorphic in a given open set $\Omega \subset \mathbb{C}$ and the series

$$F(z) := \sum_{n=1}^{\infty} f_n(z)$$

converges uniformly in compact subsets of Ω. Then F is holomorphic in Ω.

Section: Holomorphic functions defined in terms of integrals.

Theorem. Let $F(z, s)$ be defined for $(z, s) \in \Omega \times [0, 1]$ where $\Omega \subset \mathbb{C}$ is an open set. Suppose F satisfies the following properties:

- $F(z, s)$ is holomorphic in Ω for each s.
- F is continuous on $\Omega \times [0, 1]$.

Then the function f defined on Ω by

$$f(z) = \int_0^1 F(z, s)\, ds$$

is holomorphic.

Proof. To prove this result, it suffices to prove that f is holomorphic in any disc D contained in Ω. By Moreras theorem this could be achieved by showing that for any triangle T contained in D we have

$$\oint_T \int_0^1 F(z, s)\, ds dz = 0.$$

The proof would be trivial if we could change the order of integration that is not clear. In order to go around this problem we consider for each $n \geq 1$ the Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{k=1}^{n} F(z, k/n).$$

Then by the first assumption f_n is holomorphic in Ω.
We can now show that on any disc D such that $\overline{D} \subset \Omega$, the sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f .
Indeed, since F is continuous on $\Omega \times [0, 1]$ for a given $\varepsilon > 0$ there exists $\delta > 0$ such that as soon $|s_1 - s_2| < \delta$ we have

$$\sup_{z \in D} |F(z, s_1) - F(z, s_2)| < \varepsilon.$$

Then if $n > 1/\delta$ and $z \in D$ we find

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} \left(F(z, k/n) - F(z, s)\right)\, ds \right| \\ &\leq \sum_{k=1}^{n} \int_{(k-1)/n}^{k/n} |F(z, k/n) - F(z, s)|\, ds < \sum_{k=1}^{n} \frac{\varepsilon}{n} = \varepsilon. \end{aligned}$$

By the previous theorem we conclude that f is holomorphic in D and thus in Ω.

Section: Schwarz reflection principle.

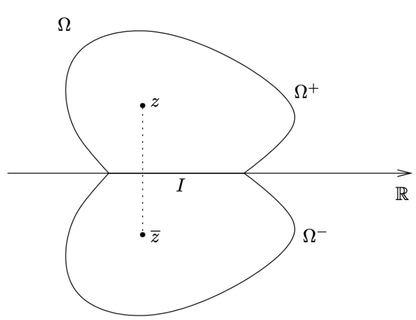
In this section we deal with a simple extension problem for holomorphic functions that is very useful in applications. It is the Schwarz reflection principle that allows one to extend a holomorphic function to a larger domain.

Let $\Omega \subset \mathbb{C}$ be open and symmetric with respect to the real line, that is

$$z \in \Omega \quad \text{iff} \quad \bar{z} \in \Omega.$$

Let

$$\Omega^+ = \{z \in \Omega : \operatorname{Im} z > 0\}, \quad \Omega^- = \{z \in \Omega : \operatorname{Im} z < 0\}$$
$$\text{and} \quad I = \{z \in \Omega : \operatorname{Im} z = 0\}.$$



The only interesting case of the next theorem occurs when I is non-empty.

Theorem. (Symmetry principle)

If f^+ and f^- are holomorphic functions in Ω^+ and Ω^- respectively, that extend continuously to I such that

$$f^+(x) = f^-(x) \quad \text{for all} \quad x \in I,$$

then the function f defined in Ω by

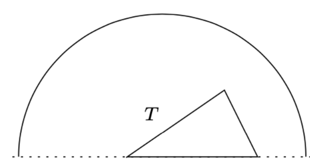
$$f(z) = \begin{cases} f^+(z), & z \in \Omega^+, \\ f^+(z) = f^-(z), & z \in I, \\ f^-(z), & z \in \Omega^-, \end{cases}$$

is holomorphic in Ω.

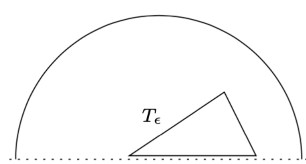
Proof. We only need to prove that f is holomorphic at points of I. Suppose D is a disc centred at a point on I and entirely contained in Ω. We prove that f is holomorphic in D by Moreras theorem. Suppose T is a triangle in D. If T does not intersect I, then

$$\oint_T f(z)\,dz = 0.$$

Suppose now that one side or vertex of T is contained in I, and the rest of T is in, for ex., the upper half-disc.



If T_ε is the triangle obtained from T by slightly raising the edge or vertex which lies on I



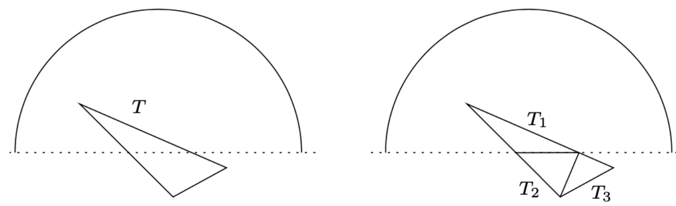
then we have

$$\oint_{T_\varepsilon} f(z)\,dz = 0.$$

since T_ε is entirely contained in the upper half-disc. Letting $\varepsilon \to 0$, by continuity we conclude that

$$\oint_T f(z)\, dz = 0.$$

If the interior of T intersects I, we can reduce the situation to the previous one by splitting T as the union of triangles each of which has an edge or vertex on I



By Moreras theorem we conclude that f is holomorphic in D.
Using the notation introduced before we prove the Schwarz reflection principle.

Theorem. (Schwarz reflection principle)

Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in Ω such that $F|_{\Omega^+} = f$.

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Lecture 11

Last time:

Theorem. (Schwarz reflection principle)

Suppose that f is a holomorphic function in Ω^+ that extends continuously to I and such that f is real-valued on I. Then there exists a function F holomorphic in Ω such that $F|_{\Omega^+} = f$.

Proof. Let us define $F(z)$ for $z \in \Omega^-$ by

$$F(z) = \overline{f(\bar{z})}.$$

To prove that F is holomorphic in Ω^- we note that if $z, z_0 \in \Omega^-$ then $\bar{z}, \bar{z}_0 \in \Omega^+$ and since f is holomorphic in Ω^+ we have

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z}_0)^n.$$

Therefore

$$F(z) = \sum_{n=0}^{\infty} \overline{a_n}(z - z_0)^n$$

and thus F is holomorphic in Ω^-.
Since f is real valued on I we have $\overline{f(x)} = f(x)$ whenever $x \in I$ and hence F extends continuously up to I.

Section: The complex logarithm.

We have seen that to make sense of the logarithm as a single-valued function, we must restrict the set on which we define it. This is the so-called choice of a branch or sheet of the logarithm.

Theorem. Suppose that Ω is simply connected with $1 \in \Omega$, and $0 \notin \Omega$. Then in Ω there is a branch of the logarithm $F(z) = \log_{\Omega}(z)$ so that:

(i) F is holomorphic in Ω,

(ii) $e^{F(z)} = z, \quad \forall z \in \Omega$,

(iii) $F(r) = \log r$ whenever r is a real number and near 1.

In other words, each branch $\log_{\Omega}(z)$ is an extension of the standard logarithm defined for positive numbers.

Proof.
We shall construct F as a primitive of the function $1/z$. Since $0 \notin \Omega$, the function $f(z) = 1/z$ is holomorphic in Ω. We define

$$\log_{\Omega}(z) = F(z) = \int_{\gamma} f(z)\, dz,$$

where γ is any curve in Ω connecting 1 to z. Since Ω is simply connected, this definition does not depend on the path chosen. Then F is holomorphic and $F'(z) = 1/z$ for all $z \in \Omega$. This proves (i).

To prove (ii), it suffices to show that $ze^{-F(z)} = 1$. Indeed,

$$\frac{d}{dz}\left(ze^{-F(z)}\right) = e^{-F(z)} - zF'(z)e^{-F(z)} = (1 - zF'(z))e^{-F(z)} = 0.$$

Thus $ze^{-F(z)}$ is a constant. Using $F(1) = 0$ we find that this constant must be 1.

Section: Zeros of holomorphic functions.

Definition. We say that f has a zero of order m at $z_0 \in \mathbb{C}$ if

$$f^{(k)}(z_0) = 0, \qquad k = 0, 1, \dots m-1,$$

and $f^{(m)}(z_0) \neq 0$.

Theorem. A holomorphic function f has a zero of order m at z_0 if and only if it can be written in the form

$$f(z) = (z - z_0)^m \, g(z),$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof.

$$\begin{aligned} f(z) &= \frac{f^{(m)}(z_0)}{m!}(z-z_0)^m + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0)^{m+1} + \dots \\ &= (z-z_0)^m \left(\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots \right). \end{aligned}$$

Then $f(z) = (z-z_0)^m \, g(z)$ where g is defined by

$$g(z) = \frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots.$$

The above series converges and thus g is holomorphic at z_0.

Conversely, if $f(z) = (z-z_0)^m \, g(z)$, where $g(z_0) \neq 0$, then $f^{(k)}(z_0) = 0$, $k = 0, 1 \dots, m-1$ and $f^{(m)}(z_0) = m! \, g(z_0) \neq 0$.

Corollary. The zeros of a non-constant holomorphic function are isolated; that is every zero has a neighbourhood inside of which it is the only zero.

Proof.

If z_0 is a zero of f of order m, then $f(z) = (z-z_0)^m \, g(z)$, where g is holomorphic at z_0 and $g(z_0) \neq 0$. This means that g is continuous and therefore there is a neighbourhood of z_0 in which $g(z) \neq 0$. Thus $f(z) \neq 0$ except for $z = z_0$.

Section: Laurent Series.

Definition. The series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n = \cdots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \ldots$$

is called Laurent series for f at z_0 where the series converges.

Example.

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!\, z^n} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n, \qquad z \neq 0.$$

Theorem. (Laurent Expansion Theorem) Let f be holomorphic in the annulus $D = \{z : r < |z - z_0| < R\}$.
Then $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z_0)^{n+1}}\, d\eta,$$

and where γ is any simple, closed, piecewise-smooth curve in D that contains z_0 in its interior.

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Lecture 12

Section: Laurent Series.

Definition. The series

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n = \cdots + a_{-2}(z - z_0)^{-2} + a_{-1}(z - z_0)^{-1}$$
$$+ a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

is called Laurent series for f at z_0 where the series converges.

Theorem. (Laurent Expansion Theorem) Let f be holomorphic in the annulus $D = \{z : r < |z - z_0| < R\}$.
Then $f(z)$ can be expressed in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{(\eta - z_0)^{n+1}} \, d\eta,$$

and where γ is any simple, closed, piecewise-smooth curve in D that contains z_0 in its interior.

Proof. Let us for simplicity assume that $z_0 = 0$ and consider

$$\gamma_1 = \{z : |z| = R' < R\} \quad \text{and} \quad \gamma_2 = \{z : |z| = r' > r\}$$

and such that $z \in D' = \{z : r' < |z| < R'\}$. Then

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta - z} \, d\eta - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\eta)}{\eta - z} \, d\eta := I_1 - I_2.$$

If $\eta \in \gamma_1$ then $|\eta| > |z|$ and we have

$$I_1 = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta - z}\, d\eta = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta(1 - z/\eta)}\, d\eta$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_{\gamma_1} \frac{f(\eta)}{\eta^{n+1}}\, d\eta\, z^n.$$

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta - z}\, d\eta - \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\eta)}{\eta - z}\, d\eta := I_1 - I_2.$$

If $\eta \in \gamma_2$ then $|\eta| < |z|$ and thus

$$-I_2 = -\frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\eta)}{\eta - z}\, d\eta = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\eta)}{z(1 - \eta/z)}\, d\eta$$

$$= \frac{1}{2\pi i} \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} \oint_{\gamma_2} f(\eta)\, \eta^n\, d\eta = [n + 1 = -k]$$

$$= \frac{1}{2\pi i} \sum_{k=-\infty}^{-1} \oint_{\gamma_2} \frac{f(\eta)}{\eta^{k+1}}\, d\eta\, z^k.$$

Finally we obtain

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n,$$

where

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_2} \frac{f(\eta)}{\eta^{n+1}}\, d\eta, \quad n = -1, -2, \ldots,$$

and

$$a_n = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{f(\eta)}{\eta^{n+1}}\, d\eta, \quad n = 0, 1, 2, \ldots.$$

It remains to show that

$$a_n = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{\eta^{n+1}}\, d\eta, \quad n = 0, \pm 1, \pm 2, \ldots.$$

Indeed,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{f(\eta)}{\eta^{n+1}}\, d\eta = \frac{1}{2\pi i} \sum_{k=-\infty}^{\infty} a_k \oint_{\gamma} \frac{\eta^k}{\eta^{n+1}}\, d\eta = a_n.$$

Example.

Find Laurent series at $z_0 = 0$ for $f(z) = 1/(z-1)$ for $z: |z| > 1$.

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} = \sum_{k=1}^{\infty}\frac{1}{z^k}.$$

This series converges for $|z| > 1$.

Example.

Find Laurent series at $z_0 = 0$ for $f(z) = \frac{1}{z(z+2)}$ for $0 < |z| < 2$.

$$\frac{1}{z(z+2)} = \frac{1}{2}\Big(\frac{1}{z} - \frac{1}{z+2}\Big) = \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{4(1+z/2)}$$
$$= \frac{1}{2}\cdot\frac{1}{z} - \frac{1}{4}\sum_{n=0}^{\infty}\Big(-\frac{z}{2}\Big)^n = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{z^n}{2^{n+2}} + \frac{1}{2}\cdot\frac{1}{z}.$$

Section: Poles of holomorphic functions.

Definition. A point z_0 is called a singularity of a complex function f if f is not holomorphic at z_0, but every neighbourhood of z_0 contains at least one point at which f is holomorphic.

Definition. A singularity z_0 of a complex function is said to be isolated if there exists a neighbourhood of z_0 in which z_0 is the only singularity of f.

Example. $f(z) = \frac{1}{1-z}$, $z_0 = 1$, $f(z) = e^{1/z^2}$, $z_0 = 0$; $f(z) = \frac{1}{(z+2)^2}$, $z_0 = -2$.

Definition. Suppose a holomorphic function f has an isolated singularity at z_0 and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

is the Laurent expansion of f valid in some annulus $0 < |z - z_0| < R$. Then

- If $a_n = 0$ for all $n < 0$, z_0 is called a removable singularity
- If $a_n = 0$ for $n < -m$ where m a fix positive integer, but $a_{-m} \neq 0$, z_0 is called a pole of order m.
- If $a_n \neq 0$ for infinitely many negative n's, z_0 is called an essential singularity.

Example.
$f(z) = \frac{\sin z}{z}$; $f(z) = e^{1/z}$; $f(z) = \frac{1}{z^3(z+2)^2}$.

Theorem. A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

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Lecture 13

Section: Poles of holomorphic functions.

Definition. Suppose a holomorphic function f has an isolated singularity at z_0 and

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

is the Laurent expansion of f valid in some annulus $0 < |z - z_0| < R$. Then

- If $a_n = 0$ for all $n < 0$, z_0 is called a removable singularity
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Example.
$f(z) = \frac{\sin z}{z}$; $f(z) = e^{1/z}$; $f(z) = \frac{1}{z^3(z+2)^2}$.

Theorem. A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z - z_0)^m},$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof. If g is holomorphic at z_0 and $g(z_0) \neq 0$ then for some $R > 0$

$$g(z) = a_0 + a_1(z - z_0) + \ldots, \qquad |z - z_0| < R,$$

where $a_0 = g(z_0) \neq 0$. Then

$$f(z) = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \ldots, \qquad 0 < |z - z_0| < R.$$

This implies that z_0 is a pole of order m.

Conversely, if f has a pole of order m at z_0, then the Laurent expansion of f about z_0 equals

$$\begin{aligned} f(z) = \frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \dots \\ + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \dots \\ = \frac{1}{(z-z_0)^m}\Big(a_{-m} + a_{-m+1}(z-z_0) + \dots\Big). \end{aligned}$$

Section: Residue Theory.

Definition. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n\,(z-z_0)^n, \qquad 0 < |z-z_0| < R.$$

be the Laurent series for f at z_0. The residue of f at z_0 is

$$\operatorname{Res}\,[f, z_0] = a_{-1}.$$

Theorem. Let $\gamma \subset \{z : 0 < |z-z_0| < R\}$ be a simple, closed, piecewise-smooth curve that contains z_0. Then

$$\operatorname{Res}\,[f, z_0] = \frac{1}{2\pi i}\oint_\gamma f(z)\,dz.$$

Proof. Let $0 < r < R$. By using the Deformation theorem we obtain

$$\begin{aligned} \frac{1}{2\pi i}\oint_\gamma f(z)\,dz &= \frac{1}{2\pi i}\oint_{|z-z_0|=r} f(z)\,dz \\ &= \frac{1}{2\pi i}\oint_{|z-z_0|=r}\sum_{n=-\infty}^{\infty} a_n\,(z-z_0)^n\,dz \\ &= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty}\int_0^{2\pi} a_n\,r^n\,e^{in\theta}\,i\,r\,e^{i\theta}\,d\theta = a_{-1}. \end{aligned}$$

Theorem. Let f be holomorphic function inside and on a simple, closed, piecewise-smooth curve γ except at the singularities $z_1, \dots, z_n$ in its interior. Then

$$\oint_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\,[f, z_j].$$

Proof. Let $\gamma_j = \{z : |z - z_j| = r_j\} \subset \Omega$. Then by using the Deformation theorem we find

$$\oint_{\gamma} f(z)\, dz = \sum_{j=1}^{n} \oint_{\gamma_j} f(z)\, dz.$$

Example. Evaluate $\oint_{|z|=1} e^{1/z}\, dz$.
Clearly

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\, z^2} + \frac{1}{3!\, z^3} + \dots$$

Therefore

$$\oint_{|z|=1} e^{1/z}\, dz = 2\pi i.$$

Let

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \dots$$

and let $g(z) = (z - z_0)^m f(z)$.

$m = 1$. Then $g(z) = a_{-1} + a_0\,(z - z_0) + \dots$ and therefore

$$\text{Res}\,[f, z_0] = a_{-1} = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0)\, f(z).$$

$m = 2$. Then $g(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$ and

$$\text{Res}\,[f, z_0] = a_{-1} = \frac{d}{dz} g(z)\Big|_{z=z_0} = \lim_{z \to z_0} \frac{d}{dz}((z - z_0)^2\, f(z)).$$

m.

$$\text{Res}\,[f, z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m\, f(z)\right).$$

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Lecture 14

Section: Residue Theory.

Definition. Let

$$f(z) = \sum_{-\infty}^{\infty} a_n (z - z_0)^n, \qquad 0 < |z - z_0| < R.$$

be the Laurent series for f at z_0. The residue of f at z_0 is

$$\text{Res}\,[f, z_0] = a_{-1}.$$

Theorem. Let $\gamma \subset \{z : 0 < |z - z_0| < R\}$ be a simple, closed, piecewise-smooth curve that contains z_0. Then

$$\text{Res}\,[f, z_0] = \frac{1}{2\pi i} \oint_\gamma f(z)\, dz.$$

Theorem. Let f be holomorphic function inside and on a simple, closed, piecewise-smooth curve γ except at the singularities $z_1, \dots, z_n$ in its interior. Then

$$\oint_\gamma f(z)\, dz = 2\pi i \sum_{j=1}^{n} \text{Res}\,[f, z_j].$$

Let

$$f(z) = a_{-m}(z - z_0)^{-m} + a_{-m+1}(z - z_0)^{-m+1} + \dots$$

and let $g(z) = (z - z_0)^m f(z)$.

$m = 1$. Then $g(z) = a_{-1} + a_0 (z - z_0) + \dots$ and therefore

$$\text{Res}\,[f, z_0] = a_{-1} = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0)\, f(z).$$

$m = 2$. Then $g(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$ and

$$\operatorname{Res}[f, z_0] = a_{-1} = \frac{d}{dz} g(z)\Big|_{z=z_0} = \lim_{z\to z_0} \frac{d}{dz}\left((z - z_0)^2 f(z)\right).$$

m.

$$\operatorname{Res}[f, z_0] = \lim_{z\to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right).$$

Example.
Evaluate

$$\oint_\gamma \frac{1}{z^5 - z^3}\, dz, \qquad \gamma = \{z : |z| = 1/2\}$$

Clearly

$$\frac{1}{z^5 - z^3} = \frac{1}{z^3(z-1)(z+1)}.$$

Since $z = \pm 1$ is outside γ we obtain

$$\oint_\gamma \frac{1}{z^5 - z^3}\, dz = 2\pi i \operatorname{Res}[f, 0] = 2\pi i \frac{1}{2!} \lim_{z\to 0} (z^3 f(z))''$$

$$= \pi i \lim_{z\to 0} \left(\frac{1}{z^2 - 1}\right)'' = \pi i \lim_{z\to 0} \left(\frac{-2z}{(z^2-1)^2}\right)'$$

$$= \pi i \lim_{z\to 0} \left(\frac{-2(z^2-1)^2 - (-2z)\, 2(z^2-1)\, 2z}{(z^2-1)^4}\right) = -2\pi i.$$

Example.

Evaluate

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\, dz, \qquad \gamma = \{z : |z| = 2\}.$$

Because the integrand has singularities at $z = -5$ and $z = \pm 1$ only the last two are interior to γ, we have

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\, dz$$

$$= 2\pi i \left\{ \operatorname{Res}\left[\frac{1}{(z+5)(z^2-1)}, -1\right] + \operatorname{Res}\left[\frac{1}{(z+5)(z^2-1)}, 1\right] \right\}.$$

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\, dz, \qquad \gamma = \{z : |z| = 2\}.$$

Now $z = 1$ is a pole of order 1 and therefore

$$\operatorname{Res}\left[\frac{1}{(z+5)(z^2-1)}, 1\right] = \lim_{z\to 1}\frac{z-1}{(z+5)(z^2-1)} = \lim_{z\to 1}\frac{1}{(z+5)(z+1)} = \frac{1}{12}.$$

Similarly, $z = -1$ is a simple pole and

$$\operatorname{Res}\left[\frac{1}{(z+5)(z^2-1)}, -1\right] = \lim_{z\to -1}\frac{z+1}{(z+5)(z^2-1)} = \lim_{z\to -1}\frac{1}{(z+5)(z-1)} = -\frac{1}{8}.$$

Thus,

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,dz = 2\pi i\left(\frac{1}{12} - \frac{1}{8}\right) = -\frac{\pi i}{12}.$$

Section: The argument principle.

Theorem. (Principle of the Argument)

Let f be holomorphic in an open set Ω except for a finite number of poles and let γ be a simple, closed, piecewise-smooth curve in Ω that does not pass through any poles or zeros of f. Then

$$\oint_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(N - P),$$

where N and P are the sums of the orders of the zeros and poles of f inside γ.

Remark. Why Principle of the Argument?

Indeed, let γ be a closed curve. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{d}{dz}\log f(z)\,dz = \frac{1}{2\pi i}\log f(z)\Big|_{z_1}^{z_2}$$
$$= \frac{1}{2\pi i}\Big(\ln|f(z_2)| - \ln|f(z_1)| + i(\arg f(z_2) - \arg f(z_1))\Big) = \frac{1}{2\pi}\Delta \arg f(z).$$

Example. Let $f(z) = z^3$ and let $\gamma = \{z : z = e^{i\theta},\ \theta \in [0, 2\pi]\}$, then $f(z) = e^{i3\theta}$ and $\frac{1}{2\pi}\Delta_\gamma \arg f = 3$.

Example. Let $f(z) = 1/z$ and let $\gamma = \{z : z = e^{i\theta},\ \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi}\Delta_\gamma \arg f = -1$.

Example. Let $f(z) = z + 2$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi} \Delta_\gamma \arg f = 0$.

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Lecture 15

Section: The argument principle.

Theorem. (Principle of the Argument)

Let f be holomorphic in an open set Ω except for a finite number of poles and let γ be a simple, closed, piecewise-smooth curve in Ω that does not pass through any poles or zeros of f. Then

$$\oint_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(N-P),$$

where N and P are the sums of the orders of the zeros and poles of f inside γ.

Remark. Why Principle of the Argument?

Indeed, let γ be a closed curve. Then

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{d}{dz}\log f(z)\,dz = \frac{1}{2\pi i}\log f(z)\Big|_{z_1}^{z_2}$$
$$= \frac{1}{2\pi i}\Big(\ln|f(z_2)| - \ln|f(z_1)| + i(\arg f(z_2) - \arg f(z_1))\Big) = \frac{1}{2\pi}\Delta \arg f(z).$$

Proof of Theorem.

Step 1. If z_1 is a zero of order n, then

$$f(z) = (z-z_1)^n g(z),$$

where g is holomorphic at z_1 and $g(z_1) \neq 0$. Consequently

$$f'(z) = n\,(z-z_1)^{n-1}\,g(z) + (z-z_1)^n\,g'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{n}{z-z_1} + \frac{g'(z)}{g(z)}.$$

Since $g(z_1) \neq 0$ it follows that $g(z) \neq 0$ in some neighbourhood of z_1. Therefore there is $r > 0$ such that $g'(z)/g(z)$ is holomorphic for $z : |z - z_1| \leq r$ and we have

$$\oint_{|z-z_1|=r} \frac{f'(z)}{f(z)}\, dz = \oint_{|z-z_1|=r} \frac{n}{z - z_1}\, dz + \oint_{|z-z_1|=r} \frac{g'(z)}{g(z)}\, dz = 2\pi i\, n.$$

Step 2. If z_2 is a pole of order p at z_2, then

$$f(z) = \frac{g(z)}{(z - z_2)^p},$$

where g is holomorphic at z_2 and $g(z_2) \neq 0$. Consequently

$$f'(z) = \frac{-p\, g(z)}{(z - z_2)^{p+1}} + \frac{g'(z)}{(z - z_2)^p}$$

and

$$\frac{f'(z)}{f(z)} = \frac{-p}{z - z_2} + \frac{g'(z)}{g(z)}.$$

Since $g(z_2) \neq 0$ it follows that $g(z) \neq 0$ in some neighborhood of z_2. Therefore there is $r > 0$ such that $g'(z)/g(z)$ is holomorphic for $z : |z - z_2| \leq r$ and we have

$$\oint_{|z-z_2|=r} \frac{f'(z)}{f(z)}\, dz = \oint_{|z-z_2|=r} \frac{-p}{z - z_2}\, dz + \oint_{|z-z_2|=r} \frac{g'(z)}{g(z)}\, dz = -2\pi i\, p.$$

Finally we complete the proof by locating finite number of zeros and poles and using the Deformation theorem.

Example. Let $f(z) = (1 + z)/z = 1 + 1/z$, where $\gamma = \{z : z = 2\, e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $N - P = 0$. Indeed,

$$w = f(z) = 1 + \frac{1}{2}\, e^{-i\theta} = 1 + \frac{1}{2} \cos\theta - \frac{i}{2} \sin\theta$$

and finally we have $\frac{1}{2\pi}\, \Delta_\gamma \arg f = 0$.

Example. The same problem with $\gamma = \{z : |z| = 1/2\}$ implies $w = f(z) = 1 + 2\cos\theta - 2i \sin\theta$. Thus $\frac{1}{2\pi}\, \Delta_\gamma \arg f = -1$.

Theorem. (Rouche's Theorem)

Let f and g be holomorphic in an open set Ω and let $\gamma \subset \Omega$ be a simple, closed, piecewise-smooth curve that contains in its interior only points of Ω.

If $|g(z)| < |f(z)|$, $z \in \gamma$, then the sums of the orders of the zeros of $f + g$ and f inside γ are the same.

Proof.

Let us consider the function

$$f_t(z) = f(z) + t\, g(z), \qquad t \in [0, 1].$$

Clearly $f_0(z) = f(z)$ and $f_1(z) = f(z) + g(z)$. Let $n(t)$ be the number of zeros of f_t inside γ counted with multiplicities. The inequality $|f(z)| > |g(z)|$, $z \in \gamma$, implies that f_t has no zeros on γ and hence

$$F_t(z) = \frac{f_t'(z)}{f_t(z)}$$

has no poles on γ. Therefore the argument principle implies

$$n(t) = \frac{1}{2\pi i} \oint_\gamma F_t(z)\, dz = \frac{1}{2\pi i} \oint_\gamma \frac{f_t'(z)}{f_t(z)}\, dz.$$

Since $n(t) \in \mathbb{Z}$, in order to prove that $N(f) = N(f+g)$ it is enough to show that $n(t)$ is continuous.

Indeed, from $|f(z)| > |g(z)|$ we obtain that there is $\delta > 0$ such that $|f_t| = |f + tg| > \delta$, $z \in \gamma$, $t \in [0, 1]$. Thus for any $t_1, t_2 \in [0, 1]$ we have

$$\begin{aligned} |n(t_2) - n(t_1)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{f'(z) + t_2\, g'(z)}{f(z) + t_2\, g(z)} - \frac{f'(z) + t_1\, g'(z)}{f(z) + t_1\, g(z)} \right) dz \right| \\ &\le \frac{1}{2\pi} \max_\gamma \left| \frac{(t_2 - t_1)(f(z)g'(z) - f'(z)g(z))}{(f(z) + t_2\, g(z))f((z) + t_1\, g(z))} \right| \cdot \text{length}\, \gamma \\ &\le C\, \frac{1}{\delta^2}\, |t_2 - t_1|. \end{aligned}$$

MATH50001 COMPLEX ANALYSIS 2021
LECTURES

Lecture 16

1. The argument principle

Theorem 1.1 (Principle of the Argument). *Let* f *be holomorphic in an open set* Ω *except for a finite number of poles and let* γ *be a simple, closed, piecewise-smooth curve in* Ω *that does not pass through any poles or zeros of* f. *Then*

$$\oint_\gamma \frac{f'(z)}{f(z)}\,dz = 2\pi i(N - P),$$

where N *and* P *are the sums of the orders of the zeros and poles of* f *inside* γ.

Remark 1.1. *Why Principle of the Argument?*
Indeed, let γ *be a closed curve. Then*

$$\frac{1}{2\pi i}\oint_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\oint_\gamma \frac{d}{dz}\,\log f(z)\,dz = \frac{1}{2\pi i}\,\log f(z)\Big|_{z_1}^{z_2}$$
$$= \frac{1}{2\pi i}\Big(\ln|f(z_2)| - \ln|f(z_1)| + i(\arg f(z_2) - \arg f(z_1))\Big) = \frac{1}{2\pi}\Delta \arg f(z).$$

Example 1.1. Let $f(z) = z^3$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$, then $f(z) = e^{i3\theta}$ and $\frac{1}{2\pi}\Delta_\gamma \arg f = 3$.

Example 1.2. Let $f(z) = 1/z$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi}\Delta_\gamma \arg f = -1$.

Example 1.3. Let $f(z) = z + 2$ and let $\gamma = \{z : z = e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $\frac{1}{2\pi}\Delta_\gamma \arg f = 0$.

Proof. Step 1. If z_1 is a zero of order n, then

$$f(z) = (z - z_1)^n g(z),$$

where g is holomorphic at z_1 and $g(z_1) \neq 0$. Consequently

$$f'(z) = n\,(z - z_1)^{n-1}\,g(z) + (z - z_1)^n\,g'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_1} + \frac{g'(z)}{g(z)}.$$

Since $g(z_1) \neq 0$ it follows that $g(z) \neq 0$ in some neighborhood of z_1. Therefore there is $r > 0$ such that $g'(z)/g(z)$ is holomorphic for $z : |z - z_1| \leq r$ and we have

$$\oint_{|z-z_1|=r} \frac{f'(z)}{f(z)}\, dz = \oint_{|z-z_1|=r} \frac{n}{z - z_1}\, dz + \oint_{|z-z_1|=r} \frac{g'(z)}{g(z)}\, dz = 2\pi i\, n.$$

Step 2. If z_2 is a pole of order p at z_2, then

$$f(z) = \frac{g(z)}{(z - z_2)^p},$$

where g is holomorphic at z_2 and $g(z_2) \neq 0$. Consequently

$$f'(z) = \frac{-p\, g(z)}{(z - z_2)^{p+1}} + \frac{g'(z)}{(z - z_2)^p}$$

and

$$\frac{f'(z)}{f(z)} = \frac{-p}{z - z_2} + \frac{g'(z)}{g(z)}.$$

Since $g(z_2) \neq 0$ it follows that $g(z) \neq 0$ in some neighborhood of z_2. Therefore there is $r > 0$ such that $g'(z)/g(z)$ is holomorphic for $z : |z - z_2| \leq r$ and we have

$$\oint_{|z-z_2|=r} \frac{f'(z)}{f(z)}\, dz = \oint_{|z-z_2|=r} \frac{-p}{z - z_2}\, dz + \oint_{|z-z_2|=r} \frac{g'(z)}{g(z)}\, dz = -2\pi i\, p.$$

Finally we complete the proof by locating finite number of zeros and poles and using the Deformation theorem. $\square$

Example 1.4. Let $f(z) = (1 + z)/z = 1 + 1/z$, where $\gamma = \{z : z = 2\, e^{i\theta}, \theta \in [0, 2\pi]\}$. Then $N - P = 0$. Indeed,

$$w = f(z) = 1 + \frac{1}{2}\, e^{-i\theta} = 1 + \frac{1}{2}\, \cos\theta - \frac{i}{2}\, \sin\theta$$

and finally we have $\frac{1}{2\pi}\, \Delta_\gamma \arg f = 0$.

Example 1.5. The same problem with $\gamma = \{z : |z| = 1/2\}$ implies $w = f(z) = 1 + 2\cos\theta - 2i\sin\theta$. Thus $\frac{1}{2\pi}\, \Delta_\gamma \arg f = -1$.

Theorem 1.2. *(Rouche's Theorem) Let* f *and* g *be holomorphic in an open set* Ω *and let* $\gamma \subset \Omega$ *be a simple, closed, piecewise-smooth curve that contains in its interior only points of* Ω. *If* $|g(z)| < |f(z)|$, $z \in \gamma$, *then the sums of the orders of the zeros of* $f + g$ *and* f *inside* γ *are the same.*

Proof. Let us consider the function

$$f_t(z) = f(z) + t\,g(z), \qquad t \in [0, 1].$$

Clearly $f_0(z) = f(z)$ and $f_1(z) = f(z) + g(z)$. Let $n(t)$ be the number of zeros of f_t inside γ counted with multiplicities. The inequality $|f(z)| > |g(z)|$, $z \in \gamma$, implies that f_t has no zeros on γ and hence

$$F_t(z) = \frac{f_t'(z)}{f_t(z)}$$

has no poles on γ. Therefore the argument principle implies

$$n(t) = \frac{1}{2\pi i} \oint_\gamma F_t(z)\,dz = \frac{1}{2\pi i} \oint_\gamma \frac{f_t'(z)}{f_t(z)}\,dz.$$

Since $n(t) \in \mathbb{Z}$, in order to prove that $N(f) = N(f + g)$ it is enough to show that $n(t)$ is continuous.

Indeed, from $|f(z)| > |g(z)|$ we obtain that there is $\delta > 0$ such that $|f_t| = |f + tg| > \delta$, $z \in \gamma$, $t \in [0, 1]$. Thus for any $t_1, t_2 \in [0, 1]$ we have

$$\begin{aligned}
|n(t_2) - n(t_1)| &= \left| \frac{1}{2\pi i} \int_\gamma \left(\frac{f'(z) + t_2\,g'(z)}{f(z) + t_2\,g(z)} - \frac{f'(z) + t_1\,g'(z)}{f(z) + t_1\,g(z)} \right) dz \right| \\
&\le \frac{1}{2\pi} \max_\gamma \left| \frac{(t_2 - t_1)(f(z)g'(z) - f'(z)g(z))}{(f(z) + t_2\,g(z))f((z) + t_1\,g(z))} \right| \cdot \text{length}\,\gamma \\
&\le C\,\frac{1}{\delta^2}\,|t_2 - t_1|.
\end{aligned}$$

□

Example 1.6. Show that $N(z^5 + 3z^2 + 6z + 1) = 1$ inside the curve $|z| = 1$.

Proof. Let $f(z) = 6z + 1$ and $g(z) = z^5 + 3z^2$. If $|z| = 1$, then $|g(z)| < |f(z)|$. Indeed

$$|g(z)| = |z^5 + 3z^2| \le |z^5| + 3|z^2| = 4.$$
$$|f(z)| = |6z + 1| \ge 6|z| - 1 = 5 > 4 \ge |g(z)|.$$

Since $6z + 1 = 0$ has only one zero $z = -1/6$, then $N(f) = N(f + g) = 1$. □

Example 1.7. Show that all roots of $w(z) = z^7 - 2z^2 + 8 = 0$ are inside the annulus $1 < |z| < 2$.

Proof. 1. Consider first $\gamma = \{z : |z| = 2\}$. Let $f(z) = z^7$ and $g(z) = -2z^2 + 8$. If $|z| = 2$, then $|f(z)| = 2^7 = 128$ and

$$|g(z)| = |-2z^2 + 8| \leq 2|z^2| + 8 = 2\,2^2 + 8 = 16 < 128 = |f(z)|.$$

Since $|f(z)| > |g(z)|$, $|z| = 2$, then the number of roots of w inside the curve $|z| = 2$ coincides with the number of roots of $f(z) = z^7 = 0$ and equals 7.

2. Let now $\gamma = \{z : |z| = 1\}$ and let $f(z) = 8$ and $g(z) = z^7 - 2z^2$. Then

$$|z^7 - 2z^2| \leq |z^7| + 2|z|^2 \leq 3 < 8.$$

The equation $f(z) = 0$ has no solutions. This implies that all zeros of $f + g$ are outside $\gamma = \{z : |z| = 1\}$. □

2. Open mapping theorem and Maximum modulus principle

Definition 2.1. A mapping is said to be open if it maps open sets to open sets.

Theorem 2.1. *(Open mapping theorem) If f is holomorphic and non-constant in an open set $\Omega \subset \mathbb{C}$, then f is open.*

Proof. Let w_0 belong to the image of f, $w_0 = f(z_0)$. We must prove that all points for $while$ near w_0 also belong to the image of f.
Define $g(z) = f(z) - w$. Then

$$g(z) = (f(z) - w_0) + (w_0 - w) = F(z) + G(z).$$

Now choose $\delta > 0$ such that the disc $\{z : |z - z_0| \leq \delta$ is contained in Ω and $f(z) \neq w_0$ on the circle $|z - z_0| = \delta$.

(*WHY is it possible??*)

We then select $\varepsilon > 0$ so that we have $|f(z) - w_0| \geq \varepsilon$ on the circle $C_\delta = \{z : |z - z_0| = \delta\}$. Now if $|w - w_0| < \varepsilon$ we have $|F(z)| > |G(z)|$ on the circle C_δ, and by Rouchés theorem we conclude that $g = F + G$ has a zero inside C_δ since F has one. □

Remark 2.1. *Note that if f is open, then $|f|$ is also open.*

Theorem 2.2. *(Maximum modulus principle)*
If f *is a non-constant holomorphic function is an open set* $\Omega \subset \mathbb{C}$*, then* f *cannot attain a maximum in* Ω*.*

Proof. Suppose that f did attain a maximum at $z_0 \subset \Omega$. Since f is holomorphic it is an open mapping, and therefore, if $D \subset \Omega$ is a small open disc centred at z_0, its image $f(D)$ is open and contains $f(z_0)$. This proves that there are points $z \in D$ such that $|f(z)| > |f(z_0)|$, a contradiction. □

Corollary 2.1. *Suppose that is an open set* $\Omega \subset \mathbb{C}$ *with compact closure* $\overline{\Omega}$*. If* f *is holomorphic on* Ω *and continuous on* $\overline{\Omega}$ *then*

$$\sup_{z\in\Omega} |f(z)| \leq \sup_{z\in\overline{\Omega}\setminus\Omega} |f(z)|.$$

Remark 2.2. *The hypothesis that* $\overline{\Omega}$ *is compact (that is, bounded) is essential for the conclusion.*

WHY ??? Give an example.

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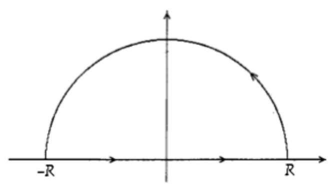
Lecture 17

Section: Evaluation of Definite integrals.

Example. Evaluate
$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

Solution. Consider
$$\oint_{\gamma} \frac{1}{1+z^2}\, dz,$$
where $\gamma = \gamma_1 \cup \gamma_2$.



$$\gamma_1 = \{z : z = x + i0,\ -R < x < R\},$$
$$\text{and} \quad \gamma_2 = \{z : z = R\, e^{i\theta},\ 0 \le \theta \le \pi\}, \quad R > 1.$$

The integrant $(1+z^2)^{-1}$ has simple poles at $\pm i$ and only the pole at i is interior to γ. Therefore
$$\oint_{\gamma} \frac{1}{1+z^2}\, dz = 2\pi i \operatorname{Res}\left[\frac{1}{1+z^2}, i\right] = 2\pi i \lim_{z\to i} \frac{z-i}{1+z^2} = 2\pi i \frac{1}{2i} = \pi.$$

Then
$$\pi = \int_{-R}^{R} \frac{1}{1+x^2}\, dx + \int_{\gamma_2} \frac{1}{1+z^2}\, dz.$$

Note that by using the ML-inequality we have
$$\left| \int_{\gamma_2} \frac{1}{1+z^2}\, dz \right| \le \frac{1}{R^2-1} R\pi \to 0, \qquad R \to \infty.$$

Finally we have
$$\pi = \lim_{R\to\infty} \left(\int_{-R}^{R} \frac{1}{1+x^2}\, dx + \int_{\gamma_2} \frac{1}{1+z^2}\, dz \right) = \int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

Example. Evaluate

$$\int_0^\infty \frac{1}{1+x^3}\,dx.$$

Solution. Consider

$$\oint_\gamma \frac{1}{1+z^3}\,dz, \qquad \gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3,$$

where

$$\gamma_1 = \{z : z = x + iy, x \in [0, R], y = 0\}, \quad R > 1,$$
$$\gamma_2 = \{z : z = R\,e^{i\theta}, 0 \le \theta \le 2\pi/3\},$$
$$\gamma_3 = \{z : z = r\,e^{i2\pi/3}, r \in [R, 0]\}$$

The function $1 + z^3$ has three zeros

$$z_1 = e^{i\pi/3}, \quad z_2 = e^{i\pi} \quad \text{and} \quad z_3 = e^{5i\pi/3},$$

of which only z_1 is internal for γ. Therefore

$$\oint_\gamma \frac{1}{1+z^3}\,dz = 2\pi i\,\mathrm{Res}\left[\frac{1}{1+z^3}, e^{i\pi/3}\right]$$
$$= 2\pi i \lim_{z\to e^{i\pi/3}} \frac{z - e^{i\pi/3}}{1+z^3}$$
$$= 2\pi i \lim_{z\to e^{i\pi/3}} \frac{1}{3z^2} = 2\pi i\,\frac{1}{3}\,e^{-2i\pi/3} = \frac{2}{3}\pi i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$
$$= \frac{\pi\sqrt{3}}{3} - i\,\frac{\pi}{3}.$$

Note that

$$\lim_{R\to\infty}\int_{\gamma_1} \frac{1}{1+z^3}\,dz = \lim_{R\to\infty}\int_0^R \frac{1}{1+x^3}\,dx = \int_0^\infty \frac{1}{1+x^3}\,dx.$$

Moreover by using that $|1 + R^3 e^{i3\theta}| > |R^3 - 1|$ and the ML-inequality we have

$$\left|\int_{\gamma_2} \frac{1}{1+z^3}\,dz\right| = \left|\int_0^{2\pi/3} \frac{1}{1+R^3\,e^{i3\theta}}\,iR\,e^{i\theta}\,d\theta\right|$$
$$\le \frac{R}{R^3-1}\cdot\frac{2\pi}{3} \to 0, \quad \text{as} \quad R \to \infty.$$

The integral over γ_3 equals

$$\int_{\gamma_3} \frac{1}{1+z^3}\,dz = \int_R^0 \frac{1}{1+r^3\,e^{i2\pi 3/3}}\,e^{i2\pi/3}\,dr$$
$$= -\Big(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\Big)\int_0^R \frac{1}{1+r^3}\,dr \to \Big(\frac{1}{2}-i\frac{\sqrt{3}}{2}\Big)\int_0^\infty \frac{1}{1+r^3}\,dr, \quad \text{as} \quad R\to\infty.$$

Finally we obtain

$$\frac{\pi\sqrt{3}}{3} - i\,\frac{\pi}{3} = \frac{\pi}{3}\,(\sqrt{3}-i)$$
$$= \int_0^\infty \frac{1}{1+x^3}\,dx + \Big(\frac{1}{2}-i\frac{\sqrt{3}}{2}\Big)\int_0^\infty \frac{1}{1+r^3}\,dr$$
$$= \Big(\frac{3}{2}-i\frac{\sqrt{3}}{2}\Big)\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{\sqrt{3}}{2}\,(\sqrt{3}-i)\int_0^\infty \frac{1}{1+x^3}\,dx.$$

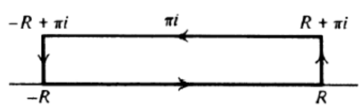
This implies

$$\int_0^\infty \frac{1}{1+x^3}\,dx = \frac{2\pi}{3\sqrt{3}}.$$

Example. Evaluate

$$\int_{-\infty}^\infty \frac{\cos x}{e^x+e^{-x}}\,dx.$$

Solution. Let introduce the contour



$$\gamma = \gamma_1\cup\gamma_2\cup\gamma_3\cup\gamma_4$$
$$= [-R,R]\cup[R,R+i\pi]\cup[R+i\pi,-R+i\pi]\cup[-R+i\pi,-R]$$

Let $f(z) = e^{iz}/(e^z+e^{-z})$. The singularities of f are solutions of the equation $e^z+e^{-z}=0$, or

$$e^{2x}e^{2iy} = -1.$$

Solutions of this equation are $x=0$, $y=\pi/2+k\pi$, $k=0,\pm1,\pm2,\dots$. The only singularity of f in the interior of the counter γ is at $z_0 = i\pi/2$ and

$$\text{Res}\left[\frac{e^{iz}}{e^z+e^{-z}}, i\pi/2\right] = \lim_{z\to i\pi/2}\frac{(z-i\pi/2)e^{-\pi/2}}{e^z+e^{-z}} = \frac{e^{i(i\pi/2)}}{2i}.$$

Therefore

$$\oint_\gamma \frac{e^{iz}}{e^z+e^{-z}}\,dz = 2\pi i\cdot\frac{e^{i(i\pi/2)}}{2i} = \pi\, e^{-\pi/2}.$$

The integral over γ_2 can be estimated as follows

$$\Big|\int_{\gamma_2}\frac{e^{iz}}{e^z+e^{-z}}\,dz\Big| \le \pi\max_{0\le y\le\pi}\Big|\frac{e^{iR}\,e^{-y}}{e^R\,e^{iy}+e^{-R}\,e^{-iy}}\Big|$$

$$\le \pi\max_{0\le y\le\pi}\frac{e^{-y}}{e^R|e^{iy}+e^{-2R}e^{-iy}|}\le\frac{1}{e^R(1-e^{-2R})}\to 0,$$

as $R\to\infty$.
A similar argument proves the same result for the integral of f over γ_4.

$$\int_{\gamma_3}\frac{e^{iz}}{e^z+e^{-z}}\,dz = \int_R^{-R}\frac{e^{ix-\pi}}{e^{x+i\pi}+e^{-x-i\pi}}\,dx$$

$$= e^{-\pi}\int_R^{-R}\frac{e^{ix}}{-e^x-e^{-x}}\,dx = e^{-\pi}\int_{-R}^{R}\frac{e^{ix}}{e^x+e^{-x}}\,dx$$

$$= e^{-\pi}\int_{-R}^{R}\frac{\cos x}{e^x+e^{-x}}\,dx.$$

Therefore

$$(1+e^{-\pi})\int_{-\infty}^{\infty}\frac{\cos x}{e^x+e^{-x}}\,dx = \pi\, e^{-\pi/2}$$

and finally

$$\int_{-\infty}^{\infty}\frac{\cos x}{e^x+e^{-x}}\,dx = \frac{\pi}{e^{\pi/2}+e^{-\pi/2}}.$$

Example. Evaluate

$$\int_0^\infty\frac{(\log x)^2}{1+x^2}\,dx.$$

Solution. Introduce the following function

$$f(z)=\frac{(\log z - i\pi/2)^2}{1+z^2}$$

and take the branch of the logarithm given by the cut $-\pi/2<\theta\le 3\pi/2$.

Consider $\gamma = \gamma_R \cup \gamma_1 \cup \gamma_r \cup \gamma_2$, where

$$\gamma_R = R\, e^{i\theta}, \quad R >> 1, \quad \theta \in [0, \pi],$$
$$\gamma_1 = \{z : \; z = x + i0, \; x \in [-R, -r]\}, \qquad r << 1,$$
$$\gamma_r = r\, e^{i\theta}, \qquad \theta \in [\pi, 0],$$
$$\gamma_2 = \{z : \; z = x + i0, \; x \in [r, R]\}.$$

The only singularity of f which is internal for γ is $z_0 = i$ and

$$\text{Res}\left[\frac{(\log z - i\pi/2)^2}{1+z^2}, i\right] = \frac{2(\log i - i\pi/2)}{2i\, i} = 0.$$

This explains why we have the strange constant $i\pi/2$ in the definition of f. So

$$\oint_\gamma \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz = 0.$$

Note that $\log z - i\pi/2 = \ln|z| + i(\theta - \pi/2)$, where $\theta \in (-\pi/2, 3\pi/2]$.
By using the ML-inequality we obtain

$$\left|\int_{\gamma_R} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz\right| \le \frac{(\ln R)^2 + \pi^2}{R^2 - 1} \cdot \pi R \to 0,$$

as $R \to \infty$.
The integral over γ_r equals

$$\left|\int_{\gamma_r} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz\right| \le \frac{(\ln r)^2 + \pi^2}{1 - r^2} \cdot \pi r \to 0,$$

as $r \to 0$.

$$\int_{\gamma_1} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz = \int_{-R}^{-r} \frac{(\ln|x| + i\pi/2)^2}{1+x^2}\, dx = \int_r^R \frac{(\ln|x| + i\pi/2)^2}{1+x^2}\, dx$$

and

$$\int_{\gamma_2} \frac{(\log z - i\pi/2)^2}{1+z^2}\, dz = \int_r^R \frac{(\ln|x| - i\pi/2)^2}{1+x^2}\, dx.$$

Letting $R \to \infty$ and $r \to 0$ we get

$$2\int_0^\infty \frac{(\ln|x|)^2}{1+x^2}\, dx - 2\,\frac{\pi^2}{4}\int_0^\infty \frac{dx}{x^2+1} = 0.$$

Therefore

$$\int_0^\infty \frac{(\log x)^2}{1+x^2}\, dx = \frac{\pi^2}{4}\int_0^\infty \frac{dx}{x^2+1} = \frac{\pi^2}{4}\, \arctan x\Big|_0^\infty = \frac{\pi^3}{8}.$$

MATH50001 Complex Analysis 2021

Lecture 18

Section: Harmonic functions.

Definition. Let $\varphi = \varphi(x, y)$, $x, y \in \mathbb{R}^2$ be a real function of two variables. It said to be *harmonic* in an open set $\Omega \subset \mathbb{R}^2$ if

$$\Delta\varphi(x, y) := \frac{\partial^2 \varphi}{\partial x^2}(x, y) + \frac{\partial^2 \varphi}{\partial y^2}(x, y) = \varphi''_{xx}(x, y) + \varphi''_{yy}(x, y) = 0.$$

Usually Δ is called the Laplace operator.

Theorem. Let $f(z) = u(x, y) + iv(x, y)$ be holomorphic in an open set $\Omega \subset \mathbb{C}$. Then u and v are harmonic.

Proof.
Since $f = u + iv$ is holomorphic it is infinitely differentiable. In particular, the functions u and v have continuous second derivatives that allows us to change the order of the second derivatives and using the Cauchy-Riemann equations to obtain

$$u''_{xx} = (u'_x)'_x = (v'_y)'_x = (v'_x)'_y = (-u'_y)'_y = -u''_{yy}.$$

Therefore

$$u''_{xx} + u''_{yy} = 0.$$

Similarly we find that $\Delta v = 0$.

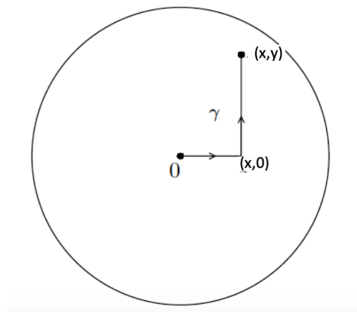
Theorem. (Harmonic conjugate)

Let u be harmonic in an open disc $D \subset \mathbb{C}$. Then there exists a harmonic function v such that $f = u + iv$ is holomorphic in D. In this case v is called harmonic conjugate to u.

Proof.

We can assume that $D = D_R = \{(x, y) \in \mathbb{R}^2 : |z| < R\}$, $R > 0$. Let $(x, y) \in D_R$ and let $\gamma = \gamma_1 \cup \gamma_2$, where

$$\gamma_1 = \{(t, s) \in \mathbb{R}^2 : t \in (0, x), \ s = 0\},$$
$$\gamma_2 = \{(t, s) : t = x, \ s \in (0, y)\},$$



We now define

$$v(x,y) = \int_\gamma \left(-\frac{\partial u}{\partial y}\,dt + \frac{\partial u}{\partial x}\,ds \right) = -\int_0^x \frac{\partial u(t,0)}{\partial y}\,dt + \int_0^y \frac{\partial u(x,s)}{\partial x}\,ds.$$

Using $u''_{xx} = -u''_{yy}$ we obtain

$$v'_x(x,y) = -u'_y(x,0) + \int_0^y \frac{\partial^2 u(x,s)}{\partial x^2}\,ds = -u'_y(x,0) - \int_0^y \frac{\partial^2 u(x,s)}{\partial s^2}\,ds$$
$$= -u'_y(x,0) + u'_y(x,0) - u'_y(x,y) = -u'_y(x,y).$$

Differentiating v with respect to y we have

$$v'_y(x,y) = \frac{\partial}{\partial y}\left(-\int_0^x \frac{\partial u(t,0)}{\partial y}\,dt + \int_0^y \frac{\partial u(x,s)}{\partial x}\,ds \right) = 0 + u'_x(x,y).$$

Thus the C-R equations are satisfied and we conclude that $f(z) = u(x,y) + iv(x,y)$ is holomorphic inside D.

Remark.

In a simply connected domain $\Omega \subset \mathbb{R}^2$ every harmonic function u has a harmonic conjugate v defined by the line integral

$$v(x,y) = \int_\gamma \left(-\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \right),$$

where the path of integration γ is a curve starting at a fixed base-point $(x_0, y_0) \in \Omega$ with the end point at $(x,y) \in \Omega$. The integral in independent of path by Green's theorem because u is harmonic and Ω is simply connected.

We leave this statement without the proof because it requires Green's theorem that we did not have in our course.

Example. Let $u(x,y) = \ln(x^2 + y^2)$ defined in $\mathbb{R}^2 \setminus \{0\}$ and let

$$\Omega = \mathbb{C} \setminus \{z = x + iy : x \in (-\infty, 0],\ y = 0\}.$$

Find in Ω a harmonic conjugate v to u and thus a holomorphic function $f = u + iv$.

Step 1. We first check that $\ln(x^2+y^2)$ is harmonic in $\mathbb{R} \setminus \{0\}$. Indeed,

$$u'_x = \frac{2x}{x^2+y^2}, \qquad u''_{xx} = \frac{2}{x^2+y^2} - \frac{4x^2}{(x^2+y^2)^2}$$

and

$$u'_y = \frac{2y}{x^2+y^2}, \qquad u''_{yy} = \frac{2}{x^2+y^2} - \frac{4y^2}{(x^2+y^2)^2}.$$

Thus $\Delta u = 0$.

Step 2. In order to find u's harmonic conjugate we use the Cauchy-Riemann equations.

a) $v'_y = u'_x = 2x/(x^2+y^2)$ implies

$$v(x,y) = \int \frac{2x}{x^2+y^2} dy = 2 \arctan \frac{y}{x} + C(x).$$

b) $u'_y = -v'_x$ implies

$$\frac{2y}{x^2+y^2} = -\frac{2}{1+y^2/x^2} \cdot \frac{-y}{x^2} + C'(x) \implies C'(x) = 0$$

and thus $C(x) = C \in \mathbb{R}$.

Solution: $v = 2\arctan \frac{y}{x} + C$ and hence

$$f(z) = \ln(x^2+y^2) + 2i \arctan \frac{y}{x} + iC = 2(\ln|z| + i \operatorname{Arg} z) + iC.$$

Example. Let $u(x,y) = x^3 - 3xy^2 + y$.

i. Verify that the function u is harmonic.

ii. Find all harmonic conjugates v of u.

iii. Find the holomorphic function f, $\operatorname{Re} f = u$, as a function of z, s.t. $f(1) = 1 + i$.

Step 1. For $u = x^3 - 3xy^2 + y$ we have $u'_x = 3x^2 - 3y^2$, $u''_{xx} = 6x$ and $u'_y = -6xy + 1$, $u''_{yy} = -6x$. Thus we have

$$\Delta u(x,y) = u''_{xx} + u''_{yy} = 6x - 6x = 0.$$

Step 2. Cauchy-Riemann equations imply

$$v'_y = u'_x = 3x^2 - 3y^2.$$

Integrating the latter w.r.t. y we find

$$v = 3x^2y - y^3 + F(x),$$

and differentiating it w.r.t. x we have

$$v'_x = 6xy + F'(x) = -u'_y = 6xy - 1.$$

So $F'(x) = -1$ and $F(x) = -x + c$, $c \in \mathbb{R}$. This implies

$$v = 3x^2y - y^3 - x + c,$$

$$\begin{aligned} f = u + iv = x^3 - 3xy^2 + y + 3ix^2y - iy^3 - ix + ic \\ = (x+iy)^3 - i(x+iy) + ic. \end{aligned}$$

Step 3.

We find $f(z) = z^3 - iz + ic$. Solving the equation

$$f(1) = 1 + i = \left(z^3 - iz + ic\right)_{z=1} = 1 - i + ic$$

we find $c = 2$.

Section: Properties of real and imaginary parts of holomorphic functions.

Theorem.

Assume that $f = u + iv$ is a holomorphic function defined on an open connected set $\Omega \subset \mathbb{C}$. Consider two equations

$$a) \quad u(x,y) = C \qquad \text{and} \qquad b) \quad v(x,y) = K,$$

where C, K are two real constants.

Assume that the equations $a)$ and $b)$ have the same solution (x_0, y_0) and that $f'(z_0) \neq 0$ at $z_0 = x_0 + iy_0$. Then the curve defined by the equation $a)$ is orthogonal to the curve defined by the equation $b)$ at (x_0, y_0).

Proof. It is enough to show that the gradient ∇u and ∇v are orthogonal at z_0. We use C-R equations and obtain

$$\nabla u \cdot \nabla v = u'_x v'_x + u'_y v'_y = v'_y v'_x - v'_x v'_y = 0.$$

Example. Let $f(z) = \ln(x^2 + y^2) + 2i \arctan \frac{y}{x}$. Consider

$$\ln(x^2 + y^2) = C \quad \Longrightarrow \quad x^2 + y^2 = e^C.$$

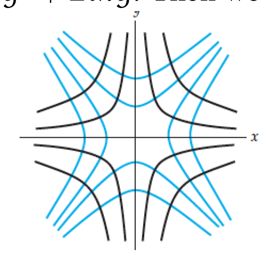
This is a circle whose radius is $e^{C/2}$.

The second equation

$$2\arctan\frac{y}{x} = K \implies \frac{y}{x} = \tan(K/2) \implies y = \tan(K/2)\cdot x$$

and this equation describes a straight line going through the origin.

Example. Let $f(z) = z^2 = x^2 - y^2 + 2ixy$. Then we have



MATH50001 COMPLEX ANALYSIS 2021
LECTURES

Conformal mappings.

Section: Preservation of angles.

Let us considers a smooth curve $\gamma \subset \mathbb{C}$ parametrised by $z(t) = x(t) + iy(t)$, $t \in [a, b]$. For each $t_0 \in [a, b]$ there is the direction vector

$$\begin{aligned} L_{t_0} &= \{(z(t_0) + tz'(t_0) : t \in \mathbb{R}\} \\ &= \{(x(t_0) + tx'((t_0) + i\,(y(t_0) + ty'(t_0)) : t \in \mathbb{R}\}. \end{aligned}$$

Consider now two curves γ_1 and γ_2 parametrised by the functions $z_1(t)$ and $z_2(t)$, $t \in [0, 1]$, respectively intersecting in the point $t = 0$, namely, $z_1(0) = z_2(0)$.

We then define the angle between the curves γ_1 and γ_2 to be the angle between the tangents, namely

$$\arg z_2'(0) - \arg z_1'(0).$$

We have the following result:

Theorem. (Angle preservation theorem)
Let f be holomorphic in an open subset set $\Omega \subset \mathbb{C}$. Suppose that two curves γ_1 and γ_2 lying inside Ω are parametrised by $z_1(t)$ and $z_2(t)$, $t \in [0, 1]$. Assume that $z_0 = z_1(0) = z_2(0)$ is their intersecting point and $z_1'(0)$, $z_2'(0)$ and also $f'(z_0)$ are all non-zero.

Then the angles between the curves $(z_1(t), z_2(t))$ and $(f(z_1(t)), f(z_2(t))$ at $t = 0$ satisfy

$$\arg z_2'(t) - \arg z_1'(t)\Big|_{t=0} = \arg\,(f(z_2(t)))' - \arg\,(f(z_1(t))')\Big|_{t=0} \mod (2\pi).$$

Proof. Indeed,

$$\frac{(f(z_1(t)))'}{(f(z_2(t)))'}\Big|_{t=0} = \frac{f'(z_1(0))z_1'(0)}{f'(z_2(0))z_2'(0)} = \frac{f'(z_0)z_1'(0)}{f'(z_0)z_2'(0)} = \frac{z_1'(0)}{z_2'(0)}.$$

This implies

$$\arg\left(f\circ z_2)\right)'(0)-\arg\left(f\circ z_1\right)'(0)\big)=\arg z_2'(0)-\arg z_1'(0) \mod (2\pi).$$

Remark.

The condition $f'(z_0)\neq 0$ in the Theorem is essential. For example, consider the holomorphic function $f(z)=z^2$ at $z_0=0$. The positive x-axis maps to itself, and the line $\theta=\pi/4$ maps to the positive y-axis. The angle between the lines doubles.

Remark.

The theorem states that it is not only the value of the angle is preserved by f but also its orientation. Consider for example of a (nonholomorphic) f preserving the value of the angle but not the orientation

$$f(z)=\bar{z}$$

One can think of this mapping geometrically as reflection in the x-axis.

Definition. We say that a complex function f is conformal in an open set $\Omega\subset\mathbb{C}$ if it is holomorphic in Ω and if $f'(z)\neq 0$, $\forall z\in\Omega$.

For example, the function $f(z)=z^2$ is conformal in the open set $\mathbb{C}\setminus\{0\}$.

The angle preservation theorem tells us that conformal mappings preserve angles.

Definition. A holomorphic function is a local injection on an open set $\Omega\subset\mathbb{C}$ if for any $z_0\in\Omega$ there exists $D=\{z: |z-z_0|<r\}\subset\Omega$ such that $f: D\to f(D)$ is injection.

Theorem.

If $f:\Omega\to\mathbb{C}$ is a local injection and holomorphic, then $f'(z)\neq 0$ for all $z\in\Omega$. In particular, the inverse of f defined on its range is holomorphic, and thus the inverse of a conformal map is also holomorphic.

Proof. We argue by contradiction. Suppose that $f'(z_0)=0$ for some $z_0\in\Omega$. Then for a sufficiently small $r>0$ there is $D=\{z: |z-z_0|<r\}$, $\overline{D}\subset\Omega$, such that

$$f(z)-f(z_0)=a\,(z-z_0)^k+g(z),\qquad z\in D,$$

where $a \neq 0$, $k \geq 2$ and $g(z) = O(|z - z_0|^{k+1})$. For sufficiently small $0 \neq w \in \mathbb{C}$ denote

$$f(z) - f(z_0) - w = F(z) + G(z),$$

where

$$F(z) = a\,(z - z_0)^k - w, \quad G(z) = g(z).$$

If $r > 0$ and $|w|$ are small enough then we have

$$|G(z)| < |F(z)|, \qquad z \in \{z : |z - z_0| = r\},$$

Rouche's theorem implies that $f(z) - f(z_0) - w$ has at least two zeros in D.

Note that since the zeros of holomorphic function are isolated and $f'(z_0) = 0$ then for a sufficiently small r it follows $f'(z) \neq 0$, $z \neq z_0$. Therefore the roots of $\varkappa(z) = f(z) - f(z_0) - w$ are **distinct**. Indeed, $\varkappa(z_0) = w \neq 0$. Hence if $\varkappa(z)$ has a root of degree at least two at some z_1 then $\varkappa'(z_1) = f'(z_1) = 0$ which is impossible.

This finally implies that f is not injective and gives contradiction.

Let $g = f^{-1}$ denote the inverse of f on its range, which we can assume is $V \subset \mathbb{C}$. Suppose $w_0 \in V$ and w is closed to w_0. Assuming $w = f(z)$ and $w_0 = f(z_0)$ with $w \neq w_0$ we find

$$\frac{g(w) - g(w_0)}{w - w_0} = \frac{1}{\frac{w - w_0}{g(w) - g(w_0)}} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}}.$$

Since $f'(z_0) \neq 0$ then letting $z \to z_0$ we conclude that g is holomorphic at w_0 and $g'(w_0) = 1/f'(g(w_0))$.

Section: Möbius Transformations.

Definition.

A Möbius transformation (that is also called a bilinear transformation) is a map

$$f(z) = \frac{az + b}{cz + d}, \quad \text{where} \quad a, b, c, d \in \mathbb{C} \quad \text{and} \quad ad - bc \neq 0.$$

The condition $ad - bc \neq 0$ is necessary for the transformation to be non-trivial. Indeed, $ad - bc = 0$ gives $a/c = b/d = \text{const}$ and the transformation reduces to $f(z) = \text{const}$.

It is clear that a Möbius transformation is holomorphic except for a simple pole at $z = -d/c$. Its derivative is the function

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+b)^2} = \frac{ad-bc}{(cz+d)^2}$$

and therefore the mapping is conformal throughout $\mathbb{C} \setminus \{-d/c\}$.

Theorem.

The inverse of a Möbius transformation is a Möbius transformation. The composition of two Möbius transformations is a Möbius transformation.

Proof. It is easily to verify, that the Möbius transformation

$$g(w) = \frac{dw-b}{-cw+a}$$

is the inverse of $f(z) = \frac{az+b}{cz+d}$. Indeed,

$$g(f(z)) = \frac{d\,\frac{az+b}{cz+d} - b}{-c\,\frac{az+b}{cz+d} + a} = \frac{d(az+b) - b(cz+d)}{-c(az+b) + a(cz+d)}$$

$$= \frac{adz + db - bcz - db}{-caz - cb + acz + ad} = z.$$

Composition of two Möbius transformations.

Given two Möbius transformations

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1} \quad \text{and} \quad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

an easy calculation gives

$$f_1 \circ f_2(z) = f_1(f_2(z)) = \frac{Az+B}{Cz+D},$$

where

$$A = a_1a_2 + b_1c_2,\ B = a_1b_2 + b_1d_2,\ C = c_1a_2 + d_1c_2,\ D = c_1b_2 + d_1d_2.$$

Thus $f_1 \circ f_2$ is a Möbius transformation. A simple computation gives

$$AD - BC = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) \neq 0.$$

Remark.

The composition of Möbius transformations in effect corresponds to matrix multiplication. Indeed,

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Besides,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

This is essentially the matrix of the inverse mapping $f(z) = \frac{az+b}{cz+d}$, since multiplication of all the coefficients by a non-zero complex constant does not change a Möbius transformation.

Special Möbius transformations.

Let

$$f(z) = \frac{az + b}{cz + d}$$

and consider the following cases:

(M1) $\quad z \mapsto az \quad (b = c = 0,\ d = 1);$

if $|a| = 1$, $a = e^{i\theta}$, then this is a rotation by θ. If $a > 0$ then f corresponds to a dilation and if $a < 0$ the map consists of a dilation by $|a|$ followed by a rotation of π.

(M2) $\quad z \mapsto z + b \quad (a = d = 1, c = 0 - \text{translation by } b);$

(M3) $\quad z \mapsto \frac{1}{z} \quad (a = d = 0,\ b = c = 1 - \text{inversion}).$

In (M1), if $a = re^{i\theta}$, the geometrical interpretation is an expansion by the factor r followed by a rotation anticlockwise by the angle θ.

Theorem.

Every Möbius transformation

$$f(z) = \frac{az + b}{cz + d}$$

is a composition of transformations of type (M1), (M2) and (M3).

MATH50001 Complex Analysis 2021

Lecture 20

Section: Möbius Transformations.

Definition.

A Möbius transformation (that is also called a bilinear transformation) is a map

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Special Möbius transformations.

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(M1) $\quad z \mapsto az \quad (b = c = 0,\ d = 1);$

if $|a| = 1$, $a = e^{i\theta}$, then this is a rotation by θ. If $a > 0$ then f corresponds to a dilation and if $a < 0$ the map consists of a dilation by $|a|$ followed by a rotation of π.

(M2) $\quad z \mapsto z + b \quad (a = d = 1, c = 0 - \text{translation by } b);$

(M3) $\quad z \mapsto \frac{1}{z} \quad (a = d = 0,\ b = c = 1 - \text{inversion}).$

In (M1), if $a = re^{i\theta}$, the geometrical interpretation is an expansion by the factor r followed by a rotation anticlockwise by the angle θ.

Theorem.

Every Möbius transformation

$$f(z) = \frac{az+b}{cz+d}$$

is a composition of transformations of type (M1), (M2) and (M3).

Proof.

1. If $c = 0$ and $d \neq 0$, then

$$f(z) = \frac{az+b}{d} = g_2 \circ g_1(z),$$

where

$$g_1(z) = \frac{a}{d} z, \qquad g_2(z) = z + \frac{b}{d}.$$

2. If $c \neq 0$, then $f(z) = g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z)$, where

$$g_1(z) = cz, \qquad g_2(z) = z + d, \qquad g_3 = \frac{1}{z},$$
$$g_4(z) = \frac{1}{c}(bc - ad)z \qquad g_5(z) = z + \frac{a}{c}.$$

Indeed,

$$g_1(z) = c\,z, \qquad g_2 \circ g_1(z) = c\,z + d, \qquad g_3 \circ g_2 \circ g_1(z) = \frac{1}{cz + d},$$
$$g_4 \circ g_3 \circ g_2 \circ g_1(z) = \frac{bc - ad}{c(cz + d)},$$
$$g_5 \circ g_4 \circ g_3 \circ g_2 \circ g_1(z) = \frac{a}{c} + \frac{bc - ad}{c(cz + d)} = \frac{az + b}{cz + d} = f(z).$$

Corollary.

A Möbius transformation transforms circles into circles, and interior points into interior points. (Here we mean that straight lines are also circles whose radius equal infinity).

Proof. Each of the transformations (M1), (M2) and (M3) transform circles into circles.

Section: Cross-Ratios Möbius Transformation.

Theorem.

If $w = f(z)$ is a Möbius transformation that maps the distinct points (z_1, z_2, z_3) into the distinct points (w_1, w_2, w_3) respectively, then

$$\left(\frac{z - z_1}{z - z_3}\right) \left(\frac{z_2 - z_3}{z_2 - z_1}\right) = \left(\frac{w - w_1}{w - w_3}\right) \left(\frac{w_2 - w_3}{w_2 - w_1}\right), \quad \text{for all } z.$$

Proof.

The Möbius transformation

$$g(z) = \left(\frac{z - z_1}{z - z_3}\right) \left(\frac{z_2 - z_3}{z_2 - z_1}\right)$$

maps z_1, z_2, z_3 to $0, 1, \infty$ respectively. Similarly the Möbius transformation

$$h(w) = \left(\frac{w - w_1}{w - w_3}\right)\left(\frac{w_2 - w_3}{w_2 - w_1}\right)$$

maps w_1, w_2, w_3 to $0, 1, \infty$ respectively. Therefore $h^{-1} \circ g$ maps (z_1, z_2, z_3) into (w_1, w_2, w_3).

Example. Find a Möbius transformation $w = f(z)$ that maps the points 1, i, and -1 on the unit circle $|z| = 1$ onto the points -1, 0, 1 on the real axis. Determine the image of the interior $|z| < 1$ under this transformation.

Proof. Let $z_1 = 1$, $z_2 = i$, $z_3 = -1$ and $w_1 = -1$, $w_2 = 0$, $w_3 = 1$. The the mapping $w = f(z)$ must satisfy the Cross-Ratios Möbius Transformation

$$\begin{aligned}
&\frac{z-1}{z-(-1)} \cdot \frac{i-(-1)}{i-1} = \frac{w-(-1)}{w-1} \cdot \frac{0-1}{0-(-1)} \\
&\qquad \implies \quad \frac{z-1}{z+1} \cdot \frac{i+1}{i-1} = -\frac{w+1}{w-1} \implies \frac{z-1}{z+1}(-i) = -\frac{w+1}{w-1} \\
&\qquad \implies \quad (w-1)(z-1)\,i = (w+1)(z+1) \\
&\qquad \implies \quad w\left((z-1)i - (z+1)\right) = (z-1)i + (z+1) \\
&\qquad \implies w = \frac{iz - i + z + 1}{zi - i - z - i - 1} = \frac{z(1+i) + (1-i)}{iz(1+i) - (1+i)} = \frac{z-i}{iz-1}.
\end{aligned}$$

Note that if $z = 0$ then $f(0) = i$.

Example. Find a linear fractional transformation $w = f(z)$ that maps the points $z_1 = -i$, $z_2 = 1$, and $z_3 = \infty$ on the line $y = x - 1$ onto the points $w_1 = 1$, $w_2 = i$, and $w_3 = -1$ on the unit circle $|w| = 1$.

Proof. Note that

$$\begin{aligned}
\lim_{z_3 \to \infty} \frac{z+i}{z-z_3} \cdot \frac{1-z_3}{1+i} &= \lim_{t \to 0} \frac{z+i}{z-1/t} \cdot \frac{1-1/t}{1+i} \\
&= \lim_{t \to 0} \frac{z+i}{tz-1} \cdot \frac{t1-1}{1+i} = \frac{z+i}{1+i}.
\end{aligned}$$

Therefore in this case the cross-ratio could be written

$$\frac{z+i}{1+i} = \frac{w-1}{w+1} \cdot \frac{i+1}{i-1} \quad \implies \quad \frac{z+i}{1+i} = -i\,\frac{w-1}{w+1}$$

$$\implies \quad w = \frac{-z-1}{z+2i-1}.$$

Section: Conformal mapping of a half-plane to the unit disc.

The upper half-plane can be mapped by a holomorphic bijection to the disc, and this is given by a Möbius transformation.
Let
$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : \operatorname{Im} z = y > 0\}.$$
A remarkable surprising fact is that the unbounded set $\mathbb{H}$ is conformally equivalent to the unit disc. Moreover, an explicit formula giving this equivalence exists. Indeed, let
$$w = f(z) = \frac{i - z}{i + z}, \qquad g(w) = i\,\frac{1 - w}{1 + w}.$$

Theorem. Let $\mathbb{D} = \{z : |z| < 1\}$. Then the map $f : \mathbb{H} \mapsto \mathbb{D}$ is a conformal map with inverse $g : \mathbb{D} \mapsto \mathbb{H}$.

Proof. Clearly both functions are holomorphic in their respective domains. If $z = x + iy$, $y > 0$, then
$$\left|\frac{i - z}{i + z}\right|^2 = \left|\frac{x^2 + (y - 1)^2}{x^2 + (y + 1)^2}\right| < 1.$$
Let $w = u + iv$, $|w| < 1$. Then
$$\operatorname{Im} g(w) = \operatorname{Re}\left(\frac{1 - u - iv}{1 + u + iv}\right) = \operatorname{Re}\left(\frac{(1 - u - iv)(1 + u - iv)}{(1 + u)^2 + v^2)}\right)$$
$$= \frac{1 - u^2 - v^2}{(1 + u)^2 + v^2} > 0.$$
Finally
$$f \circ g(w) = \frac{i - i\,\frac{1-w}{1+w}}{i + i\,\frac{1-w}{1+w}} = \frac{1 + w - 1 + w}{1 + w + 1 - w} = w.$$
Similarly we also have $g \circ f(z) = z$.

Note that f is holomorphic in $\mathbb{C} \setminus \{-i\}$ and, in particular, it is continuous on the boundary of $\partial(\mathbb{H}) = \{z = x + i0 \in \mathbb{C}\}$. Clearly
$$|f(z)|_{z = x + i0} = \left|\frac{i - x}{i + x}\right| = 1.$$
Thus f maps $\mathbb{R}$ onto the boundary of the unit disc $\partial\mathbb{D}$. Moreover,
$$f(z) = \frac{i - x}{i + x} = \frac{1 - x^2}{1 + x^2} + i\,\frac{2x}{1 + x^2}.$$

$$f(z) = \frac{i - x}{i + x} = \frac{1 - x^2}{1 + x^2} + i\,\frac{2x}{1 + x^2}.$$

Let $x = \tan\theta$ with $\theta \in (-\pi/2, \pi/2)$. Since

$$\cos 2\theta = \frac{1 - \tan^2\theta}{1 + \tan^2\theta} \quad \text{and} \quad \sin 2\theta = \frac{2\tan\theta}{1 + \tan^2\theta}$$

we obtain

$$f(z) = \cos 2\theta + i\,\sin 2\theta = e^{2i\theta}.$$

$$f(z) = \cos 2\theta + i\,\sin 2\theta = e^{2i\theta}, \qquad \theta \in (-\pi/2, \pi/2).$$

Therefore the image of the real line is the arc consisting of the circle omitting the point -1. Moreover, if the value of x changes from $-\infty$ to ∞, $f(x)$ changes along that arc starting from -1 and first going through that part of the circle that lies in the lower half-plane. The point -1 on the circle corresponds to "infinity" of the upper half-plane.

Section: Riemann mapping theorem.

Definition. We say that $\Omega \subset \mathbb{C}$ is *proper* if it is non-empty and not the whole of $\mathbb{C}$.

Theorem.

Suppose Ω is proper and simply connected. If $z_0 \in \Omega$, then there exists a unique conformal map $f : \Omega \to \mathbb{D}$ such that

$$f(z_0) = 0 \quad \text{and} \quad f'(z_0) > 0.$$

Corollary

Any two proper simply connected open subsets in $\mathbb{C}$ are conformally equivalent.

Thank you and good luck with the exam