Analysis II, Term I

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Introduction to the module

This is a continuation of Analysis I module you had in year-one. In that module, you have learned about the real numbers, completeness, convergence of sequences and series, continuity and differentiability of functions on an interval or $\mathbb{R}$, integral of a function on an interval. Analysis II is a single module in year-two, delivered during term I and term II.

The content of Analysis II in term I has two parts. In the first part we complete the study of analysis on Euclidean spaces, by introducing the concepts of converges of sequences in higher dimensional Euclidean spaces $\mathbb{R}^n$, and the continuity and differentiability of maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. In the second part of the module, we generalise these notions of analysis on Euclidean spaces into a broader setting, called metric spaces and topological spaces. That is a setting where one can define the notions of converge of sequences, completeness of spaces, continuity of maps, etc. Many theorems you have learned in the previous analysis module extends into this setting, and indeed, one can give unified proofs to all those statements at once. Many theorems find a natural form in the setting of metric spaces, and you will see that the proof you already know for a statement can be adapted to the more general setting.

Any section/subsection marked with * is not examinable, but will be valuable in future courses, especially if you take pure analysis courses in your third year and beyond. You should certainly at least read through the notes on these sections, even if you choose not to attempt the questions. I will try to indicate in lectures when I'm covering those material.

Throughout this lecture notes, the definitions are numbered successively within each chapter, that is, in Chapter 1, you will see Definition 1.1, Definition 1.2, Definition 1.3, and so on. The same numbering mechanism applies to Examples, Exercises, and Remarks in each chapter. On the other hand, the results such as lemmas, propositions, corollaries, and theorems are collectively numbered in a successive fashion. That is, in Chapter 1, you will see Proposition 1.1, Theorem 1.2, Theorem 1.3, etc.

Contents

Chapter 1

Differentiation in higher dimensions

1.1 Euclidean spaces

1.1.1 Preliminaries from analysis I

In this chapter we are going to extend some of the ideas that you saw last year (such as limits and continuity) to higher dimensions. The definitions are almost identical, so this should mostly feel like a review chapter to begin with, although some of the ideas we are going to approach from a different point of view.

Throughout these notes we frequently use the standard notations for the set of natural numbers

$$\mathbb{N} = \{1, 2, 3, \ldots\},$$

the set of integers

$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\},$$

the set of rational numbers

$$\mathbb{Q} = \{p/q \mid p \in \mathbb{Z},\, q \in \mathbb{Z} \setminus \{0\}\},$$

and the set of real numbers $\mathbb{R}$. The set of real numbers is obtained as the *completion* of $\mathbb{Q}$. We may add, multiply and subtract elements of $\mathbb{R}$, and we can divide by elements of $\mathbb{R} \setminus \{0\}$. Note that some authors use the notation $\mathbb{N}$ to denote the set $\{0, 1, 2, \ldots\}$, but we will omit 0 from this set.

On $\mathbb{R}$ we have a notion of ordering $\leq$, so that we may say whether a real number is greater than, less than or equal to another. Moreover, $\mathbb{R}$ satisfies the **completeness axiom**, that is, if $A \subset \mathbb{R}$ is non-empty and bounded above, then A has a least upper bound. The standard notation for the least upper bound of A is $\sup(A)$.

An important function defined on all real numbers is the **modulus function**, defined as

$$|x| := \begin{cases} x & x \geq 0, \\ -x & x < 0. \end{cases}$$

This function has the following properties:

(i) for all $x \in \mathbb{R}$, we have $|x| \geq 0$, with $|x| = 0$ if and only if $x = 0$,

(ii) for all x and y in $\mathbb{R}$, $|xy| = |x|\,|y|$,

(iii) for all x and y in $\mathbb{R}$,

$$|x + y| \leq |x| + |y| \,.$$

The third property in the above list is called the **triangle inequality** for the modulus function.

1.1.2 Euclidean space of dimension n

For $n \geq 1$, the **n-dimensional Euclidean space**, denoted by $\mathbb{R}^n$, is defined as the set of ordered n-tuples $(x^1, x^2, \ldots, x^n)$, where each $x^i \in \mathbb{R}$, for $i = 1, 2, \ldots, n$. Each such n-tuple is denoted by a single letter $x = (x^1, x^2, \ldots, x^n)$ and will be referred to as a point in $\mathbb{R}^n$. The entries x^i are called the **coordinates** of x.

One may see each element of $\mathbb{R}^n$ as a row vector with n real components, or as a column vector with n real components. We do not make this distinction (unless when a matrix is acting on the point x. When a matrix M acts on a vector with the same components as x we use Mx^t to make it clear that x is viewed as a column vector. Here ^{t} denotes the transpose operation.)

We shall try to stick to the convention of using superscripts to label components of vectors, and subscripts to label different vectors, so that $x_1, x_2 \in \mathbb{R}^n$ are two different vectors, while $x^1, x^2 \in \mathbb{R}$ are the components of one vector.

If x and y are elements of $\mathbb{R}^n$ with

$$x = \left(x^1, \ldots, x^n\right), \qquad y = \left(y^1, \ldots, y^n\right),$$

we can add these two elements according to

$$x + y = \left(x^1 + y^1, \ldots, x^n + y^n\right).$$

Moreover, for every $\lambda \in \mathbb{R}$, we define

$$\lambda x = \left(\lambda x^1, \ldots, \lambda x^n\right).$$

With these definitions, $\mathbb{R}^n$ is a **vector space** over $\mathbb{R}$.

The **inner product**,

$$\langle \cdot, \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},$$

is defined as

$$\left\langle (x^1, \dots, x^n), (y^1, \dots, y^n) \right\rangle = \sum_{i=1}^{n} x^i y^i.$$

Using the inner product, we may define the **length**, or **norm**, function

$$\|\cdot\| : \mathbb{R}^n \to [0, \infty)$$

as

$$\|x\| = \sqrt{\langle x, x \rangle} = \langle x, x \rangle^{1/2}.$$

Note that the inner product of two vectors is a real number, not a vector.

The norm function on $\mathbb{R}^n$ has the following properties:

(i) for all $x \in \mathbb{R}^n$, we have $\|x\| \geq 0$, with $\|x\| = 0$ if and only if $x = 0$,

(ii) for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $\|\lambda x\| = |\lambda| \, \|x\|$,

(iii) for all x and y in $\mathbb{R}^n$,

$$\|x + y\| \leq \|x\| + \|y\|. \tag{1.1}$$

The third property in the above list is called the **triangle inequality** for the norm on $\mathbb{R}^n$.

Remark 1.1. As we shall see later, these properties can be used in an abstract fashion to define more general "normed vector spaces". The norm gives us a useful notion of "distance" between two points, that is, the distance from x to y is given by $\|x - y\|$. Notice that if $n = 1$ we have $|\cdot| = \|\cdot\|$, and we will use either interchangeably in this case.

Exercise 1.1. (a) Show that the inner product satisfies the following properties: for all x, y, and z in $\mathbb{R}^n$ and all $a \in \mathbb{R}$,

$$\langle x, y \rangle = \langle y, x \rangle, \qquad \langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle, \qquad \langle ax, y \rangle = a \langle x, y \rangle.$$

(b) For $t \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$, show that:

$$\|x + ty\|^2 = \|x\|^2 + 2t \langle x, y \rangle + t^2 \|y\|^2 \geq 0. \tag{1.2}$$

(c) By thinking of (1.2) as a quadratic in t, and considering its possible roots, deduce the **Cauchy-Schwartz** inequality:

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|. \tag{1.3}$$

When does equality hold?

(d) Deduce the triangle inequality (1.1).

(e) Show the reverse triangle inequality:

$$\big|\, \|x\| - \|y\| \,\big| \leq \|x - y\|$$

Exercise 1.2. Suppose $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$.

(a) Show that:

$$\max_{k=1,\ldots,n} \left|x^k\right| \leq \|x\| \,. \tag{1.4}$$

(b) Show that:

$$\|x\| \leq \sqrt{n} \max_{k=1,\ldots,n} \left|x^k\right|. \tag{1.5}$$

1.1.3 Convergence of sequences in Euclidean spaces

Now that we have a few definitions relating to $\mathbb{R}^n$, we're ready to revisit some concepts from first year analysis and see how they can be extended to higher dimensions.

A sequence in $\mathbb{R}^n$ is an ordered list

$$x_0, x_1, x_2, \ldots,$$

with each $x_i \in \mathbb{R}^n$, for $i = 0, 1, 2, \ldots$. This is often written $(x_i)_{i=0}^{\infty}$, or $(x_i)_{i\in\mathbb{N}}$. A very important concept relating to sequences is convergence.

Definition 1.1. A sequence $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ **converges** to (the vector) $x \in \mathbb{R}^n$ if the following holds: For every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have

$$\|x_i - x\| < \epsilon.$$

We then write:

$$x_i \to x, \quad \text{as } i \to \infty,$$

or

$$\lim_{i\to\infty} x_i = x.$$

One may compare the above definition to the one for convergence of a sequence of real numbers. Indeed, this notion is intimately related to convergence of real numbers, as stated in the next lemma.

Proposition 1.1. *The sequence of vectors $(x_i)_{i=0}^{\infty}$ with $x_i \in \mathbb{R}^n$ converges to the vector $x \in \mathbb{R}^n$ if and only if each component of x_i converges to the corresponding component of x. That is, if we write:*

$$x_i = (x_i^1, \ldots, x_i^n), \quad \textit{and} \quad x = (x^1, \ldots x^n),$$

then, $x_i \to x$ as $i \to \infty$ if and only if for all $k = 1, \ldots n$, $x_i^k \to x^k$ as $i \to \infty$.

Proof. Let us first assume that for all $k = 1, 2, \dots, n$,

$$x_i^k \to x^k, \quad \text{as } i \to \infty.$$

Fix an arbitrary $\epsilon > 0$. Then, for each $k = 1, \dots, n$, we apply the definition of convergence of $x_i^k \to x^k$ to $\epsilon/\sqrt{n}$ to obtain $N_k \in \mathbb{N}$ such that for all $i \geq N_k$ we have

$$\left|x_i^k - x^k\right| < \frac{\epsilon}{\sqrt{n}}.$$

Let $N = \max\{N_1 \dots, N_n\}$. Then, for every $i \geq N$, we have

$$\max_{k=1,\dots,n} \left|x_i^k - x^k\right| < \frac{\epsilon}{\sqrt{n}}.$$

Now, recall from the inequality in (1.4) that for every $y = (y^1, y^2, \dots, y^n) \in \mathbb{R}^n$,

$$\|y\| \leq \sqrt{n} \max_{k=1,\dots,n} \left|y^k\right|,$$

so we deduce

$$\|x_i - x\| \leq \sqrt{n} \max_{k=1,\dots,n} \left|x_i^k - x^k\right| < \epsilon.$$

This establishes the result in one direction.

Now assume that

$$\lim_{i\to\infty} x_i = x.$$

Fix an arbitrary integer k with $1 \leq k \leq n$, and an arbitrary $\epsilon > 0$. We aim to show that $x_i^k \to x^k$, as $i \to \infty$. The definition of convergence of $x_i \to x$, as $i \to \infty$, with ϵ, gives us $N \in \mathbb{N}$ such that for all $i \geq N$ we have

$$\|x_i - x\| < \epsilon.$$

Recall from Exercise 1.1, Equation (1.5) that for every $y = (y^1, y^2, \dots, y^n) \in \mathbb{R}^n$,

$$\max_{k=1,\dots,n} \left|y^k\right| \leq \|y\|.$$

In particular, for all $i \geq N$, we have

$$\left|x_i^k - x^k\right| \leq \max_{k=1,\dots,n} \left|x_i^k - x^k\right| \leq \|x_i - x\| < \epsilon.$$

As $\epsilon > 0$ was arbitrary, this shows that x_i^k converges to x^k, as $i \to \infty$. $\square$

Exercise 1.3. Suppose that $(x_i)_{i=0}^\infty$ and $(y_i)_{i=0}^\infty$ are two sequences in $\mathbb{R}^n$ with

$$\lim_{i\to\infty} x_i = x, \quad \lim_{i\to\infty} y_i = y.$$

(a) Show that

$$x_i + y_i \to x + y \quad \text{as } i \to \infty.$$

(b) Show that

$$\langle x_i, y_i \rangle \to \langle x, y \rangle \qquad \text{as } i \to \infty,$$

deduce that

$$\|x_i\| \to \|x\| \qquad \text{as } i \to \infty.$$

(c) Suppose that $(a_i)_{i=0}^{\infty}$ is a sequence in $\mathbb{R}$ with $a_i \to a$ as $i \to \infty$. Show that:

$$a_i x_i \to ax, \qquad \text{as } i \to \infty.$$

1.2 Continuity

Last year, you learned about the notion of continuity for functions from $\mathbb{R}$ (or subsets of $\mathbb{R}$) to $\mathbb{R}$. In this section we revisit those definitions and upgrade them to higher dimensions. In fact, the definitions we shall give are almost identical: the only thing that changes is that we use the appropriate "norm" for the domain and range.

1.2.1 Open sets in Euclidean spaces

In dimension one, you are familiar with sets of the form (a, b) and $[a, b]$, i.e. the open interval and the closed interval respectively. These form natural domains for functions in dimension one, and it is fairly general to present theorems about maps in dimension one on such intervals. In higher dimensions, one may generalise these sets to sets of the from

$$(a^1, b^1) \times (a^2, b^2) \times \cdots \times (a^n, b^n) \\ = \left\{(x^1, x^2, \dots, x^n) \in \mathbb{R}^n \mid \text{ for } 1 \leq i \leq n, a^i < x^i < b^i\right\},$$

or

$$[a^1, b^1] \times [a^2, b^2] \times \cdots \times [a^n, b^n] \\ = \left\{(x^1, x^2, \dots, x^n) \in \mathbb{R}^n \mid \text{ for } 1 \leq i \leq n, a^i \leq x^i \leq b^i\right\}.$$

But this is very restrictive and does not capture the same level of generality of intervals in dimension one. The domains of maps in higher dimensions may appear in many forms. Due to this, we present a class of subsets of $\mathbb{R}^n$, called open sets.

For $x \in \mathbb{R}^n$ and the real number $r > 0$, the **open ball** of radius r about x is defined as the set

$$B_r(x) = \{y \in \mathbb{R}^n : \|x - y\| < r\}.$$

That is, $B_r(x)$ consists of all points in $\mathbb{R}^n$ which are at distance less than r from x. We sometimes denote the open ball $B_r(x)$ by $B(x, r)$. Both notations are widely used in mathematics.

Definition 1.2. A set $U \subseteq \mathbb{R}^n$ is called **open in** $\mathbb{R}^n$, if for every $x \in U$ there exists $r > 0$ such that $B_r(x) \subseteq U$.

In other words, about any point in an open set we can find a small ball which is entirely contained in the set. Note that in this definition, the radius of the ball is allowed to depend on x. See Figure 1.2.1.

We may compare the above definition with the definition of open sets in $\mathbb{R}$ you saw in Analysis I. Recall that a set $I \subseteq \mathbb{R}$ is open in $\mathbb{R}$, if for every $x \in I$, there is $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq I$. This definition is consistent with the one we have given in $\mathbb{R}^n$, since in $\mathbb{R}^1$, $B_\delta(x) = (x - \delta, x + \delta)$.

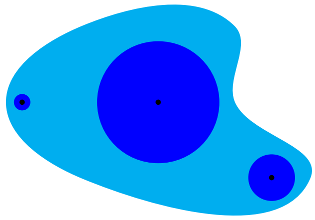

Figure 1.1: An open set in $\mathbb{R}^2$ in cyan, and some balls inside it. The radius of the ball depends on the location of the point.

Example 1.1. The ball $B_1(0)$ is open in $\mathbb{R}^n$. To see this, suppose $x \in B_1(0)$, so that $\|x\| < 1$. Let $r = (1 - \|x\|)/2$. We need to show that $B_r(x) \subseteq B_1(0)$. To that end, let $y \in B_r(x)$ be an arbitrary point. Using the triangle inequality for the norm in $\mathbb{R}^n$, we have

$$\|y\| = \|y - x + x\| \leq \|y - x\| + \|x\| < r + \|x\| = \frac{1 - \|x\|}{2} + \|x\| < \frac{1 + \|x\|}{2} < 1.$$

This means that $y \in B_1(0)$.

Observe that in the above example, one can replace 1 with any other positive real number, and the result is still valid. That is, for every $\delta > 0$, the set $B_\delta(0)$ is open in $\mathbb{R}^n$. Similarly, one can also replace 0 with any $y \in \mathbb{R}^n$. Thus, in general, for any $y \in \mathbb{R}^n$ and any $\delta > 0$, $B_\delta(y)$ is open in $\mathbb{R}^n$.

Example 1.2. The set $A = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is not open. Clearly $y := (1, 0, \ldots, 0)$ belongs to A. On the other hand, if $r > 0$ then $z = (1 + r/2, 0, \ldots, 0)$ belongs to $B_r(y)$ but not to A, so there is no $r > 0$ such that $B_r(y) \subset A$.

Exercise 1.4. Which of the following subsets of $\mathbb{R}^n$ is open:

(a) $\mathbb{R}^n$?

(b) $\emptyset$?

(c) $\big\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid x^1 > 0\big\}$?

(d) $\big\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid \forall i,\ x^i \in [0, 1)\big\}$?

(e) $\big\{x = (x^1, \ldots, x^n) \in \mathbb{R}^n \mid \forall i,\ x^i \in \mathbb{Q}\big\}$?

Exercise 1.5. Let $(x_i)_{i=0}^{\infty}$ be a sequence in $\mathbb{R}^n$ with $\lim_{i\to\infty} x_i = x \in \mathbb{R}^n$. Assume that there is $r > 0$ such that for all $i \geq 0$, we have $\|x_i\| < r$. Show that

$$\|x\| \leq r.$$

Exercise 1.6. (a) Show that if U_1 and U_2 are open sets in $\mathbb{R}^n$, then $U_1 \cup U_2$ and $U_1 \cap U_2$ are open in $\mathbb{R}^n$.

(b) Suppose that U_α, for α in an index set I, are open sets in $\mathbb{R}^n$.

(i) Show that the set $\bigcup_{\alpha\in I} U_\alpha$ is open in $\mathbb{R}^n$.

(ii) Give an example showing that $\bigcap_{\alpha\in I} U_\alpha$ need not be open.

Remark 1.2. It is worth noting that the notion of open sets in $\mathbb{R}^n$ relies on the length function $\|\cdot\|$ we have on $\mathbb{R}^n$. As we shall see in the next chapter, one can consider functions (called metric) with similar properties on a wide range of other sets (such as the set of all continuous functions from $[0,1]$ to $\mathbb{R}$ or the set of all sequences in $[0,1]$, etc). These lead to notions of open sets on such sets. We will look into this in the next chapter.

1.2.2 Continuity at a point, and continuity on an open set

We start with the simple definition

Definition 1.3. Let $A \subset \mathbb{R}^n$ be an open set, and suppose $f : A \to \mathbb{R}^m$. We say that f is **continuous at** $p \in A$ if the following holds: for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in A$ with $\|x - p\| < \delta$ we have

$$\|f(x) - f(p)\| < \epsilon.$$

If f is continuous at every p in A, we say f is **continuous on** A.

We can think of this as saying "f maps points in A close to p to points in $\mathbb{R}^m$ close to $f(p)$". Notice that in the definition above, the symbol $\|\cdot\|$ is playing two slightly different roles: as the norm on $\mathbb{R}^n$ and the norm on $\mathbb{R}^m$.

Remark 1.3. The words "function" and "map" are not identical. For $f : X \to Y$, we use the word "function" when the target space Y is the real numbers or the complex numbers (or in general a field). Otherwise, we use the word "map". Of course it is correct to refer to $f : X \to \mathbb{R}$ as a map, but it is uncommon to refer to $f : X \to Y$ as a function, when Y is not a set of numbers where one can not add and multiply elements. On the other hand, it is common in analysis and geometry to see expressions like, "let f be a function on X", which means that $f : X \to \mathbb{R}$ or $f : X \to \mathbb{C}$. In those cases, the target space is understood from the context.

Example 1.3. The map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = \|x\|$ is continuous on $\mathbb{R}^n$.

To show this, fix an arbitrary $p \in \mathbb{R}^n$. Suppose $\|x - p\| < \delta$, then by the reverse triangle inequality (see Exercise 1.1) we have:

$$|f(x) - f(p)| = \big|\, \|x\| - \|p\| \,\big| \leq \|x - p\| < \delta.$$

Thus we can take $\delta = \epsilon$ and we have satisfied the criteria for continuity of f at p.

Example 1.4. Every linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.

Let $\{e_j\}_{j=1}^n$ be the canonical basis for $\mathbb{R}^n$, that is, all entries of e_j are 0 except the j-th entry which is 1. We may define the real number

$$M = \max_{j=1,\ldots,n} \|\Lambda(e_j)\| .$$

We note that,

$$\begin{aligned}
\|\Lambda(x) - \Lambda(p)\| = \|\Lambda(x - p)\| &= \left\| \Lambda\Big(\sum_{j=1}^{n} e_j (x-p)^j \Big) \right\| \\
&= \left\| \sum_{j=1}^{n} (x-p)^j \Lambda(e_j) \right\| \\
&\leq \sum_{j=1}^{n} \left\| (x-p)^j \Lambda(e_j) \right\| \\
&\leq \sum_{j=1}^{n} \left| (x-p)^j \right| \|\Lambda(e_j)\| \\
&\leq M \sum_{j=1}^{n} \left| (x-p)^j \right|
\end{aligned}$$

Thus, using the inequality in Equation (1.4),

$$\|\Lambda(x) - \Lambda(p)\| \leq M \sum_{j=1}^{n} \|x - p\| = Mn \|x - p\| .$$

Thus, if we take $\delta = \epsilon/(2Mn)$, then for any x with $0 < \|x - p\| < \delta$, we have

$$\|\Lambda(x) - \Lambda(p)\| < \frac{\epsilon}{2Mn} Mn < \epsilon,$$

so Λ is continuous.

Example 1.5. The map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x^1, \ldots, x^n) = x^1$ is continuous on $\mathbb{R}^n$.

To see this, fix an arbitrary $p \in \mathbb{R}^n$. Suppose $\|x - p\| < \delta$, then by the inequality in (1.5) we have:

$$|f(x) - f(p)| = \left|x^1 - p^1\right| \leq \max_{k=1,\ldots,n} \left| x^k - p^k \right| \leq \|x - p\| < \delta,$$

so we may take $\delta = \epsilon$ and we have satisfied the condition for continuity. Obviously the same argument shows that all of the coordinate maps (i.e. the map taking x to x^k) are continuous.

Theorem 1.2. *Let A be an open subset of $\mathbb{R}^n$ and B be an open subset of $\mathbb{R}^m$. Suppose $f : A \to B$ is continuous at p and $g : B \to \mathbb{R}^l$ is continuous at $f(p)$. Then $g \circ f : A \to \mathbb{R}^l$ is continuous at p.*

Proof. Fix an arbitrary $\epsilon > 0$. Since g is continuous at $f(p)$, we know that there exists $\delta_1 > 0$ such that for any $y \in B$ with $\|y - f(p)\| < \delta_1$, we have $\|g(y) - g(f(p))\| < \epsilon$. Similarly, since f is continuous at p, we know that there exists $\delta > 0$ such that for any $x \in A$ with $\|x - p\| < \delta$, we have $\|f(x) - f(p)\| < \delta_1$. Combining these two statements and taking $y = f(x)$, we deduce that if $x \in A$ with $\|x - p\| < \delta$, we have $\|g(f(x)) - g(f(p))\| < \epsilon$. $\square$

It is sometimes useful to express the continuity of a map in a slightly different way, for which we need the following definition:

Definition 1.4. Let A be an open subset of $\mathbb{R}^n$ and suppose $f : A \to \mathbb{R}^m$. For $p \in A$, we say that the **limit** of f as x tends to p is equal to $q \in \mathbb{R}^m$, if the following holds: for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in A$ with $0 < \|x - p\| < \delta$ we have

$$\|f(x) - q\| < \epsilon.$$

In this case, we write

$$\lim_{x \to p} f(x) = q.$$

Note that in the above definition, we do not allow $x = p$. With this notion of a limit in hand, we can give the definition of continuity more compactly as:

"f is continuous at p, if $\lim_{x \to p} f(x) = f(p)$."

Theorem 1.3. *Suppose A is an open subset of $\mathbb{R}^n$, $p \in A$, and $f, g : A \to \mathbb{R}$ with*

$$\lim_{x \to p} f(x) = F, \qquad \lim_{x \to p} g(x) = G.$$

Then

(i) $\lim_{x \to p} (f(x) + g(x)) = F + G$,

(ii) $\lim_{x \to p} (f(x)g(x)) = FG$,

(iii) If, furthermore $G \neq 0$, then:

$$\lim_{x \to p} \frac{f(x)}{g(x)} = \frac{F}{G}.$$

Proof. (i) Fix an arbitrary $\epsilon > 0$. Since $\lim_{x\to p} f(x) = F$, we know that there exists $\delta_1 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_1$,

$$|f(x) - F| < \frac{\epsilon}{2}.$$

Similarly, there exists $\delta_2 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_2$,

$$|g(x) - G| < \frac{\epsilon}{2}.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Evidently $\delta > 0$. For every $x \in A$ with $0 < \|x - p\| < \delta$, by the triangle inequality, we have

$$|f(x) + g(x) - (F + G)| \leq |f(x) - F| + |g(x) - G| < \epsilon.$$

(ii) Fix an arbitrary $\epsilon > 0$, and assume without loss of generality that $\epsilon < 3$ (Why can see assume this?). Since $\lim_{x\to p} f(x) = F$, we know that there exists $\delta_1 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_1$,

$$|f(x) - F| < \frac{\epsilon}{3(1 + |G|)}.$$

Similarly, there exists $\delta_2 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_2$,

$$|g(x) - G| < \frac{\epsilon}{3(1 + |F|)}.$$

To control $f(x)g(x) - FG$, we add and subtract the same terms, so that we obtain terms of the form $f(x) - F$ and $g(x) - G$. That is,

$$\begin{aligned} f(x)g(x) - FG &= f(x)g(x) - f(x) \cdot G + f(x) \cdot G - F \cdot G \\ &= f(x)(g(x) - G) + (f(x) - F) \cdot G \\ &= (f(x) - F + F)(g(x) - G) + (f(x) - F) \cdot G \\ &= (f(x) - F)(g(x) - G) + F \cdot (g(x) - G) + (f(x) - F) \cdot G \end{aligned}$$

Now, take $\delta = \min\{\delta_1, \delta_2\}$. For every $x \in A$ with $0 < \|x - p\| < \delta$, by the triangle inequality, we have

$$\begin{aligned} |f(x)g(x) - FG| &\leq |f(x) - F|\,|g(x) - G| + |F|\,|g(x) - G| + |G|\,|f(x) - F| \\ &< \frac{\epsilon^2}{9(1 + |F|)(1 + |G|)} + \frac{\epsilon\,|F|}{3(1 + |F|)} + \frac{\epsilon\,|G|}{3(1 + |G|)} \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon. \end{aligned}$$

(iii) Given the previous part, it suffices to show that if $\lim_{x\to p} g(x) = G$ with $G \neq 0$, then

$$\lim_{x\to p} \frac{1}{g(x)} = \frac{1}{G}.$$

Fix an arbitrary $\epsilon > 0$. Since $\lim_{x\to p} g(x) = G$, we know that there exist $\delta_1 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_1$,

$$|g(x) - G| < \frac{\epsilon\,|G|^2}{2}.$$

Also, since $G \neq 0$, $G/2 > 0$, and hence, there is $\delta_2 > 0$ such that for every $x \in A$ with $0 < \|x - p\| < \delta_2$,

$$|g(x) - G| < \frac{|G|}{2}.$$

By the triangle inequality, this implies that

$$|g(x)| = |g(x) - G + G| \geq |G| - |g(x) - G| > |G| - \frac{|G|}{2} = \frac{|G|}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. For every $x \in A$ with $0 < \|x - p\| < \delta$, we have

$$\left|\frac{1}{g(x)} - \frac{1}{G}\right| = |G - g(x)| \cdot \frac{1}{|G|} \cdot \frac{1}{|g(x)|} < \frac{\epsilon\,|G|^2}{2} \cdot \frac{1}{|G|} \cdot \frac{2}{|G|} = \epsilon.$$

This completes the proof.

□

Corollary 1.4. *Suppose A is an open set in $\mathbb{R}^n$ and $f, g : A \to \mathbb{R}$ are continuous at $p \in A$. Then,*

(i) $f + g$ is continuous at p.

(ii) fg is continuous at p.

(iii) If, furthermore $g(p) \neq 0$, then $\dfrac{f}{g}$ is continuous at p.

Exercise 1.7. Assume that A is an open set in $\mathbb{R}^n$ and $f : A \to \mathbb{R}^m$. Show that $\lim_{x\to p} f(x) = F$, if and only if, for any sequence $(x_i)_{i=0}^{\infty}$ in $A \setminus \{p\}$ with $\lim_{i\to\infty} x_i = p$,

$$\lim_{i\to\infty} f(x_i) = F.$$

Exercise 1.8. (a) Show that the map $f : \mathbb{R} \to \mathbb{R}^n$ defined as $f(x) = (x, 0, \ldots, 0)$ is continuous on $\mathbb{R}$.

(b) Let A be an open set in $\mathbb{R}^n$ and $f^1, f^2, \ldots, f^m$ are functions from A to $\mathbb{R}$. Consider the map $f : A \to \mathbb{R}^m$ defined as

$$f(x^1, \ldots, x^n) \mapsto \left(f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n)\right).$$

Show that f is continuous at $p \in A$, if and only if, for every $k = 1, \ldots, m$ the map $f^k : A \to \mathbb{R}$ is continuous at p.

(c) Show that the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x^1, x^2, \ldots, x^n) = 3x^1(x^2)^5 + 4x^2(x^n)^7$ is continuous on $\mathbb{R}^n$. Here, $(x^j)^m$ denotes the coordinate x^j raised to power m.

With the above results, one can build many continuous maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. For example,

$$(x^1, x^2) \mapsto \left(\sin(x^1x^2), \cos(x^2)\right),$$
$$(x^1, x^2, x^3) \mapsto \left(\frac{x^1 - x^2}{1 + (x^2)^2}, e^{x^3}\right).$$

Exercise 1.9 (*)**.** (a) Suppose $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous on $\mathbb{R}^n$, and suppose $U \subset \mathbb{R}^m$ is open. Show that:

$$f^{-1}(U) := \{x \in \mathbb{R}^n : f(x) \in U\}$$

is open.

(b) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^m$ has the property that $f^{-1}(U) \subset \mathbb{R}^n$ is open for every open set $U \subset \mathbb{R}^m$. Show that f is continuous on $\mathbb{R}^n$.

1.3 Derivative of a map of Euclidean spaces

So far, when differentiating functions, we've restricted ourselves to the situation where the function depends only on one variable. This covers lots of situations that we're interested in, but of course we often wish to consider maps of more than one variable. In this chapter we will see how the idea of differentiation can be extended to maps which send (subsets of) $\mathbb{R}^n$ to $\mathbb{R}^m$. The basic idea will be that the derivative of a map at a point p should be the "best linear approximation" to the map at p.

1.3.1 Derivative as a linear map

Before we think about how to define a derivative of a map in higher dimensions, let's first note some of the potential challenges. In one dimension, we say that f is differentiable at p if the limit

$$\lim_{x \to p} \frac{f(x) - f(p)}{x - p}$$

exists. If $x, p \in \mathbb{R}^n$ and $f(x), f(p) \in \mathbb{R}^m$ then we obviously have a problem: we don't even know how to make sense of 'dividing by $x - p$', and it's not clear what sort of object we should end up with.

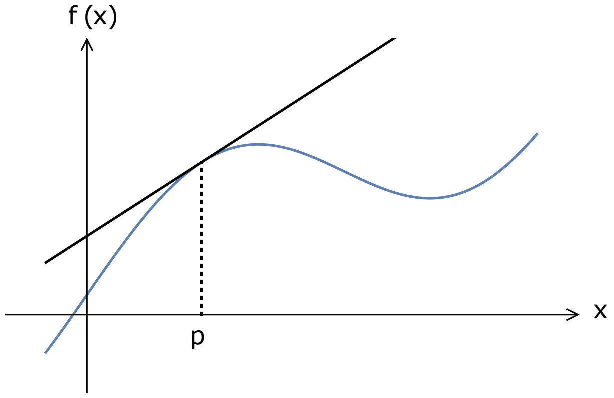

Figure 1.2: The tangent to f at p.

To try and find a way through this impasse, let's just remind ourselves how the derivative is introduced in one dimension. By approximating with successive chords, we consider the tangent to the graph of f at p (see Figure 1.2). Let us think a little about how the tangent is characterised. Any (non-vertical) straight line passing through $(p, f(p))$ is the graph of the affine map

$$A_\lambda : x \mapsto \lambda(x - p) + f(p)$$

for some $\lambda \in \mathbb{R}$. Let's consider the difference between f and such an affine map

$$f(x) - A_\lambda(x) = f(x) - f(p) - \lambda(x - p).$$

In general, from the continuity of f we know that for any $\lambda \in \mathbb{R}$,

$$\lim_{x\to p} [f(x) - A_\lambda(x)] = 0. \tag{1.6}$$

However, if f is differentiable, there is a unique choice of λ that allows us to make a stronger statement. If f is differentiable, there exists a unique $\lambda \in \mathbb{R}$ such that

$$\lim_{x\to p} \frac{|f(x) - A_\lambda(x)|}{|x-p|} = 0.$$

This is a stronger statement than (1.6) because it tells us that $f(x) - A_\lambda(x)$ is going to zero faster than $|x-p|$, as $x \to p$. We make this informal discussion more precise in the following lemma.

Lemma 1.5. *The map $f : (a,b) \to \mathbb{R}$ is differentiable at $p \in (a,b)$ if and only if there exists a map of the from $A_\lambda(x) = \lambda(x-p) + f(p)$, for some $\lambda \in \mathbb{R}$, such that*

$$\lim_{x\to p} \frac{|f(x) - A_\lambda(x)|}{|x-p|} = 0.$$

Proof. We can re-write

$$\frac{|f(x) - f(p) - \lambda(x-p)|}{|x-p|} = \left| \frac{f(x) - f(p)}{x-p} - \lambda \right|,$$

so that

$$\lim_{x\to p} \frac{|f(x) - A_\lambda(x)|}{|x-p|} = 0 \quad \iff \quad \lim_{x\to p} \frac{f(x) - f(p)}{x-p} = \lambda.$$

The expression on the right-hand side of the above equation is the definition of differentiability of f at p. $\square$

We may rewrite

$$A_\lambda(x) = \lambda(x-p) + f(p) = \lambda x + (f(p) - \lambda p).$$

Thus, $A_\lambda : \mathbb{R} \to \mathbb{R}$ is the composition of the linear map $x \mapsto \lambda x$ and the translation $x \mapsto x + (f(p) - \lambda p)$. Such maps are called affine maps of $\mathbb{R}$. By the above lemma, the map f is differentiable at p, if it is "well approximated" by an affine map at p. We may generalise this to higher dimensions.

Since we are going to frequently apply linear an nonlinear maps to variables, to distinguish between these two cases, we shall use the notation $h[v]$ when h is a linear map and v is seen as a vector, and use $h(v)$ when h is a map and v is seen as a point in the domain of h.

Let $L(\mathbb{R}^n; \mathbb{R}^m)$ denote the set of all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$. Recall that $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map if

$$\Lambda[x+y] = \Lambda[x] + \Lambda[y], \qquad \forall x, y \in \mathbb{R}^n,$$

$$\Lambda[ax] = a\Lambda[x], \qquad \forall a \in \mathbb{R} \text{ and } x \in \mathbb{R}^n.$$

In analogy to the statement in Lemma 1.5 we propose the following definition.

Definition 1.5. Suppose $\Omega \subset \mathbb{R}^n$ is open. The map $f : \Omega \to \mathbb{R}^m$ is **differentiable** at $p \in \Omega$, if there exists a linear map $\Lambda \in L(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$\lim_{x \to p} \frac{\|f(x) - (\Lambda[x-p] + f(p))\|}{\|x-p\|} = 0.$$

In this case, we write

$$Df(p) := \Lambda,$$

and call $Df(p)$ the derivative of the map f at the point p.

Note that some authors refer to the derivative of a map as **total derivative**, or **differential**. We shall refer to that as derivative.

It is often useful to have the following equivalent characterisation of differentiability in higher dimensions: $f : \Omega \to \mathbb{R}$ is differentiable at $p \in \Omega$ if and only if there exists $\Lambda \in L(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - \Lambda[h]\|}{\|h\|} = 0.$$

Note that in the above equation, $h \to 0$ in $\mathbb{R}^n$.

Recall that using a canonical basis for $\mathbb{R}^n$ and $\mathbb{R}^m$ any linear map $\Lambda \in L(\mathbb{R}^n; \mathbb{R}^m)$ can be expressed as an $m \times n$ matrix which is called the **Jacobian** of f at p. The convention is that an $m \times n$ matrix has m rows and n columns. For the purposes of this course, we won't make a big deal of the difference between a linear map and its matrix representation with respect to the canonical basis, so will use the words derivative and Jacobian essentially indistinguishably.

Lemma 1.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set. If $f : \Omega \to \mathbb{R}^m$ is differentiable at $p \in \Omega$, then it is continuous at p.*

Proof. Since

$$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - \Lambda[h]\|}{\|h\|} = 0,$$

we must have

$$\lim_{h \to 0} \|f(p+h) - f(p) - \Lambda[h]\| = 0.$$

On the other hand, since linear maps are continuous, see Example 1.4, we obtain

$$0 = \lim_{h \to 0} (f(p+h) - f(p) - \Lambda[h]) = \lim_{h \to 0} (f(p+h) - f(p)).$$

□

Example 1.6. By Lemma 1.5 any function $f : (a,b) \to \mathbb{R}$ which is differentiable at p satisfies the conditions of 1.5 with $Df(p) = f'(p)$. Notice that a 1×1 matrix is simply a real number.

Example 1.7. Let $B \in L(\mathbb{R}^n;\mathbb{R}^m)$ and $V \in \mathbb{R}^m$. Then, the map $f : \mathbb{R}^n \to \mathbb{R}^m$ defined as

$$f(x) = B(x) + V$$

is differentiable at each $p \in \mathbb{R}^n$, and $Df(p) = B$. To see this, note that

$$\begin{aligned} f(p+h) - f(p) - B(h) &= (B(p+h) + V) - (B(p) + V) - B(h) \\ &= B(p) + B(h) + V - B(p) - V - B(h) = 0. \end{aligned}$$

Thus,

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - B(h)\|}{\|h\|} = \lim_{h\to 0} 0 = 0.$$

Example 1.8. The map $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = \|x\|^2$$

is differentiable at each $p \in \mathbb{R}^n$, and $Df(p)$ is the linear map

$$Df(p)[h] = 2\langle p, h\rangle, \quad \forall h \in \mathbb{R}^n.$$

From the properties of the inner product in Exercise 1.1-(a), we can see that the map $h \mapsto 2\langle p, h\rangle$ is a linear map.

We note that

$$f(p+h) = \|p+h\|^2 = \langle p+h, p+h\rangle = \|p\|^2 + 2\langle p, h\rangle + \|h\|^2,$$

so that

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - 2\langle p, h\rangle\|}{\|h\|} = \lim_{h\to 0} \|h\| = 0.$$

As a matrix, we have that $Df(p) = 2p$, where p is viewed as a row vector with n components (this is in line with our convention that a $1 \times n$ matrix maps $\mathbb{R}^n$ to $\mathbb{R}^1$). So the Jacobian is a row vector for this map.

Example 1.9. Let $m \geq 1$ be an integer, and assume that for $i = 1, 2, \dots, m$, the map $f^i : (a, b) \to \mathbb{R}$ is differentiable at $p \in (a, b)$. Then the map $f : (a, b) \to \mathbb{R}^m$ defined as

$$f(x) = \left(f^1(x), f^2(x), \dots, f^m(x)\right),$$

is differentiable at p, and the derivative $Df(p) : \mathbb{R} \to \mathbb{R}^m$ has the matrix representation

$$Df(p) = \begin{pmatrix} (f^1)'(p) \\ \vdots \\ (f^m)'(p) \end{pmatrix}.$$

To see this, we note that

$$f(p+h)-f(p)-\begin{pmatrix}(f^1)'(p)\\ \vdots \\ (f^m)'(p)\end{pmatrix}h=\begin{pmatrix}f^1(p+h)-f^1(p)-(f^1)'(p)h\\ \vdots \\ f^m(p+h)-f^m(p)-(f^m)'(p)h\end{pmatrix}$$

so that, using the inequality in (1.5),

$$\frac{\|f(p+h)-f(p)-Df(p)\,[h]\|}{\|h\|}\leq\sqrt{m}\max_{j=1,\dots,m}\frac{\left|f^j(p+h)-f^j(p)-(f^j)'(p)h\right|}{|h|}.$$

Since each f^j is differentiable at p, the left hand side of the above equation tends to 0, as $h\to 0$. And since the left hand side of the equation is non-negative, it must tend to 0, as $h\to 0$. Notice here that the expression $Df(p)\,[h]$ means applying the linear map $Df(p)$ to the one dimensional vector h, which gives us an element of $\mathbb{R}^m$.

Implicitly in the discussion above, we've assumed that $Df(p)$, if it exists, must be unique. Of course, this is something that we need to prove.

Theorem 1.7. *The derivative, if it exists, is unique.*

Proof. Suppose $\Omega\subset\mathbb{R}^n$ is open, $f:\Omega\to\mathbb{R}^m$, $p\in\Omega$ and that Λ and Λ' satisfy:

$$\lim_{h\to 0}\frac{\|f(p+h)-f(p)-\Lambda[h]\|}{\|h\|}=\lim_{h\to 0}\frac{\|f(p+h)-f(p)-\Lambda'[h]\|}{\|h\|}=0.$$

Let e be an arbitrary vector in $\mathbb{R}^n$ with $\|e\|=1$. Then for any real number $\alpha\neq 0$ we have

$$\frac{\Lambda[\alpha e]}{\alpha}=\Lambda[e].$$

Now, let $(\alpha_j)_{j=0}^{\infty}$ be a sequence of non-zero real numbers tending to 0 as $j\to\infty$. By adding and subtracting identical terms, we see that

$$\begin{aligned}
&\|\Lambda[e]-\Lambda'[e]\|\\
&\quad=\left\|\frac{\Lambda[\alpha_j e]}{\alpha_j}-\frac{\Lambda'[\alpha_j e]}{\alpha_j}\right\|\\
&\quad=\lim_{j\to\infty}\frac{\|\Lambda[\alpha_j e]-\Lambda'[\alpha_j e]\|}{\|\alpha_j e\|}\\
&\quad=\lim_{j\to\infty}\frac{\|-f(p+\alpha_j e)+f(p)+\Lambda[\alpha_j e]+f(p+\alpha_j e)-f(p)-\Lambda'[\alpha_j e]\|}{\|\alpha_j e\|}\\
&\quad\leq\lim_{j\to\infty}\frac{\|f(p+\alpha_j e)-f(p)-\Lambda[\alpha_j e]\|}{\|\alpha_j e\|}+\lim_{j\to\infty}\frac{\|f(p+\alpha_j e)-f(p)-\Lambda'[\alpha_j e]\|}{\|\alpha_j e\|}\\
&\quad=0.
\end{aligned}$$

For the last equality in the above equation we have used that $\alpha_j e\to 0$ as $j\to\infty$. By the above equation, for any unit vector e we have $\Lambda[e]=\Lambda'[e]$, which implies that (as linear maps) $\Lambda=\Lambda'$. □

Exercise 1.10. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is given by $f(x) = x$. Show that f is differentiable at each $p \in \mathbb{R}^n$ and

$$Df(p) = \mathrm{id},$$

where $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map.

Exercise 1.11. Show that the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) \mapsto x^2 + y^2,$$

is differentiable at all points $p = (\xi, \eta) \in \mathbb{R}^2$ with Jacobian

$$Df(p) = \begin{pmatrix} 2\xi & 2\eta \end{pmatrix}.$$

Exercise 1.12. One might hope that the derivative can be calculated by finding

$$\lim_{x \to p} \frac{f(x) - f(p)}{\|x - p\|}.$$

By considering the example of Exercise 1.10 or otherwise, show that this limit may not always exist, even if f is differentiable at p.

Exercise 1.13. Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f, g : \Omega \to \mathbb{R}^m$ are differentiable at $p \in \Omega$. Show that $h = f + g$ is differentiable at p and

$$Dh(p) = Df(p) + Dg(p)$$

1.3.2 Chain rule

In dimension one there is a simple "algorithm" which allows us to calculate the derivative of more complicated maps using the derivative of simpler ones. That algorithm is the chain rule. If $f, g : \mathbb{R} \to \mathbb{R}$, with g differentiable at p and f differentiable at $g(p)$, then $f \circ g$ is differentiable at p with

$$(f \circ g)'(p) = f'(g(p))g'(p).$$

Now, suppose that $g : \mathbb{R}^n \to \mathbb{R}^m$ and $f : \mathbb{R}^m \to \mathbb{R}^l$, with g differentiable at p and f differentiable at $g(p)$. Let $h = f \circ g$. We know that $Dg(p) : \mathbb{R}^n \to \mathbb{R}^m$ and $Df(g(p)) : \mathbb{R}^m \to \mathbb{R}^l$ are linear maps, so it certainly makes sense to consider $Df(g(p)) \circ Dg(p)$, where "$\circ$" denotes the composition of linear maps (corresponding to matrix multiplication). This will be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^l$, which is the right kind of object to be $Dh(p)$. In fact, it is the case that $h = f \circ g$ is differentiable at p with

$$Dh(p) = Df(g(p)) \circ Dg(p)$$

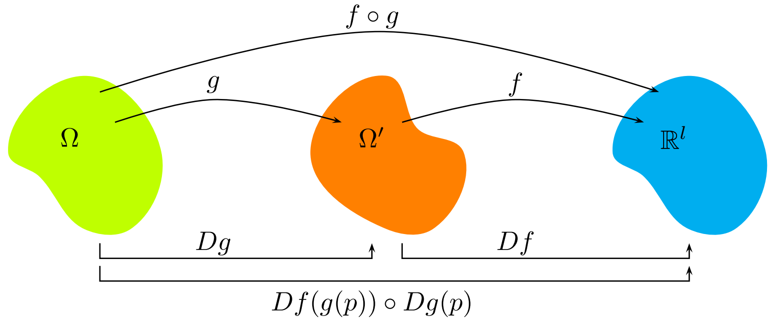

Figure 1.3: Illustration of Theorem 1.8.

Theorem 1.8. *Assume $\Omega \subseteq \mathbb{R}^n$ and $\Omega' \subseteq \mathbb{R}^m$ are open sets, with $g : \Omega \to \Omega'$ differentiable at $p \in \Omega$ and $f : \Omega' \to \mathbb{R}^l$ differentiable at $g(p) \in \Omega'$. Then $h = f \circ g : \Omega \to \mathbb{R}^l$ is differentiable at p with derivative*

$$Dh(p) = Df(g(p)) \circ Dg(p).$$

(*) *Proof.* Let $g(p) = q$, $A = Dg(p)$, $B = Df(q)$. We define the map

$$\begin{aligned}
\phi(x) &= g(x) - g(p) - A(x - p), \quad \forall x \in \Omega \\
\psi(y) &= f(y) - f(q) - B(y - q), \quad \forall y \in \Omega' \\
\tau(x) &= f(g(x)) - f(g(p)) - B\left(A(x - p)\right), \quad \forall x \in \Omega.
\end{aligned}$$

By the assumptions in the theorem we know that

$$0 = \lim_{x \to p} \frac{\phi(x)}{\|x - p\|}, \tag{1.7}$$

$$0 = \lim_{y \to q} \frac{\psi(y)}{\|y - q\|}, \tag{1.8}$$

and we need to show that

$$\lim_{x \to p} \frac{\tau(x)}{\|x - p\|} = 0.$$

We may rewrite the map τ as

$$\begin{aligned}
\tau(x) &= f(g(x)) - f(g(p)) - B\left(A(x - p)\right) \\
&= f(g(x)) - f(g(p)) - B\left(g(x) - g(p) - \phi(x)\right) \\
&= f(g(x)) - f(g(p)) - B\left(g(x) - g(p)\right) + B(\phi(x)) \\
&= \psi(g(x)) + B(\phi(x)).
\end{aligned}$$

On the other hand, we recall from Example 1.4 that there is a real number M such that

$$\|A(x)\| \leq M \|x\|, \quad \forall x \in \mathbb{R}^n.$$

Since B is linear, and hence continuous by Example 1.4, we have that

$$\lim_{x\to p}\frac{B(\phi(x))}{\|x-p\|}=\lim_{x\to p}B\left(\frac{\phi(x)}{\|x-p\|}\right)=B\left(\lim_{x\to p}\frac{\phi(x)}{\|x-p\|}\right)=0.$$

Fix an arbitrary $\epsilon>0$. It follows from (1.8) that there exists $\delta>0$ such that for $y\in\Omega'$ with $\|y-q\|<\delta$ we have

$$\frac{\|\psi(y)\|}{\|y-q\|}<\epsilon$$

which implies

$$\|\psi(y)\|<\epsilon\,\|y-q\|\,.$$

On the other hand, since g is continuous, there exists δ_1 such that if $x\in\Omega$ with $\|x-p\|<\delta_1$ then

$$\|g(x)-g(p)\|=\|g(x)-q\|<\delta.$$

Thus, for every $x\in\Omega$ with $\|x-p\|<\delta_1$, we have

$$\begin{aligned}\|\psi(g(x))\|&<\epsilon\,\|g(x)-q\|\\&=\epsilon\,\|\phi(x)+A(x-p)\|\\&\le\epsilon\,\|\phi(x)\|+\epsilon M\,\|x-p\|\,.\end{aligned}$$

Dividing through by $\|x-p\|$ and taking the limit, we deduce that

$$\lim_{x\to p}\frac{\|\psi(g(x))\|}{\|x-p\|}\le\epsilon M.$$

Since $\epsilon>0$ was arbitrary, we conclude

$$\lim_{x\to p}\frac{\|\psi(g(x))\|}{\|x-p\|}=0,$$

and we are done. □

Example 1.10. Let $m\ge 1$ be an integer, and assume that for $i=1,2,\dots,m$, the functions $g^i:(a,b)\to\mathbb{R}$ are differentiable at some $p\in(a,b)$. Then, the map $k:(a,b)\to\mathbb{R}$, defined as

$$k(x)=\left\|\left(g^1(x),g^2(x),\dots,g^m(x)\right)\right\|^2$$

is differentiable at p, and its Jacobian matrix has one real entry

$$2g^1(p)(g^1)'(p)+2g^2(p)(g^2)'(p)\cdots+2g^m(p)(g^m)'(p).$$

We note that by Example 1.9, the map $g:(a,b)\to\mathbb{R}^m$ defined as

$$g(x)=(g^1(p),g^2(p),\dots,g^m(p))$$

is differentiable at p with derivative

$$Dg(p) = \begin{pmatrix} (g^1)'(p) \\ \vdots \\ (g^m)'(p) \end{pmatrix}.$$

On the other hand, in Example 1.8, we saw that the map $f(x) = \|x\|^2$ is differentiable at every point in $\mathbb{R}^m$ with derivative $Df(q)[h] = 2\langle q, h\rangle$. We have $k = f \circ g$ on (a, b). Thus, by the chain rule, the map h is differentiable at p, with derivative

$$\begin{aligned} Dk(p)[h] &= Df(g(p)) \circ Dg(p)[h] \\ &= D(f(g(P))\left[((g^1)'(p)h, \dots, (g^m)'(p)h)\right] \\ &= 2\left\langle g(p), ((g^1)'(p)h, \dots, (g^m)'(p)h)\right\rangle \\ &= 2\left\langle g(p), h((g^1)'(p), \dots, (g^m)'(p))\right\rangle \\ &= 2\langle g(p), Dg(p)\rangle\, h. \end{aligned}$$

Thus, the Jacobian of k at p is the one by one matrix with real entry

$$2\langle g(p), Dg(p)\rangle = 2g^1(p)(g^1)'(p) + 2g^2(p)(g^2)'(p) \cdots + 2g^m(p)(g^m)'(p).$$

Exercise 1.14. Assume Ω and Ω' are open sets in $\mathbb{R}^n$, $g : \Omega \to \Omega'$ differentiable at $p \in \Omega$ and $f : \Omega' \to \Omega$ differentiable at $g(p) \in \Omega'$. Moreover,

$$\begin{aligned} f \circ g(x) &= x, \qquad \forall\, x \in \Omega. \\ g \circ f(x) &= x, \qquad \forall\, x \in \Omega'. \end{aligned}$$

Show that

$$Df(g(p)) = (Dg(p))^{-1}.$$

Exercise 1.15 (*)**.** (a) Show that the map $P : \mathbb{R}^2 \to \mathbb{R}$ given by

$$P(x, y) = xy$$

is differentiable at each point $p = (\xi, \eta) \in \mathbb{R}^2$, with Jacobian

$$DP(p) = \begin{pmatrix} \eta & \xi \end{pmatrix}.$$

(b) Suppose that $f, g : \mathbb{R}^n \to \mathbb{R}$ are differentiable at $q \in \mathbb{R}^n$. Show that the map $Q : \mathbb{R}^n \to \mathbb{R}^2$ defined as

$$Q(z) = (f(z), g(z))$$

is differentiable at q, with derivative

$$DQ(q) = \begin{pmatrix} Df(q) \\ Dg(q) \end{pmatrix}$$

(c) Show that the map $F : \mathbb{R}^n \to \mathbb{R}$ defined as $F(z) = f(z)g(z)$, for all $z \in \mathbb{R}^n$, is differentiable at q, with derivative

$$DF(q) = g(q)Df(q) + f(q)Dg(q)$$

1.4 Directional derivatives

1.4.1 Rates of change and partial derivatives

Although the definitions of differentiability in dimension one and in higher dimensions appear similar, there is a major difference which makes the latter a more difficult concept. In dimension one, to see if a map : $f(a,b) \to \mathbb{R}$ is differentiable at some $x \in (a,b)$, we only need to verify that the limit of $(f(x) - f(p)/(x-p))$ exists as $x \to p$. To verify this, we do not need to know the value of the limit beforehand, that is, the value of the limit does not appear in this ratio. However, in higher dimensions, to verify if a map $f : \Omega \to \mathbb{R}^n$ is differentiable at some $p \in \Omega$, we need to know the derivative at that point. In other words, the derivative of the map at p appears in the criteria for differentiability. For basic maps, it is possible to guess the derivative, but in general, it may not be obvious what the derivative is. See for instance the map in Example 1.8. The purpose of this section is to present a simple approach to identify a candidate for the derivative in higher dimensions.

For a function $f : (a,b) \to \mathbb{R}$, we are familiar with the idea of $f'(p)$ telling us something about the rate of change of $f(x)$ as we vary x near $p \in (a,b)$. We can connect the derivative to this sort of concept with the directional derivative. Let us suppose that we are given a function $f : \mathbb{R}^3 \to \mathbb{R}$, which is supposed to represent the temperature of some three dimensional body which is not changing in time. Suppose we start at the origin $0 \in \mathbb{R}^3$ and travel along the curve $t \mapsto vt$, for some fixed $v \in \mathbb{R}^3$, that is we move along a straight line with velocity v passing through the origin at time 0. We can record the temperature of our surroundings as a function of time, $\theta(t)$ and we will find $\theta(t) = f(vt)$. Suppose we ask what the rate of change of temperature is at $t = 0$. This will of course be $\theta'(0)$. Now, we notice that we can write:

$$\theta = f \circ V$$

where V is the linear map $V : \mathbb{R} \to \mathbb{R}^3$ given by $V(t) = vt$. Now, we can use the chain rule to calculate $\theta'(0) = D\theta(0)$ and we find:

$$\theta'(0) = D\theta(0) = Df(0) \circ DV(0).$$

Now, since V is a linear map, we have $DV(0) = v$ and we conclude:

$$\theta'(0) = Df(0)[v].$$

This gives us a nice interpretation of the derivative $Df(0)$. When we apply $Df(0)$ to a vector v, we find the rate of change of f at 0 as we travel along a line with velocity v. More generally, we can consider travelling along the line given by $V(t) = p + tv$ for some $p \in \mathbb{R}^3$. Then at $t = 0$, we are passing through the point $p \in \mathbb{R}^3$. Setting $\theta(t) = f(p + tv)$, We call the quantity:

$$\theta'(0) = D\theta(p) = Df(p)[v]$$

the **directional derivative** of f at p in the direction v. Sometimes the notation

$$\frac{\partial f}{\partial v}(p) := \lim_{t\to 0} \frac{1}{t}\left[f(p+vt) - f(p)\right] = Df(p)[v]$$

is used for the directional derivative.

Now, if we take $\{e_1, e_2, e_3\}$ to be the canonical basis vectors for $\mathbb{R}^3$, then we can write $v = v^1 e_1 + v^2 e_2 + v^3 e_3$ for $v^i \in \mathbb{R}$. Doing this, and recalling that $Df(p)$ is a linear map, we have:

$$\begin{aligned}\frac{\partial f}{\partial v}(p) &= Df(p)\left[v^1 e_1 + v^2 e_2 + v^3 e_3\right] && (1.9)\\ &= v^1 Df(p)\left[e_1\right] + v^2 Df(p)\left[e_2\right] + v^3 Df(p)\left[e_3\right] \\ &= v^1 D_1 f(p) + v^2 D_2 f(p) + v^3 D_3 f(p). && (1.10)\end{aligned}$$

In other words, we can find any directional derivative at p, provided we know the three numbers:

$$D_i f(p) = \frac{\partial f}{\partial e_i}(p), \qquad i = 1, 2, 3.$$

called the **partial derivatives** of f at p. Equivalently, these can be defined as

$$D_i f(p) := \lim_{t\to 0} \frac{f(p + te_i) - f(p)}{t}.$$

If $f : \mathbb{R}^3 \to \mathbb{R}$, then for x, y and z in $\mathbb{R}$,

$$D_1 f(x, y, z) = \lim_{t\to 0} \frac{f(x + t, y, z) - f(x, y, z)}{t} =: \frac{\partial f}{\partial x}(x, y, z),$$

where we've introduced yet more notation. The expression $\frac{\partial f}{\partial x}$ you should think of as meaning 'differentiate f with respect to x, while treating y, z as constants. Returning to (1.10), we see that for any $v = (v^1, v^2, v^3)$, we have

$$Df(p)[v] = \begin{pmatrix} D_1 f(p) & D_2 f(p) & D_3 f(p) \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix},$$

so that the Jacobian of f at p is given by

$$Df(p) = \begin{pmatrix} D_1 f(p) & D_2 f(p) & D_3 f(p) \end{pmatrix}.$$

To introduce even more notation, we sometimes write

$$\nabla f(p) = \begin{pmatrix} D_1 f(p) \\ D_2 f(p) \\ D_3 f(p) \end{pmatrix},$$

which is called the **gradient** of f at p, and with this notation

$$Df(p) = (\nabla f(p))^t.$$

We can extend all of these notions to more general range and domains, which leads us to the following definition.

Definition 1.6. Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$ is differentiable at $p \in \Omega$. For any vector $v \in \mathbb{R}^n$ with $\|v\| = 1$, the **directional derivative** of f at p in the direction v is given by

$$\frac{\partial f}{\partial v}(p) = \lim_{t\to 0} \frac{f(p+tv) - f(p)}{t} = Df(p)[v]$$

The partial derivatives of f at p are given by

$$D_i f(p) = \frac{\partial f}{\partial e_i}(p) = \lim_{t\to 0} \frac{f(p+te_i) - f(p)}{t}, \quad i = 1, \dots, n.$$

Notice that $f(x)$ is now a vector in $\mathbb{R}^m$, so expressions like $\lim_{t\to 0} \frac{f(p+tv)-f(p)}{t}$ have to be understood as limits in $\mathbb{R}^m$, so that $\frac{\partial f}{\partial v}(p)$ will be an $m-$dimensional column vector. That is, if

$$f(x) = (f^1(x), f^2(x), \dots, f^m(x)),$$

then

$$D_i f(p) = \begin{pmatrix} D_i f^1(p) \\ \vdots \\ D_i f^m(p) \end{pmatrix}.$$

Theorem 1.9. *Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$ is of the form*

$$f(x) = (f^1(x), f^2(x), \dots, f^m(x)).$$

If f is differentiable at some $p \in \Omega$, then the Jacobian of f at p is

$$Df(p) = \begin{pmatrix} D_1 f^1(p) & \dots & D_n f^1(p) \\ \vdots & \ddots & \vdots \\ D_1 f^m(p) & \dots & D_n f^m(p) \end{pmatrix}.$$

Proof. Let $\{e_i\}$ be the canonical basis for $\mathbb{R}^n$. For any $v \in \mathbb{R}^n$, we write $v = \sum_{i=1}^n v^i e_i$. Then by the linearity of $Df(p)$ we have:

$$\begin{aligned} Df(p)[v] = Df(p)\left[\sum_{i=1}^n v^i e_i\right] &= \sum_{i=1}^n v^i Df(p)\,[e_i] = \sum_{i=1}^n v^i D_i f(p). \\ &= \begin{pmatrix} \sum_{i=1}^n v^i D_i f^1(p) \\ \vdots \\ \sum_{i=1}^n v^i D_i f^m(p) \end{pmatrix} \\ &= \begin{pmatrix} D_1 f^1(p) & \dots & D_n f^1(p) \\ \vdots & \ddots & \vdots \\ D_1 f^m(p) & \dots & D_n f^m(p) \end{pmatrix} \begin{pmatrix} v^1 \\ \vdots \\ v^n \end{pmatrix} \end{aligned}$$

□

This allows us to restate the chain rule in terms of the partial derivatives of the functions.

Corollary 1.10. *Suppose $\Omega \subset \mathbb{R}^n$ and $\Omega' \subset \mathbb{R}^m$ are open sets, $g : \Omega \to \Omega'$ is differentiable at $p \in \Omega$, and $f : \Omega' \to \mathbb{R}^l$ is differentiable at $g(p)$. Then $h = f \circ g$ is differentiable at p with Jacobian*

$$Dh(p) = \begin{pmatrix} D_1 f^1(g(p)) & \dots & D_m f^1(g(p)) \\ \vdots & \ddots & \vdots \\ D_1 f^l(g(p)) & \dots & D_m f^l(g(p)) \end{pmatrix} \begin{pmatrix} D_1 g^1(p) & \dots & D_n g^1(p) \\ \vdots & \ddots & \vdots \\ D_1 g^m(p) & \dots & D_n g^m(p) \end{pmatrix}$$

In the one dimensional case, we often use the derivative to search for turning points, i.e. maxima and minima, since a differentiable function will have vanishing derivative at a local maximum or minimum. A similar result holds in the higher dimensional case.

Lemma 1.11. *Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$ be differentiable at each point in Ω. Suppose that f has a local maximum at $p \in \Omega$. Then:*

$$Df(p) = 0.$$

Similarly if p is a local minimum.

Proof. Pick $v \in \mathbb{R}^n$. Since Ω is open, there exists $\epsilon > 0$ such that $p + tv \in \Omega$ for $t \in (-\epsilon, \epsilon)$. Consider the function $g_v : (-\epsilon, \epsilon) \to \mathbb{R}$ defined as

$$g_v(t) = f(p + tv).$$

Since f has a local maximum at p, g_v has a local maximum at 0 and moreover, g_v is differentiable by the chain rule, so we deduce

$$0 = g_v'(0) = Df(p)[v].$$

Since v was arbitrary, we have that $Df(p) = 0$. A similar argument deals with the case where p is a minimum. □

Exercise 1.16. (i) Let the function $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x, y) = (x^2 + e^{x+y}, x - \log y, 2xy + 1).$$

Assuming f is differentiable at a point (x, y), what is its derivative?

(ii) Let $g : \mathbb{R}^3 \to \mathbb{R}^1$ be given by

$$g(x, y, z) = x + y + z.$$

Compute the derivative of $g \circ f$ assuming it exists. Compute it in 2 ways, with and without the chain rule.

1.4.2 Relation between partial derivatives and differentiability

We have seen above that for a function $f : \mathbb{R}^n \to \mathbb{R}$ which is differentiable at some point p, the limits

$$D_i f(p) := \lim_{t\to 0} \frac{f(p+te_i) - f(p)}{t} \tag{1.11}$$

exist for $i = 1, \dots n$, and moreover these limits completely determine the derivative of f at p. One might hope, based on this, that in order for f to be differentiable at p it is enough to know that the partial derivatives (i.e. the limits in (1.11)) of f at p all exist. Unfortunately, this is not the case, as we show in the following example.

Example 1.11. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x,y) = \begin{cases} 0 & x = y = 0 \\ \frac{xy}{\sqrt{x^2+y^2}} & \text{otherwise} \end{cases}$$

See Figure 1.4 for the graph of the function f.

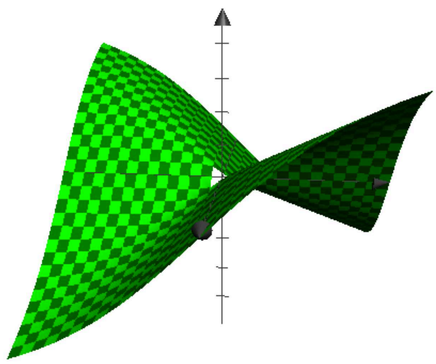

Figure 1.4: The graph of the function in Example 1.11.

First note that this function is continuous at the origin. Since $|xy| \le \frac{1}{2}\left(x^2+y^2\right)$, we have that for $p = (x,y) \neq (0,0)$:

$$|f(p)| \le \frac{1}{2}\sqrt{x^2+y^2},$$

so that

$$\lim_{p\to 0} f(p) = 0.$$

Now consider the partial derivatives. We have

$$D_1 f(0) = \lim_{t\to 0} \frac{1}{t}\left[f(te_1) - f(0)\right] = \lim_{t\to 0} \frac{0-0}{t} = 0$$

since $f(te_1) = 0$ for all t. Similarly, we also have

$$D_2 f(0) = \lim_{t\to 0} \frac{1}{t}\left[f(te_2) - f(0)\right] = \lim_{t\to 0} \frac{0-0}{t} = 0$$

Thus, if f is differentiable, then it must be that $Df = 0$, so all directional derivatives at 0 exist and are equal to zero. However, let $h = \frac{1}{\sqrt{2}}(1,1)$. For $t > 0$, we have

$$\frac{f(th) - f(0)}{t} = \frac{t^2/2}{t^2} = \frac{1}{2},$$

which contradicts the differentiability of f at the origin. Thus, even though the partial derivatives exist for this function, the function is not differentiable.

Away from the origin, the function is a composition of smooth functions so is differentiable. We can calculate the partial derivatives at a point $p = (x, y) \neq (0,0)$ and we find

$$D_1 f(p) = \frac{y}{\sqrt{x^2+y^2}} - \frac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}},$$

and by symmetry:

$$D_2 f(p) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}.$$

We claim that the function $g : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ given by

$$g(x,y) = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$$

has no limit as $p = (x, y)$ converges to $(0,0)$. To see this, let $p = (r\cos\theta, r\sin\theta)$ for some $r \in (0,\infty)$, $\theta \in [0, 2\pi)$. Then

$$g(p) = \cos^3\theta,$$

so there can be no limit as $r \to 0$, since g approaches a different value depending on which angle we approach from.

As it happens, the fact that the partial derivatives are not continuous in a neighbourhood of the origin is the only barrier to differentiability there.

Theorem 1.12. *Let $\Omega \subset \mathbb{R}^n$ be open and $f : \Omega \to \mathbb{R}$. Suppose the partial derivatives*

$$D_i f(x) := \lim_{t\to 0} \frac{f(x+te_i) - f(x)}{t}$$

exist for all $x \in \Omega$, and moreover suppose that the maps

$$x \mapsto D_i f(x)$$

are continuous at $p \in \Omega$ for all $i = 1, \dots, n$. Then f is differentiable at p.

(*) *Proof.* Since Ω is open, there exists $r > 0$ such that $B_r(p) \subset \Omega$. Suppose $h \in B_r(0)$ has components h^i, so that $h = \sum_{i=1}^n h^i e_i$. We consider

$$\begin{aligned} f(p+h) - f(p) &= f\left(p + \sum_{i=1}^n h^i e_i\right) - f(p) \\ &= f\left(p + \sum_{i=1}^n h^i e_i\right) - f\left(p + \sum_{i=1}^{n-1} h^i e_i\right) \\ &\quad + f\left(p + \sum_{i=1}^{n-1} h^i e_i\right) - f\left(p + \sum_{i=1}^{n-2} h^i e_i\right) \\ &\quad + \dots \\ &\quad + f(p + h^1 e_1) - f(p). \end{aligned}$$

Let's consider a typical line in the right hand side of the above equation, that is,

$$f\left(p + \sum_{i=1}^k h^i e_i\right) - f\left(p + \sum_{i=1}^{k-1} h^i e_i\right) = f(q + h^k e_k) - f(q),$$

where $k \in \{1, \dots, n\}$ and $q = p + \sum_{i=1}^{k-1} h^i e_i$. Now, applying the mean value theorem to the function $g(t) = f(q + te_k)$, which is differentiable by assumption, there exists $s \in \left[-\left|h^k\right|, \left|h^k\right|\right)$ such that:

$$f(q + h^k e_k) - f(q) = h^k D_k f(q + se_k) = h^k D_k f(p + c_k),$$

where $c_k = \sum_{i=1}^{k-1} h^i e_i + se_k$. One has to consider separately the cases $h^k > 0$, $h^k < 0$ and $h^k = 0$. Now, note that since $|s| \leq \left|h^k\right|$, we have

$$\|c_k\| \leq \|h\|.$$

Putting this together, we conclude that there exists $c_1, \dots, c_n \in \mathbb{R}^n$ with $\|c_k\| \leq \|h\|$ such that

$$f(p+h) - f(p) = \sum_{k=1}^n h^k D_k f(p + c_k).$$

From here we can estimate using the Cauchy-Schwartz identity

$$\begin{aligned} \left| f(p+h) - f(p) - \sum_{k=1}^n h^k D_k f(p) \right| &\leq \sum_{k=1}^n h^k \left| D_k f(p + c_k) - D_k f(p) \right| \\ &\leq \|h\| \left(\sum_{k=1}^n \left| D_k f(p + c_k) - D_k f(p) \right|^2 \right)^{\frac{1}{2}}, \end{aligned}$$

so that

$$\frac{\left| f(p+h) - f(p) - \sum_{k=1}^n h^k D_k f(p) \right|}{\|h\|} \leq \left(\sum_{k=1}^n \left| D_k f(p + c_k) - D_k f(p) \right|^2 \right)^{\frac{1}{2}}.$$

Now, fix $\epsilon > 0$. Since $x \mapsto D_k f(x)$ is continuous at p, for each $k = 1, \dots, n$, there exists δ_k such that if $\|c\| < \delta_k$ we have:

$$|D_k f(p+c) - D_k f(p)| < \frac{\epsilon}{\sqrt{n}}.$$

Suppose $\|h\| < \min\{\delta_1, \dots, \delta_n\} =: \delta$. Then as $\|c_k\| \le \|h\|$, we deduce

$$\frac{\left|f(p+h) - f(p) - \sum_{k=1}^{n} h^k D_k f(p)\right|}{\|h\|} < \left(\sum_{k=1}^{n} \frac{\epsilon^2}{n}\right)^{\frac{1}{2}} = \epsilon.$$

As ϵ was arbitrary, we conclude that f is differentiable at p, with derivative

$$Df(p)\,[h] = \sum_{k=1}^{n} D_k f(p) h^k.$$

□

Exercise 1.17. Show that each of the following maps $f : \mathbb{R}^2 \to \mathbb{R}$ is everywhere differentiable

(a) $f(x,y) = x^2 + y^2 - x - xy$,

(b) $f(x,y) = \frac{1}{\sqrt{1+x^2+y^2}}$,

(c) $f(x,y) = x^5 y^2$.

For maps $f : (a,b) \to \mathbb{R}$ we have learned that when f is differentiable at some $p \in (a,b)$, then there is a tangent line to the graph of f that passes through $(p, f(p))$ and approximates the graph of f near p. This is an intuitive picture that is only valid when we consider the graph of a function from one dimension to one dimension. By an example below, we show that this intuition should not be employed for maps of higher dimensions.

Example 1.12. Let $f : (-1, +1) \to \mathbb{R}^2$ be define as

$$f(x) = \begin{cases} (x^2, 0) & \text{if } x \ge 0 \\ (0, x^2) & \text{if } x < 0. \end{cases}$$

See Figure 1.12 for the image of the map f.

Clearly, f is continuous at 0 with

$$\lim_{x \to 0} f(x) = (0,0) = f(0).$$

The map f is differentiable at 0 with derivative equal to the constant linear map $\Lambda = 0$. To see this, note that

$$\lim_{h \to 0} \frac{\|f(0+h) - f(0) - \Lambda[h]\|}{\|h\|} = \lim_{h \to 0} \frac{\|f(h)\|}{\|h\|} = \lim_{h \to 0} \frac{h^2}{|h|} = \lim_{h \to 0} |h| = 0.$$

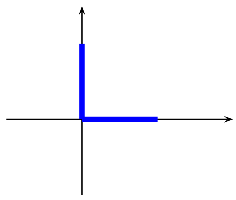

Figure 1.5: The image of the map f in Example 1.12

In fact, it is not possible to understand just by looking at the image of a map whether it is differentiable or not. As the example below shows, maps with the same image may or may not be differentiable.

Example 1.13. Define the maps k and g from $(-1,+1)$ to $\mathbb{R}^2$ as

$$k(x) = (x, x^3), \quad g(x) = (x^{1/3}, x).$$

See Figure 1.6 for the images of the maps f and g.

The maps k and g are continuous at 0 with

$$\lim_{x\to 0} k(x) = (0,0) = k(0),$$

and

$$\lim_{x\to 0} g(x) = (0,0) = g(0).$$

The maps k and g have the same image, that is, they map the interval $(-1,+1)$ to the same curve, which is the graph of the function $t \mapsto t^3$ on the interval $(-1,+1)$. However, k is differentiable at 0, but g is not differentiable at 0, as we show below.

We claim that the derivative of the map k at 0 is equal to the linear map $\Lambda(h) = (h, 0)$. To see this, note that

$$\begin{aligned}\lim_{h\to 0} \frac{\|k(0+h) - k(0) - \Lambda[h]\|}{\|h\|} &= \lim_{h\to 0} \frac{\big\|(h, h^3) - (h, 0)\big\|}{\|h\|} \\ &= \lim_{h\to 0} \frac{\big\|(0, h^3)\big\|}{\|h\|} \\ &= \lim_{h\to 0} \frac{|h|^3}{|h|} = 0.\end{aligned}$$

To prove that g is not differentiable at 0, we need to show that there is no linear map $\Lambda : \mathbb{R} \to \mathbb{R}^2$ which is the derivative of the map g at 0. In contrary assume that there is a linear map $\Lambda : \mathbb{R} \to \mathbb{R}^2$ such that

$$\lim_{h\to 0} \frac{\|g(0+h) - g(0) - \Lambda[h]\|}{\|h\|} = 0.$$

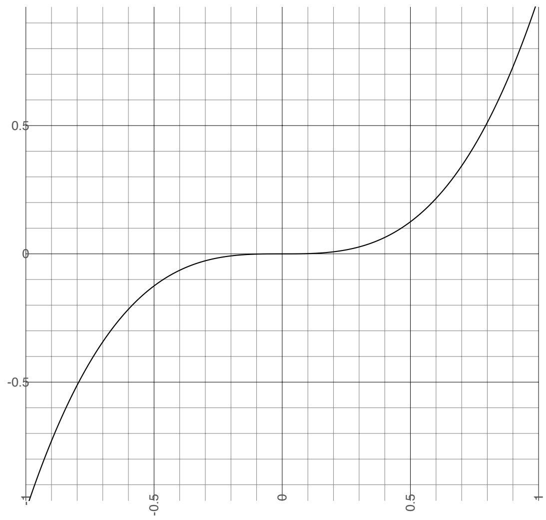

Figure 1.6: The image of the maps f and g in Example 1.13. The differentiability at $(0,0)$ depends on "how fast" we pass through the point $(0,0)$.

Let $\Lambda(1) = (a,b) \in \mathbb{R}^2$, for some real constants a and b in $\mathbb{R}$. It follows that for every $h \in \mathbb{R}$ we have

$$\Lambda(h) = \Lambda(h \cdot 1) = h\Lambda(1) = h(a,b) = (ha, hb).$$

Therefore,

$$\begin{aligned} 0 = \lim_{h\to 0} \frac{\|g(0+h) - g(0) - \Lambda[h]\|}{\|h\|} &= \lim_{h\to 0} \frac{\left\|(h^{1/3} - ah, h - bh)\right\|}{|h|} \\ &= \lim_{h\to 0} \frac{\left\|h(h^{-2/3} - a, 1 - b)\right\|}{|h|} \\ &= \lim_{h\to 0} \left\|(h^{-2/3} - a, 1 - b)\right\| \\ &= \left\|\lim_{h\to 0}(h^{-2/3} - a, 1 - b)\right\| \end{aligned}$$

In the last line of the above equation we have used that $\|\cdot\|$ is a continuous function, so we may interchange the limit and the norm. Now recall that $\|y\| = 0$, if and only if $y = 0$. Thus we must have

$$\lim_{h\to 0}(h^{-2/3} - a, 1 - b) = (0,0)$$

which implies that

$$\lim_{h\to 0} h^{-2/3} - a = 0, \qquad \text{and } \lim_{h\to 0} 1 - b = 0.$$

This is a contradiction, since for any real number a we have

$$\lim_{h\to 0} h^{-2/3} - a = \infty.$$

This contradiction shows that there is no linear map $\Lambda : \mathbb{R} \to \mathbb{R}^2$ satisfying the definition of differentiability for g at 0.

Note that the value of the other limit does not lead to any contradiction, it only says that b must be equal to 1.

1.5 Higher derivatives

1.5.1 Higher derivatives as linear maps

Suppose that $\Omega \subset \mathbb{R}^n$ is open, and $f : \Omega \to \mathbb{R}^m$ is differentiable at every point $p \in \Omega$. We may think of the differential of f as a map

$$\begin{array}{rcccl} Df & : & \Omega & \to & L(\mathbb{R}^n; \mathbb{R}^m) \\ & & p & \mapsto & Df(p). \end{array}$$

Recall that every member of $L(\mathbb{R}^n; \mathbb{R}^m)$ may be expressed as an m by n matrix, using the standard basis for $\mathbb{R}^n$ and $\mathbb{R}^m$. We can think of each m by n matrix as a point in $\mathbb{R}^{mn}$, for example, by

$$(a_{i,j})_{1\le i\le m, 1\le j\le n} \mapsto (a_{1,1}, \dots, a_{1,n}, a_{2,1}, \dots, a_{2,n}, \dots, a_{m,1}, \dots, a_{m,n}).$$

Thus, we may think of Df as a map from Ω to $\mathbb{R}^{mn}$. We can consider whether this map Df is continuous, or differentiable at a point $p \in \Omega$. If the map $Df : \Omega \to \mathbb{R}^{mn}$ is continuous, we say $f : \Omega \to \mathbb{R}^m$ is **continuously differentiable**. If $Df : \Omega \to \mathbb{R}^{mn}$ is differentiable at p, the derivative at p, denoted by $DDf(p)$, is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^{mn}$. That is,

$$DDf(p) \in L(\mathbb{R}^n; \mathbb{R}^{nm}) = L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m)).$$

Thus, $DDf(p)$ takes an n-vector to an $(m \times n)$ matrix. The above notation may appear complicated, but you have already seen some examples of maps in the right hand of the above equation. For example, the map $h \mapsto \langle h, \cdot \rangle$ is an element of $L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^1))$, that is, for every $h \in \mathbb{R}^n$, the map $u \mapsto \langle h, u \rangle$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^1$.

In terms of our definition of derivative, $DDf(p)$ is a linear map $\mathcal{L} \in L(\mathbb{R}^n; L(\mathbb{R}^n; \mathbb{R}^m))$ such that the following holds

$$\lim_{x\to p} \frac{\|Df(x) - Df(p) - \mathcal{L}[x-p]\|}{\|x-p\|} = 0.$$

Note that in the above equation, the norm on the numerator is the norm $\|\cdot\|$ on $\mathbb{R}^{mn}$ and the norm on the denominator is $\|\cdot\|$ on $\mathbb{R}^n$.

Obviously, we can generalise this to consider a map which is $k-$times differentiable. In practice, the condition of $k-$times differentiable at a point can be difficult to establish. However, if $f : \Omega \to \mathbb{R}^m$ is k times differentiable with all those derivatives continuous, we say f is **k-times continuously differentiable**.

Assume that $f = (f^1, f^2, \dots, f^m)$. We know from the previous results that if f is differentiable at $p \in \Omega$, the partial derivative maps

$$\begin{array}{rcccl} D_i f^j & : & \Omega & \to & \mathbb{R} \\ & & x & \mapsto & D_i f^j(x). \end{array}$$

exist for all $x \in \Omega$. If moreover, Df is differentiable at $p \in \Omega$, then the second partial derivatives

$$D_k D_i f^j(p) := \lim_{t\to 0} \frac{D_i f^j(p + te_k) - D_i f^j(p)}{t}$$

will exist.

It is easier to ask if all of the $k-$th partial derivatives exist and are continuous in a neighbourhood of p. This is a slightly stronger condition, which implies $k-$times differentiability at p by Theorem 1.12.

Example 1.14. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f : (x, y) = x^3 + y^3 + 5x^2 y.$$

This is differentiable at each point $p = (x, y) \in \mathbb{R}^2$, and the partial derivatives are

$$D_1 f(p) = 3x^2 + 10xy, \qquad D_2 f(p) = 3y^2 + 5x^2.$$

To find the second partial derivatives, we consider the maps

$$D_1 f(x, y) = 3x^2 + 10xy,$$

and

$$D_2 f(x, y) = 3y^2 + 5x^2,$$

and differentiate them. The second partial derivatives are thus

$$\begin{aligned} D_1 D_1 f(p) &= 6x + 10y \\ D_2 D_1 f(p) &= 10x \\ D_1 D_2 f(p) &= 10x \\ D_2 D_2 f(p) &= 6y \end{aligned}$$

Notice that

$$D_2 D_1 f(p) = D_1 D_2 f(p).$$

This is a coincidence!

1.5.2 Symmetry of mixed partial derivatives

We will state a result here, but do not give a proof. This is not the optimal result in this direction, but it is perfectly adequate for most purposes in the next section.

Theorem 1.13 (Schwartz' Theorem)**.** *Suppose $\Omega \subset \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}$ is differentiable at every $p \in \Omega$. Suppose further that for some $i, j \in \{1, \ldots, n\}$ the second partial derivatives $D_i D_j f$ and $D_j D_i f$ exist and are continuous at all $p \in \Omega$. Then, at every $p \in \Omega$,*

$$D_i D_j f(p) = D_j D_i f(p).$$

If $f : \Omega \to \mathbb{R}$, the matrix of second partial derivatives at the point p,

$$\operatorname{Hess} f(p) = [D_i D_j f(p)]_{i,j=1,\ldots,n}$$

is called the **Hessian** of f at p. Assuming the hypotheses on the second partial derivatives hold, Schwartz' Theorem states that the Hessian is a symmetric matrix.

Exercise 1.18. Suppose A is a symmetric $(n \times n)$ matrix. Consider the map $f : \mathbb{R}^n \to \mathbb{R}$ defined as

$$f(x) = xAx^t.$$

(a) Show that f is differentiable at all points $p \in \mathbb{R}^n$, with

$$Df(p) = 2pA$$

(b) Find

$$\operatorname{Hess} f(p).$$

Exercise 1.19. Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$ given by:

$$f : (x, y, z) = xy^2 + x^2 + xze^y.$$

(i) Compute the first and second partial derivatives. Observe the properties of the second partial derivative.

(ii) Write the terms of the Taylor expansion of f at zero up to and including the second-order terms.

(iii) Without computation, write the same Taylor expansion up to and including the fourth-order terms.

Exercise 1.20 (*)**.** Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as

$$f(x, y) = \begin{cases} \frac{xy^3 - x^3 y}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

(a) Show that

$$D_1 f(x,y) = \begin{cases} \frac{y^3 - 3x^2 y}{x^2+y^2} - \frac{2x\left(xy^3 - x^3 y\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

and

$$D_2 f(x,y) = \begin{cases} \frac{3y^2 x - x^3}{x^2+y^2} - \frac{2y\left(xy^3 - x^3 y\right)}{\left(x^2+y^2\right)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Show that both of these functions are continuous at $(0,0)$.

(b) Show that

$$\lim_{t\to 0} \frac{1}{t}\left(D_1 f(te_2) - D_1 f(0)\right) = 1$$

and

$$\lim_{t\to 0} \frac{1}{t}\left(D_2 f(te_1) - D_2 f(0)\right) = -1$$

(c) Conclude that both $D_2 D_1 f(0)$ and $D_1 D_2 f(0)$ exist, but

$$D_2 D_1 f(0) \neq D_1 D_2 f(0)$$

1.5.3 Taylor's theorem

The differentiability of a map of higher dimensions allows us to approximate the map near a point with a linear map which is a simpler object. This has significant consequences which we discus in Section 1.6. However, when thinking of differentiabilities of higher orders, one may wonder if those lead to better approximations than the ones by a linear map, perhaps, by more complicated objects than linear maps. In terms of complexity, the next class of maps after linear ones are polynomial maps in several variables. We look into such approximations in this section.

A powerful result concerning differentiable functions of one variable is Taylor's theorem, which permits us to approximate a function in a neighbourhood of a point p by a polynomial, with an error term that goes to zero at a controlled rate as we approach p. In order to state Taylor's theorem for higher dimensions, it's useful to introduce some new notation.

When dealing with partial derivatives of high orders, the notation can get rather messy. To mitigate this, it's convenient to introduce "multi-indices". We define a multi-index α to be an element of $(\mathbb{N})^n$, i.e. an n-vector of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_n)$. We define $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and

$$D^\alpha f := (D_1)^{\alpha_1} (D_2)^{\alpha_2} \cdots (D_n)^{\alpha_n} f,$$

It's convenient to also introduce, for a vector $h = (h^1, \ldots, h^n)$,

$$h^\alpha := \left(h^1\right)^{\alpha_1} \left(h^2\right)^{\alpha_2} \cdots \left(h^n\right)^{\alpha_n}$$

as well as the multi-index factorial $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$,

Theorem 1.14. *Suppose that $p \in \mathbb{R}^n$ and $f : B_r(p) \to \mathbb{R}$ is $k-$times continuously differentiable at all points $q \in B_r(p)$, for some integer $k \geq 1$. Then, for every $h \in \mathbb{R}^n$ with $\|h\| < r$, we have*

$$f(p+h) \quad = \quad \sum_{|\alpha| \leq k-1} \frac{h^\alpha}{\alpha!} D^\alpha f(p) + R_k(p,h).$$

where the sum is taken over all multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \leq k-1$ and the remainder term is given by:

$$R_k(p,h) = \sum_{|\alpha|=k} \frac{h^\alpha}{\alpha!} D^\alpha f(x)$$

for some x with $0 < \|x-p\| < \|h\|$.

(*) *Proof.* The result follows from the one-dimensional Taylor's theorem. First, we note that there exists $\epsilon > 0$ such that $\|h\| < \frac{r}{1+\epsilon}$. Let us define the function $g : (-1-\epsilon, 1+\epsilon) \to \mathbb{R}$ defined as

$$g(t) = f(p+th).$$

By the chain rule, this function is k-times differentiable on the interval $(-1-\epsilon, 1+\epsilon)$, and $[0,1] \subset (-1-\epsilon, 1+\epsilon)$, so by one dimensional Taylor's theorem we have

$$g(1) = g(0) + g'(0) + \frac{g''(0)}{2!} + \ldots + \frac{g^{(k-1)}(0)}{(k-1)!} + R_k,$$

where

$$R_k = \frac{g^{(k)}(\xi)}{k!},$$

for some $\xi \in (0,1)$. We will be done if we can show that for $j = 0, 1, \ldots k$ we have

$$g^{(j)}(t) = j! \sum_{|\alpha|=j} \frac{h^\alpha}{\alpha!} D^\alpha f(p+th). \tag{1.12}$$

This is certainly true for $j = 0$. Suppose it's true for some $j \geq 0$. Then we have

$$\begin{aligned} g^{(j+1)}(t) &= \sum_{l=1}^n h^l D_l \left[j! \sum_{|\alpha|=j} \frac{h^\alpha}{\alpha!} D^\alpha f \right] (p+th) \\ &= j! \sum_{l=1}^n \sum_{|\alpha|=j} \frac{h^\alpha h^l}{\alpha!} D_l D^\alpha f(p+th) \end{aligned}$$

Clearly, the right-hand side of the above equation is a sum of terms proportional to $h^\beta D^\beta f(p+th)$ where $|\beta| = j+1$. Suppose $\beta = (\beta_1, \ldots, \beta_n)$, then the coefficient of the term proportional to $h^\beta D^\beta f(p+th)$ is

$$\begin{aligned} &\frac{j!}{(\beta_1-1)!\beta_2! \cdots \beta_n!} + \frac{j!}{\beta_1!(\beta_2-1)! \cdots \beta_n!} + \ldots + \frac{j!}{\beta_1!\beta_2! \cdots (\beta_n-1)!} \\ &\qquad = \frac{(j+1)!}{\beta_1!\beta_2! \cdots \beta_n!} = \frac{(j+1)!}{\beta!}, \end{aligned}$$

by a result from combinatorics (you do not need to verify this). Thus we have

$$\begin{aligned} g^{(j+1)}(t) &= j! \sum_{l=1}^{n} \sum_{|\alpha|=j} \frac{h^\alpha h^l}{\alpha!} D_l D^\alpha f(p+th) \\ &= (j+1)! \sum_{|\beta|=j+1} \frac{h^\beta}{\beta!} D^\beta f(p+th) \end{aligned}$$

By induction we conclude that (1.12) holds for all $j = 0, \dots, k$ and the result follows. $\square$

Exercise 1.21. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined as $f(x, y) = e^x \sin(y)$.

a) Compute the degree 1 and degree 2 Taylor polynomial of f near the point $(x_0, y_0) = (0, \pi/2)$ and use those to approximate the value of f at $(x_1, y_1) = (0, \pi/2 + 1/4)$. Compare your results with the values you obtain from a calculator.

b) How precise is the degree 1 approximation in the closed ball of radius $1/4$ around (x_0, y_0). Find a rigorous upper bound for the approximation error.

1.6 Inverse and Implicit function theorems

1.6.1 Inverse function theorem

Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in an interval around $p \in \mathbb{R}$, with $f'(p) \neq 0$, say $f'(p) > 0$. Then there is an open interval I with $p \in I$ such that $f'(x) > 0$ for all $x \in I$. This (by the mean value theorem) implies that f is strictly monotone increasing on I and hence $f : I \to f(I)$ is bijective. In particular, there exists an inverse function $f^{-1} : f(I) \to I$. With a little work, one can establish that f^{-1} is differentiable, and moreover, by an application of the chain rule, obtain the following formula for the derivative of the inverse map,

$$f'(p) = \frac{1}{(f^{-1})'(f(p))}.$$

This result can be generalised to higher dimensions.

Theorem 1.15 (Inverse Function Theorem)**.** *Let Ω be an open set in $\mathbb{R}^n$, $f : \Omega \to \mathbb{R}^n$ continuously differentiable on Ω, and there is $q \in \Omega$ such that $Df(q)$ invertible.*

Then, there are open sets $U \subset \Omega$ and $V \subset \mathbb{R}^n$ with $q \in U$ and $f(q) \in V$ such that

(i) $f : U \to V$ is a bijection,

(ii) $f^{-1} : V \to U$ is continuously differentiable,

(iii) for all $y \in V$,
$$Df^{-1}(y) = \left[Df(f^{-1}(y))\right]^{-1}.$$

Recall that since the Jacobian $Df(q)$ is an $n \times n$ matrix, the statement that it is invertible is equivalent to the statement that $\det Df(q) \neq 0$.

Example 1.15. Consider the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$f(x, y) = (x + y + 5xy, y - x^2)$$

The partial derivatives of f are

$$D_1 f(x, y) = (1 + 5y, -2x), \qquad D_2 f(x, y) = (1 + 5x, 1).$$

Evidently, both of these maps are continuous from $\mathbb{R}^2$ to $\mathbb{R}^2$. Thus, by Theorem 1.12, f is differentiable at every point in $\mathbb{R}^2$. Moreover, by Theorem 1.9, the Jacobian of f at $(x, y) \in \mathbb{R}^2$ is given by the matrix

$$Df(x, y) = \begin{pmatrix} 1 + 5y & 1 + 5x \\ -2x & 1 \end{pmatrix}.$$

This is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^{2\times 2} = \mathbb{R}^4$.

We note that

$$Df(0,0) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

with $\det Df(0,0) = 1 \neq 0$, and hence $Df(0,0)$ is invertible. By the Inverse Function Theorem, f is invertible on some neighbourhood of the origin, with

$$Df^{-1}(0,0) = [Df(0,0)]^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

It is worth noting that obtaining an explicit formula for the inverse map is not easy, and hence the derivative of the inverse map is out of reach using the direct approach.

Exercise 1.22. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y - xy \\ x^2 \end{pmatrix}$$

Determine the set of points in $\mathbb{R}^2$ such that f is invertible near those points, and compute the derivative of the inverse map.

The Inverse Function Theorem has applications to solving systems of equations. Assume that we have n equations in n unknowns $x^1, x^2, \dots x^n$, given in the form

$$\begin{aligned} f^1(x^1, x^2, \dots, x^n) &= y^1, \\ f^2(x^1, x^2, \dots, x^n) &= y^2, \\ &\vdots \\ f^n(x^1, x^2, \dots, x^n) &= y^n. \end{aligned}$$

where $y^1, y^2, \dots y^n$ are given real numbers, and $f^1, f^2, \dots, f^n$ are some functions of $x^1, x^2, \dots, x^n$.

For arbitrary values of $x_0^1, x_0^2, \dots x_0^n$, we obtain real numbers $y_0^1, y_0^2, \dots y_0^n$ satisfying the above equation. That is, we define the values of $y_0^1, y_0^2, \dots y_0^n$ using the above functions. The Inverse Function Theorem can be used here and guarantees that if the map $F : \mathbb{R}^n \to \mathbb{R}^n$ is defined as

$$F(x^1, x^2, \dots, x^n) = \big(f^1(x^1, x^2, \dots, x^n), f^2(x^1, x^2, \dots, x^n), \dots, f^n(x^1, x^2, \dots, x^n)\big)$$

is continuously differentiable, and DF at $(x_0^1, x_0^2, \dots, x_0^n)$ is invertible, then for all values of $y^1, y^2, \dots y^n$ sufficiently close enough to $y_0^1, y_0^2, \dots y_0^n$ the above system of equations has unique solutions. Indeed the solution is given by the inverse of the map F.

For example, by the previous example, we conclude that for a and b close to 0, the equations

$$\begin{aligned} x + y + 5xy &= 0 \\ y - x^2 &= 0 \end{aligned}$$

has unique solutions for x and y.

This is a fairly powerful statement, but the issue here is that the theorem does not immediately say how close one must have a and b to 0 in order for the solutions exist. It only says that for close enough a and b, there are solutions. However, since there is a constructive proof of the theorem, one can follow the steps in the proof, and obtain an explicit neighbourhood of $(0, 0)$ such that for all (a, b) in that neighbourhood, the solutions exit.

Let Ω and Ω' be open subsets of $\mathbb{R}^n$. We say that a map $f : \Omega \to \Omega'$ is a **C^1-diffeomorphism**, if $f : \Omega \to \Omega'$ is a bijection (i.e. injective and surjective), $f : \Omega \to \Omega'$ is continuously differentiable, and for every $x \in \Omega$, $Df(x)$ is invertible.

Example 1.16. Let Ω be an open sets in $\mathbb{R}^n$, and define $\mathcal{D}$ as the set of all C^1-diffeomorphisms from Ω to Ω. Then $\mathcal{D}$ is a group, with the operation

$$f * g = f \circ g.$$

To see this, first we show that for every f and g in $\mathcal{D}$, $f * g$ belongs to $\mathcal{D}$. So we need to show that $f \circ g$ is a C^1-diffeomorphism from Ω to Ω. We need to verify three properties for $f \circ g$.

• Since f and g belong to $\mathcal{D}$, $f : \Omega \to \Omega$ and $g : \Omega \to \Omega$ are bijections. Hence, $f \circ g : \Omega \to \Omega$ is a bijection.

• Since f and g belong to $\mathcal{D}$, they are continuously differentiable at every point in Ω. Thus, by the chain rule, the map $f \circ g : \Omega \to \Omega$ is differentiable at every point in Ω, with

$$D(f \circ g)(x) = D(f(g(x)) \circ Dg(x).$$

Thus $f \circ g$ is differentiable on Ω. Also, since the maps $y \mapsto Df(y)$ and $x \mapsto Dg(x)$ are continuous on Ω, and the composition of continuous maps is a continuous map, the above formula shows that $D(f \circ g)$ is continuous on Ω. Thus, $f \circ g$ is continuously differentiable on Ω.

• Since f and g belong to $\mathcal{D}$, both $Df(y)$ and $Dg(x)$ are invertible at all x and y in Ω. The composition of invertible matrixes is invertible. Thus, the above formula shows that $D(f * g)$ must be invertible at every point.

The associativity of the operation $*$ is obtained from the associativity of the composition operation for functions. That is, for all f, g and h in $\mathcal{D}$, we have

$$(f * g) * h = (f \circ g) \circ h = f \circ (g \circ h) = f * (g * h).$$

The identity map $\mathrm{id} : \Omega \to \Omega$ is a C^1-diffeomorphism and hence belongs to $\mathcal{D}$. It is the identity element in $\mathcal{D}$, since for every $f \in \mathcal{D}$, we have

$$f * \mathrm{id} = f \circ \mathrm{id} = f, \qquad \mathrm{id} * f = \mathrm{id} \circ f = f.$$

Finally, for every $f \in \mathcal{D}$ we need to show that f^{-1} belongs to $\mathcal{D}$. First we note that $f^{-1} : \Omega \to \Omega$ is a bijection. Since, f is continuously differentiable on Ω and $Df(x)$ is invertible, by the Inverse Function Theorem, f^{-1} is invertible on some neighbourhood of $f(x)$, and $D(f^{-1})(f(x)) = [Df(x)]^{-1}$ is invertible. This is true on a neighbourhood of $f(x)$ for every $x \in \Omega$. So, since f is surjective, this is true on a neighbourhood of every point in $f(\Omega) = \Omega$.

When $\Omega = B_1(0)$ is the open ball of radius 1 about the origin, every rotation about 0 is an element of $\mathcal{D}$. However, there are many other maps in $\mathcal{D}$. It forms a very large group, as seen, for example when $\Omega = (-1, 1)$ is the open interval in $\mathbb{R}$.

Exercise 1.23. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable in a neighbourhood of the origin, and $f'(0) = 0$. Give an example to show that f may nevertheless be bijective.

(b) Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is bijective, differentiable at the origin, and $\det Df(0) = 0$. Show that f^{-1} is not differentiable at $f(0)$.

Exercise 1.24. The non-linear system of equations

$$\begin{aligned} e^{xy} \sin(x^2 - y^2 + x) &= 0 \\ e^{x^2+y} \cos(x^2 + y^2) &= 1 \end{aligned}$$

admits the solution $(x, y) = (0, 0)$. Prove that there exists $\varepsilon > 0$ such that for all (ξ, η) with $\xi^2 + \eta^2 < \varepsilon^2$, the perturbed system of equations

$$\begin{aligned} e^{xy} \sin(x^2 - y^2 + x) &= \xi \\ e^{x^2+y} \cos(x^2 + y^2) &= 1 + \eta \end{aligned}$$

has a solution $(x(\xi, \eta), y(\xi, \eta))$ which depends continuously on (ξ, η).

1.6.2 Implicit Function Theorem

In the previous section, we saw that the Inverse Function Theorem has applications to systems of n equations with n unknowns. What if there are more unknowns than equations. That is for some $n > m$, we have

$$\begin{aligned} f^1(x^1, x^2, \dots, x^n) &= y^1, \\ f^2(x^1, x^2, \dots, x^n) &= y^2, \\ &\vdots \\ f^m(x^1, x^2, \dots, x^n) &= y^m. \end{aligned}$$

We look into this through a simple example. Consider the equation

$$x^2 + y^2 - 1 = 0.$$

We can consider the map $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined as

$$F(x, y) = x^2 + y^2 - 1$$

and think of the above equation as

$$F(x, y) = 0.$$

Suppose (a, b) satisfies $F(a, b) = 0$, and $a \neq 1, -1$. Then there is an open interval A containing a and an open interval B containing b with the property that for each $x \in A$ there is a unique $y \in B$ such that $F(x, y) = 0$. This permits us to define a map $g : A \to B$ by $g(x) = y$, so that $F(x, g(x)) = 0$. We can think of this as 'locally solving for y in terms of x'. If $b > 0$ then $g(x) = \sqrt{1 - x^2}$. For the problem at hand, there is in fact another number b_1 such that $F(a, b_1) = 0$. Associated to this point there is an open interval B_1 containing b_1 and a map $g_1 : A \to B_1$ such that $F(x, g_1(x)) = 0$. (If $b > 0$, then $b_1 < 0$ and $g_1(x) = -\sqrt{1 - x^2}$). Both g, g_1 are differentiable. See Figure 1.7.

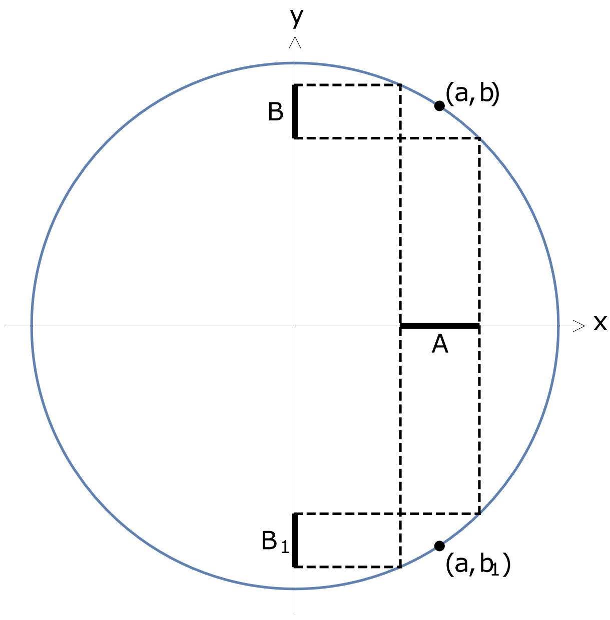

Figure 1.7: The set $x^2 + y^2 - 1 = 0$, and the intervals A, B, B_1.

In contrast when $a = \pm 1$ we must have $b = 0$ in order to have $a^2 + b^2 = 1$. Assume that $a = +1$. There are no open sets $A \subset \mathbb{R}$ containing a and $B \subset \mathbb{R}$ containing b satisfying the following property for every $x \in A$ there is a unique $y \in B$ satisfying $x^2 + y^2 = 1$.

This is because, since B is open, there is $\delta > 0$ such that $(-\delta, \delta) \subset B$. Now, for every $x \in A$ close enough to $a = 1$, there are two points $\pm\sqrt{1 - x^2}$ that belong to B. Of course one might wish to rectify this problem with choosing A as an interval of the $(1-c, 1]$, and B an interval of the from $[0, \sqrt{1-c^2})$ so that for every $x \in A$ there is a unique $y \in B$ satisfying $x^2 + y^2 = 1$. But, when we go to higher dimensions, it is not clear what is the correct analogue of the intervals of the form $[z, w)$ or $(z, w]$.

The main question here is to identify the conditions on F which allows us to write the solutions of the equation $F(x, y) = 0$ as graphs of maps. The Implicit Function Theorem gives us a sufficient condition for this property to be true in a more general setting. We first state a relatively easier version of the theorem.

Theorem 1.16 (Implicit Function Theorem–simple version)**.** *Assume that $\Omega \subset \mathbb{R}^2$ is open, $F : \Omega \to \mathbb{R}$ is continuously differentiable, and there is $(x', y') \in \Omega$ such that*

(i) $F(x', y') = 0$, *and*

(ii) $D_2F(x', y') \neq 0$.

Then, there are open sets $A \subset \mathbb{R}$ and $B \subset \mathbb{R}$ with $x' \in A$ and $y' \in B$, and a map $f : A \to B$ such that

$$(x, y) \in A \times B \text{ satisfies } F(x, y) = 0 \quad \text{iff} \quad y = f(x) \text{ for some } x \in A.$$

Moreover, the map $f : A \to B$ is continuously differentiable.

Roughly speaking, the above theorem states that for each solution x_0, y_0 of the equation

$$F(x, y) = 0,$$

the nearby solutions x, y of the above equation, look like the graph of a map from x unknown to the y unknown.

Exercise 1.25. For each of the following equations determine at which points one cannot find a function $y = f(x)$ which describes the graph in this neighbourhood. Sketch the graphs.

(a)

$$\frac{1}{3}y^3 - 2y + x = 1$$

(b)

$$x^2\left(\frac{\cos^2\phi}{a^2} + \frac{\sin^2\phi}{b^2}\right) - xy\left(\frac{1}{a^2} - \frac{1}{b^2}\right)\sin(2\phi) + y^2\left(\frac{\sin^2\phi}{a^2} + \frac{\cos^2\phi}{b^2}\right) = 1,$$

where $a > 0$, $b > 0$, $0 \leq \phi \leq \pi/2$ are fixed parameters. Note the cases $a = b$, $\phi = 0$, $\phi = \pi/2$.

Exercise 1.26. Consider the equation

$$2x^2 + 4xy + y^2 = 3x + 4y$$

(a) Show that this system of equations (implicitly) defines a function $y = f(x)$ with $f(1) = 1$.

(b) Compute $f'(1)$ without knowing f explicitly.

(c) Find an explicit formula for f and check your result from b).

1.6.3 * Sketch of the proof of the Implicit Function Theorem

There is an intuitive argument which explains why the conditions in Theorem 1.16 are sufficient. With careful attention to details, one may turn this into a proof. The argument is fairly elementary, but since it is long, you may treat it as optional.

Consider a map $F : \Omega \to \mathbb{R}$ which satisfies the hypothesis in Theorem 1.16. We break the argument into several steps. Note that $D_2F(x', y') \neq 0$. Without loss of generality we may assume that $D_2F(x', y') > 0$ (the other case is similar).

Step 1. There is $\delta > 0$ such that for every $x \in [x' - \delta, x' + \delta]$ and every $y \in [y' - \delta, y' + \delta]$, we have $D_2F(x, y) > 0$.

To see this, note that since F is continuously differentiable, the map

$$(x, y) \mapsto D_2F(x, y)$$

is continuous from Ω to $\mathbb{R}$. As this function is positive at (x', y'), it must be positive on a neighbourhood of that point. Thus, there is $\delta > 0$ satisfying the property in Step 1.

Step 2. There are δ' with $0 < \delta' < \delta$ such that on the set $(x' - \delta', x' + \delta') \times \{y' - \delta\}$ we have $F < 0$, and on the set $(x' - \delta', x' + \delta') \times \{y' + \delta\}$ we have $F > 0$.

To see this, consider the map $h : [y' - \delta, y' + \delta] \to \mathbb{R}$ defined as

$$h(y) = F(x', y).$$

By the property in Step 1 we note that $h'(y) = D_2F(x', y) > 0$, for all $y \in (y' - \delta, y' + \delta)$. This implies that h is strictly increasing on the interval $(y' - \delta, y' + \delta)$. As $h(y') = 0$, we must have $h(y' - \delta) < 0$ and $h(y' + \delta) > 0$.

By the above paragraph, $F(x', y' - \delta) < 0$ and $F(x', y' + \delta) > 0$. Since F is continuous, there is $\delta' > 0$ such that F is negative on $(x' - \delta', x' + \delta') \times \{y' - \delta\}$, and is positive on $(x' - \delta', x' + \delta') \times \{y' + \delta\}$.

Step 3. For every $x \in (x' - \delta', x' + \delta')$, there is a unique $y \in (y' - \delta, y' + \delta)$ such that $F(x, y) = 0$.

Fix an arbitrary $x \in (x' - \delta', x' + \delta')$, and consider the map $g : [y' - \delta, y' + \delta] \to \mathbb{R}$ defined as

$$g(y) = F(x, y).$$

The map g is continuous on $[y' - \delta, y' + \delta]$, with $g(y') = F(x, y' - \delta) < 0$ and $g(y' + \delta) = F(x, y' + \delta) > 0$. By the intermediate value theorem, there must be $y \in [y' - \delta, y' + \delta]$ such that $g(y) = 0$. So $F(x, y) = 0$.

On the other hand, since $g'(y) = D_2 F(x, y) > 0$ for all $y \in [y' - \delta, y' + \delta]$, g is strictly increasing on $[y' - \delta, y' + \delta]$. This implies that there is a unique point in $(y' - \delta, y' + \delta)$ where g becomes 0. This proves the uniqueness.

With the above argument, we can introduce $A = (x' - \delta', x' + \delta')$ and $B = (y' - \delta, y' + \delta)$.

1.6.4 The general from of the Implicit Function Theorem

There is a more general version of the Implicit Function Theorem for arbitrary dimensions.

Theorem 1.17 (Implicit Function Theorem). *Let $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^m$ be open sets, and $f : \Omega \times \Omega' \to \mathbb{R}^m$ be continuously differentiable on $\Omega \times \Omega'$. Suppose there is $p = (a, b) \in \Omega \times \Omega'$ such that*

(i) $f(p) = 0$, and

(ii) the $m \times m$ matrix

$$\left(D_{n+j} f^i(p)\right), \qquad 1 \leq i, j \leq m.$$

is invertible.

Then, there are open sets $A \subset \Omega$ and $B \subset \Omega'$ with $a \in \Omega$ and $b \in \Omega'$, as well as a map $g : A \to B$ such that

$$f(x, y) = 0 \text{ for some } (x, y) \in A \times B \qquad \text{iff} \qquad y = g(x) \text{ for some } x \in A.$$

The map g is continuously differentiable.

1.6.5 * Equivalence of the two theorems

In this section we prove that the Inverse Function Theorem and the Implicit Function Theorem are equivalent.

Inverse Function Theorem implies the Implicit Function Theorem: Assume that f satisfies the assumptions in Theorem 1.17. We define a new map

$$F : \Omega \times \Omega' \to \mathbb{R}^n \times \mathbb{R}^m$$

as

$$F(x,y) = (x, f(x,y)).$$

The Jacobian of F at $p = (a,b)$ is

$$DF(p) = \left(\begin{array}{c|c} I & 0 \\ \hline N & M \end{array}\right)$$

Here I is the $n \times n$ identity matrix, M is the matrix in Theorem 1.17, and N is the $m \times n$ matrix with components:

$$\left(D_j f^i(p)\right), \qquad 1 \le i \le m, \;\; 1 \le j \le n.$$

Since $\det M \neq 0$, we must have $\det DF(p) \neq 0$. Note that $F(a,b) = (a,0)$. Therefore, we can apply the Inverse Function Theorem to deduce the existence of open sets $U \subset \Omega \times \Omega'$ and $V \subset \mathbb{R}^n \times \mathbb{R}^m$ with $(a,b) \in U$, $(a,0) \in V$ such that $F : U \to V$ has a continuously differentiable inverse $h : V \to U$. By shrinking U, if necessary, we can assume that $U = A \times B$ for some open sets $A \subset \Omega$ and $B \subset \Omega'$.

Note that the map h must be of the form $h(x,y) = (x, k(x,y))$ for some continuously differentiable map k (since F has this form). Let $\pi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the projection map $\pi(x,y) = y$. Then $f = \pi \circ F$. Now, by the associativity of composition of maps,

$$\begin{aligned} f(x, k(x,y)) &= f \circ h(x,y) = (\pi \circ F) \circ h(x,y) \\ &= \pi \circ (F \circ h)(x,y) = \pi(x,y) = y. \end{aligned}$$

Thus $f(x, k(x,0)) = 0$, so we can take $g(x) = k(x,0)$. □

Implicit Function Theorem implies the Inverse Function Theorem. Let $f : \Omega \to \mathbb{R}^n$ be the map in Theorem 1.15. Let us consider the map

$$F : \mathbb{R}^n \times \Omega \to \mathbb{R}^n$$

defined as

$$F(y,x) = y - f(x).$$

Let us also define $p = (f(q), q) \in \mathbb{R}^n \times \Omega$. We have

$$F(p) = 0.$$

We note that the matrix

$$D_{n+j} F^i(p), \qquad 1 \le i,j \le n$$

is $-Df(q)$. So, by the assumption in inverse function theorem, the above matrix is invertible. Therefore, by the Implicit Function Theorem, there is an open set $U \subset \mathbb{R}^n$ and $B \subset \Omega$ with $f(q) \in A$ and $q \in B$, and a map $g : A \to B$ such that

$$F(y,x) = 0 \text{ for some } (x,y) \in A \times B \qquad \text{iff} \qquad x = g(y) \text{ for some } y \in A.$$

In particular, for all $y \in A$, $F(y, g(y)) = 0$. By the definition of F, this means that $y = f(g(y))$, for all $y \in A$. The if and only in the above statement, implies that f is invertible on B, and g is the inverse of f on B. □

Chapter 2

Metric and topological spaces

2.1 Metric spaces

2.1.1 Motivation and definition

The notions of modulus function on $\mathbb{R}$ and the norm function on $\mathbb{R}^n$ allow us to develop the analysis on Euclidean spaces. We would like to extend the fundamental notions of analysis, such as convergence of sequences, continuity of maps, etc, to more general settings. We have already seen that most concepts in higher dimensional Euclidean spaces are analogous to the corresponding concepts in one dimensional Euclidean space; replacing the modulus function with the norm function. Over all, all those concepts rely on a notion of "distance" on the ambient space.

We have all been using the concept of "distance" in our everyday life, for example, by asking

- how much time does it take to walk from my apartment to the maths department,
- how long does it take to travel from South Kensington tube station to Cambridge by public transport,
- how much does the cheapest public transport from South Kensington tube station to Heathrow airport cost,
- what is the distance, in kilometres, from London to Edinburgh.

What should be the correct way of defining "distance" in more general settings. From the above examples we can see that the notion of distance should be a function of two variables, that is, we give it two elements. There has been a long historical development on this question, with various properties proposed and refined. Here we present the outcome of those developments, and define what is now standard.

Definition 2.1. Let X be an arbitrary set. A **metric** on X is a function

$$\mathrm{d} : X \times X \to \mathbb{R}$$

satisfying the following three properties:

(M1) for all x and y in X we have $\mathrm{d}(x, y) \geq 0$, and $\mathrm{d}(x, y) = 0$ if and only if $x = y$;

(M2) for all x and y in X, $\mathrm{d}(x, y) = \mathrm{d}(y, x)$;

(M3) for all x, y and z in X, we have $\mathrm{d}(x, y) \leq \mathrm{d}(x, z) + \mathrm{d}(z, y)$.

Property M1 is called **positivity**, property M2 is called **symmetry**, and property M3 is called **triangle inequality**.

Remark 2.1. The triangle inequality in Euclidean spaces has a rather simple interpretation. That is, in any triangle, the length of each side is bounded from above by the sum of the lengths of the other two sides. In an arbitrary set, triangles may not make sense. But the interpretation still makes sense, and is the reason behind requiring condition M3. We think of $\mathrm{d}(x, y)$ as "the length of the shortest way from x to y". So the length of the shortest way from x to y should be bounded from above by the length of the shortest way from x to y passing through z. See Figure 2.1.

On the other hand, property M1 tells us that the metric "separates" points. That is, the distance between distinct points is strictly positive.

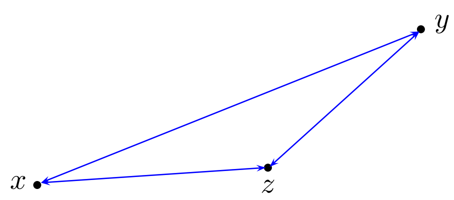

Figure 2.1: The triangle inequality.

Definition 2.2. By a **metric space** we mean a pair of a set and a metric on that set. That is often denoted as $M = (X, \mathrm{d})$, where X is a set, and $d : X \times X \to \mathbb{R}$ is a metric. We refer to M as the metric space. The elements of X are called **points**. Given two points x and y in X, the real number $\mathrm{d}(x, y)$ is called the **distance** between x and y with respect to the metric d.

In the above definition, when it is clear what metric is involved, we simply refer to $\mathrm{d}(x, y)$ as the distance between x and y.

It is customary to use the same notation for M and X, that is, the metric space $X = (X, \mathrm{d})$.

Remark 2.2. The reason that we refer to the elements of X as points, is because we would like a unified approach to all metric spaces. That is, to present statements and proofs so that it applies to a variety of settings. We understand that when $X = \mathbb{R}$, then elements of X are numbers, when $X = \mathbb{R}^n$, the elements of X are vectors, and when X is the set of all 5×5 matrices, then each element of X is a matrix. We refer to all those elements as points in X.

2.1.2 Examples of metric spaces

There are many examples of metrics. You are already familiar with some of them, although you did not use the terminology of metric spaces.

Example 2.1. Let $X = \mathbb{R}$ and $\mathrm{d}_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined as

$$\mathrm{d}_1(x, y) = |x - y|.$$

From the properties of the modulus function, see Section 1.1.1, we immediately see that d_1 satisfies the properties M1, M2, and M3. For example, for M2, we see that

$$\mathrm{d}_1(x, y) = |x - y| = |y - x| = \mathrm{d}_1(y, x).$$

Example 2.2. Let $X = \mathbb{R}^n$, and for $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$, let

$$\mathrm{d}_2(x, y) = \|x - y\| = \left(\sum_{j=1}^{n} (x^j - y^j)^2 \right)^{1/2}.$$

By the properties of the norm function on $\mathbb{R}^n$, see Section 1.1.2, d_2 satisfies the properties M1, M2, and M3 in Definition 2.1. For example, to see property M3, we note that for every x, y, and z in $\mathbb{R}^n$, by the triangle inequality for the norm function, we have

$$\mathrm{d}_2(x, y) = \|x - y\| \leq \|x - z\| + \|z - y\| = \mathrm{d}_2(x, z) + \mathrm{d}_2(z, y).$$

The metric d_2 on $\mathbb{R}^n$ is called the **Euclidean metric** on $\mathbb{R}^n$.

Example 2.3. Let $X = \mathbb{R}^n$, and for $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$, let

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j|.$$

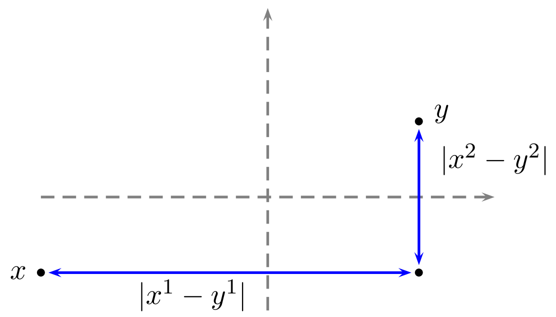

Figure 2.2: Illustration of the metric d_1 on $\mathbb{R}^2$.

We need to verify that the properties M1, M2 and M3 in Definition 2.1 hold.

M1: Fix arbitrary $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$. Since the modulus function only produces non-negative values, for every $j = 1, 2, \dots, n$, we have $|x^j - y^j| \geq 0$. Thus,

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| \geq 0.$$

On the other hand, if

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| = 0,$$

then, for all $j = 1, 2, \dots, n$, we must have $|x^j - y^j| = 0$ (because each of the numbers in the above sum is non-negative). By the first property of the modulus function, this implies that for all $j = 1, 2, \dots, n$, we have $x^j = y^j$. Hence, $x = y$.

M2: For every $x = (x^1, x^2, \dots, x^n)$ and $y = (y^1, y^2, \dots, y^n)$ in $\mathbb{R}^n$, we have

$$\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| = \sum_{j=1}^{n} |y^j - x^j| = \mathrm{d}_1(y, x).$$

M3: For every $x = (x^1, x^2, \dots, x^n)$, $y = (y^1, y^2, \dots, y^n)$ and $z = (z^1, z^2, \dots, z^n)$ in $\mathbb{R}^n$, we have

$$\begin{aligned}
\mathrm{d}_1(x, y) = \sum_{j=1}^{n} |x^j - y^j| &\leq \sum_{j=1}^{n} \big(|x^j - z^j| + |z^j - y^j| \big) \\
&= \sum_{j=1}^{n} |x^j - z^j| + \sum_{j=1}^{n} |z^j - y^j| \\
&= \mathrm{d}_1(x, z) + \mathrm{d}_1(z, y).
\end{aligned}$$

In the first line of the above equation we have used the triangle inequality for the modulus function n times (i.e. $|x^j - y^j| \leq |x^j - z^j| + |z^j - y^j|$, for $j = 1, 2, \dots, n$).

Intuitively, the metric d_1 on $\mathbb{R}^2$ means that we are only allowed to travel along horizontal and vertical directions to go from $x \in \mathbb{R}^2$ to $y \in \mathbb{R}^2$. See Figure 2.2.

Exercise 2.1. Let $X = \mathbb{R}^n$ and define the function $\mathrm{d}_\infty : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$\mathrm{d}_\infty(x, y) = \max\{|x^1 - y^1|, \dots, |x^n - y^n|\}.$$

Show that d_∞ is a metric on $\mathbb{R}^n$.

The above examples show that there can be more than one metric on $\mathbb{R}^n$. The following exercise shows that indeed, there can be many metrics on $\mathbb{R}^n$.

Exercise 2.2. Show that each of the following functions is a metric on $\mathbb{R}$:

(i) $\mathrm{d}(x, y) = |x^3 - y^3|$, (here x^3 means x raised to power 3)

(ii) $\mathrm{d}(x, y) = |e^x - e^y|$,

(iii) $\mathrm{d}(x, y) = |\tan^{-1}(x) - \tan^{-1}(y)|$.

Which property of the maps $x \mapsto x^3$, $x \mapsto e^x$, and $x \mapsto \tan^{-1}(x)$ makes these functions a metric.

We will need the following property of the integral later on.

Lemma 2.1. *Assume that $a < b$ are real numbers, and $f : [a, b] \to \mathbb{R}$ is a continuous function such that $f \geq 0$ on $[a, b]$, and f is not identically equal to 0. Then,*

$$\int_a^b f(t)\, dt > 0.$$

Proof. Since f is not identically equal to 0, there must be $c \in [a, b]$ such that $f(c) > 0$. Let $h = f(c)$. Since f is continuous at c, for $\epsilon = h/2 > 0$ there is $\delta > 0$ such that for all $t \in [a, b]$ with $|t - c| < \delta$, we have $|f(t) - h| \leq h/2$. This implies that for all $t \in (c - \delta, c + \delta) \cap [a, b]$, we have

$$f(t) = h + (f(t) - h) \geq h - h/2 = h/2.$$

Without loss of generality we may assume that $\delta < (b - a)/2$.

Consider the function $g : [a, b] \to \mathbb{R}$ defined as

$$g(t) = \begin{cases} 0 & \text{if } t \notin (c - \delta, c + \delta) \cap [a, b], \\ h/2 & \text{if } t \in (c - \delta, c + \delta) \cap [a, b]. \end{cases}$$

We note that $f \geq g$ on $[a, b]$. Also, since g is only discontinuous at two points (finite number of points is ok), it is integrable on $[a, b]$. Moreover,

$$\int_a^b f(t)\, dt \geq \int_a^b g(t)\, dt \geq \delta \cdot h/2 > 0.$$

Note that since $c \in [a, b]$, the length of the interval $[c - \delta, c + \delta] \cap [a, b]$ is at least δ, with the minimum length happening when $c = a$ or $c = b$.

□

Exercise 2.3. Assume that $a < b$ are real numbers, and $h : (a, b) \to (0, \infty)$ be a continuous function. For x and y in (a, b), we define

$$\mathrm{d}_h(x, y) = \int_{\min\{x,y\}}^{\max\{x,y\}} h(t)\, dt.$$

Show that d_h is a metric on (a, b).

Intuitively, in the above exercise, the function h determines the cost of travelling from x to y.

Exercise 2.4. Consider the function $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined as

$$g(x, y) = |x - y|^2.$$

Show that g is not a metric on $\mathbb{R}$.

Below is an example of a metric on an slightly different set.

Example 2.4. Let S^1 be the circle of radius 1 about 0 in $\mathbb{R}^2$, that is,

$$S^1 = \left\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| = 1\right\}.$$

Any pair of points a and b in S^1 divides the circle S^1 into two arcs. We assume the convention that the end points of the arcs are included in the arcs (this does not make any difference when calculating the arc length). We define $\mathrm{d}(a, b)$ as the length of the shortest arc between a and b. When the points a and b are antipodal (diametrically opposite of one another), the shortest arc is not unique, but those arcs have the same length. Thus, the function $\mathrm{d} : S^1 \times s^1 \to \mathbb{R}$ is well-defined.

M1: The length of any arc is non-negative, and when the end points are distinct, the length is strictly positive.

M2: Since the shortest arc between two points does not depend on the order at which we choose the end points, M2 holds as well. When the end points lie on opposite sides, the shortest arc is not unique, but the length is unique. So in that case we have symmetry as well.

M3: Let θ_1, θ_2 and θ_3 be arbitrary points on S^1. If the points θ_1, θ_2 and θ_3 are not pairwise disjoint, then we obviously have

$$\mathrm{d}(\theta_1, \theta_3) \leq \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

That is because, if $\theta_1 = \theta_3$, the left hand side of the above inequality is 0, and the right hand side is non-negative by the definition of metric. Also, if $\theta_2 \in \{\theta_1, \theta_3\}$, the value on the left hand side also appears on the right hand side of the inequality, with the other term on the right hand side non-negative. So we may assume that the points θ_1, θ_2, and θ_3 are pairwise disjoint. Let $\ell_{i,j}$ denote the shortest arc between θ_i and θ_j, for i and j in $\{1, 2, 3\}$. We consider few cases:

(i) θ_2 belongs to $\ell_{1,3}$: Then $\ell_{1,2} \cup \ell_{2,3} = \ell_{1,3}$, and hence

$$\mathrm{d}(\theta_1, \theta_3) = \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3) \le \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

(ii) θ_1 belongs to $\ell_{2,3}$. Then, $\ell_{1,3} \subset \ell_{2,3}$, and hence

$$\mathrm{d}(\theta_1, \theta_3) \le \mathrm{d}(\theta_2, \theta_3) \le \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

(iii) θ_3 belongs to $\ell_{1,2}$. Then, $\ell_{1,3} \subset \ell_{1,2}$, and hence

$$\mathrm{d}(\theta_1, \theta_3) \le \mathrm{d}(\theta_1, \theta_2) \le \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

(iv) neither of the cases (i)-(iii) holds. Then, $\ell_{1,2} \cup \ell_{1,3} \cup \ell_{2,3} = S^1$, and hence

$$\mathrm{d}(\theta_1, \theta_3) = \text{“length of”}\ \ell_{1,3} \le \text{“length of”}\ (S^1 \setminus \ell_{1,3}) = \mathrm{d}(\theta_1, \theta_2) + \mathrm{d}(\theta_2, \theta_3).$$

See Figure 2.3.

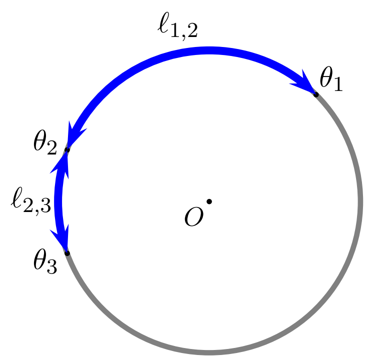

Figure 2.3: The circle of radius 1 about 0 in $\mathbb{R}^2$, and the distance of arc length.

All the examples of metrics we have seen so far are on the of real numbers and Euclidean spaces. But the purpose of giving an axiomatic definition of metric is to generalises analysis. Here are few examples of metric spaces which shows the generality of this notion.

Example 2.5. Let E be a finite set, and let $\mathscr{P}(E)$ denote the set of all subsets of E. Given $A \in \mathscr{P}(E)$, we define $\mathrm{Card}(A)$ as the number of elements in A. Also, for A and B in $\mathscr{P}(E)$, we define the symmetric difference of A and B as

$$A \Delta B = (A \setminus B) \cup (B \setminus A).$$

The function $\mathrm{d}_{\mathrm{card}} : \mathscr{P}(E) \times \mathscr{P}(E) \to \mathbb{R}$ defined as

$$\mathrm{d}_{\mathrm{card}}(A, B) = \mathrm{Card}(A \Delta B)$$

is a metric on $\mathscr{P}(E)$.

This metric is called the Hamming metric, and plays important role in information theory and cryptography.

Although we all have an intuitive way of thinking about distances, we need to be cautious when dealing with metrics in general. The axiomatic description of the metric in Definition 2.1 captures a wide range of settings, as we discuss in the next two examples.

Example 2.6. Let X be an arbitrary non-empty set. Define, $\mathrm{d}_{\text{disc}} : X \times X \to \mathbb{R}$ as

$$\mathrm{d}_{\text{disc}}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

You can see that this is a metric on X. In this metric all distinct points lie at distance 1 from each other (you may wish to imagine this for some sets). The metric d_{disc} is called the **discrete metric**.

Another counter intuitive example of a metric is presented in the next Exercise.

Exercise 2.5. Let $X = \mathbb{R}^2$, and define $\mathrm{d}_{\text{rail}} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ as

$$\mathrm{d}_{\text{rail}}(x, y) = \begin{cases} \|x - y\| & \text{if } x = ky \text{ for some } k \in \mathbb{R} \\ \|x\| + \|y\| & \text{otherwise} \end{cases}$$

Show that d_{rail} is a metric on $\mathbb{R}^2$.

This is called the British rail metric. The intuition behind this metric is that if two towns are on the same rail line, then we travel between them, but if the towns are on distinct lines, we travel via London (represented as as the origin in $\mathbb{R}^2$).

Example 2.7. We say that a sequence $(x^1, x^2, x^3, \dots)$ is bounded, if there is $M \in \mathbb{R}$ such that for all $i \geq 1$, $|x^i| \leq M$. Let X be the set of all bounded sequences, and consider the function $\mathrm{d}_\infty : X \times X \to \mathbb{R}$ defined as

$$\mathrm{d}_\infty(x, y) = \sup_{k \geq 1} |x^k - y^k|.$$

M1: Since the supremum of a collection of non-negative numbers is a non-negative number, $\mathrm{d}_\infty(x, y) \geq 0$ for all x and y in X. On the other hand, if $\mathrm{d}_\infty(x, y) = \sup_{k \geq 1} |x^k - y^k| = 0$, we must have $|x^k - y^k| = 0$ for all $k \geq 1$. Therefore, $x = y$.

M2: Evidently, since $|t| = |-t|$ for all $t \in \mathbb{R}$, we have

$$\mathrm{d}_\infty(x, y) = \sup_{k \geq 1} |x^k - y^k| = \sup_{k \geq 1} |y^k - x^k| = \mathrm{d}_\infty(y, x).$$

M3: Fix arbitrary elements of X:

$$x = (x^1, x^2, \dots,), \quad y = (y^1, y^2, \dots,), \quad z = (z^1, z^2, \dots).$$

For every $j \geq 1$, we have

$$|x^j - y^j| \leq |x^j - z^j| + |z^j - y^j| \leq \left(\sup_{k\geq 1} |x^k - z^k|\right) + \left(\sup_{k\geq 1} |z^k - y^k|\right)$$
$$= \mathrm{d}_\infty(x, z) + \mathrm{d}_\infty(z, y).$$

The right hand side of the above equation is a constant independent of j. Thus, for all $j \geq 1$, $|x^j - y^j|$ is bounded from above by that constant. Therefore, their supremum must be bounded by that constant. That is,

$$\mathrm{d}_\infty(x, y) = \sup_{j\geq 1} |x^j - y^j| \leq \mathrm{d}_\infty(x, z) + \mathrm{d}_\infty(z, y).$$

The metric space (X, d_∞) is called the l_∞ space.

Assume that a and b are real numbers with $a < b$. Define the set

$$C([a, b]) = \big\{f : [a, b] \to \mathbb{R} \mid f : [a, b] \to \mathbb{R} \text{ is continuous.}\big\}$$

Example 2.8. For f and g in $C([a, b])$, define

$$\mathrm{d}_\infty(f, g) = \max_{a\leq t\leq b} |f(t) - g(t)|.$$

Since f and g are continuous on $[a, b]$, they are bounded so there exists k_1 and k_2 in $\mathbb{R}$ such that for all $t \in [a, b]$, $|f(t)| \leq k_1$ and $|g(t)| \leq k_2$. Therefore, $\mathrm{d}_\infty(f, g) \leq k_1 + k_2$, so d_∞ is well defined on $C([a, b])$.

As in the previous example, one can see that d_∞ is a metric on $C([a, b])$. This is called the **supremum metric**, or the **uniform metric**.

Example 2.9. For f and g in $C([a, b])$, define

$$\mathrm{d}_1(f, g) = \int_a^b |f(t) - g(t)|\, dt.$$

The function d_1 is a metric on $C([a, b])$. To see this, first note that since the modulus of a continuous function is a continuous function, the integral in the above definition is defined.

M1: For every f and g in X, and every $t \in [a, b]$, $|f(t) - g(t)| \geq 0$. Thus,

$$\mathrm{d}_1(f, g) = \int_a^b |f(t) - g(t)|\, dt \geq 0.$$

On the other hand, if $\mathrm{d}_1(f, g) = 0$, by Lemma 2.1, we must have $|f - g|$ is identically equal to 0. Thus, $f = g$ as functions on $[a, b]$.

M2: Since for all $t \in \mathbb{R}$, $|t| = |-t|$, we have

$$\mathrm{d}_1(f, g) = \int_a^b |f(t) - g(t)|\, dt = \int_a^b |g(t) - f(t)|\, dt = \mathrm{d}_1(g, f).$$

M3: Let f, g and h be continuous functions on $[a, b]$. For all $t \in [a, b]$, by the triangle inequality for the modulus function, we have

$$|f(t) - g(t)| = \big|(f(t) - h(t)) + (h(t) - g(t))\big| \leq |f(t) - h(t)| + |h(t) - g(t)|.$$

Integrating the above functions, we note that

$$\int_a^b |f(t) - g(t)|\, dt \leq \int_a^b |f(t) - h(t)|\, dt + \int_a^b |h(t) - g(t)|\, dt,$$

which gives us

$$\mathrm{d}_1(f, g) \leq \mathrm{d}_1(f, h) + \mathrm{d}_1(h, g).$$

We have already seen many examples of metric spaces. There are some ways to define new metric spaces using other metric spaces. We present two approaches below.

Definition 2.3. Let (X, d) be a metric space, and $Y \subset X$ be an arbitrary subset. Define $\mathrm{d}|_Y : Y \times Y \to \mathbb{R}$ as $d|_Y(x, y) = \mathrm{d}(x, y)$, for all x and y in Y. Clearly $\mathrm{d}|_Y$ is a metric on Y (it inherits all the properties from d). The pair $(Y, \mathrm{d}|_Y)$ is called a **metric subspace** of (X, d), and $\mathrm{d}|_Y$ is called the **induced metric** on Y from d.

Example 2.10. Consider the Euclidean metric space $(\mathbb{R}, \mathrm{d}_1)$. We may restrict this metric to the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. Also, d_1 induces a metric on the set of integers $\mathbb{Z} \subset \mathbb{R}$.

Similarly, since $\mathbb{Z}^n \subset \mathbb{R}^n$ and $\mathbb{Q}^n \subset \mathbb{R}^n$, we may restrict any of the metrics d_1, d_2, and d_∞ onto those sets.

Given arbitrary sets X_1 and X_2, we define the (set-theoretical) **product** of these two sets as

$$X_1 \times X_2 = \{(x_1, x_2) \mid x_1 \in X_1, x_2 \in X_2\}.$$

That is, the set of all ordered pairs (x_1, x_2) such that $x_1 \in X_1$ and $x_2 \in X_2$.

Definition 2.4. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. We may use the metrics d_1 and d_2 to define a metric on $X_1 \times X_2$. For example,

$$\mathrm{d}\left((x_1, x_2), (y_1, y_2)\right) = \max\{\mathrm{d}_1(x_1, y_1), \mathrm{d}_2(x_2, y_2)\},$$
$$\mathrm{d}\left((x_1, x_2), (y_1, y_2)\right) = \mathrm{d}_1(x_1, y_1) + \mathrm{d}_2(x_2, y_2).$$

Each of the above functions from $(X_1 \times X_2) \times (X_1 \times X_2)$ to $\mathbb{R}$ is a metric. For each of the above metrics d, the metric space $(X_1 \times X_2, \mathrm{d})$ is called a **product metric spaces**.

2.1.3 Normed vector spaces

Definition 2.5. Let V be a vector space on $\mathbb{R}$. We say that a function $\|\cdot\| : V \to \mathbb{R}$ is a **norm** on V, if the following properties are satisfied:

(N1) for every $v \in V$, $\|v\| \geq 0$, and $\|v\| = 0$ if and only if $v = 0$,

(N2) for every $v \in V$ and every $\lambda \in \mathbb{R}$, we have $\|\lambda V\| = |\lambda| \, \|v\|$,

(N3) for all u and v in V, $\|u + v\| \leq \|u\| + \|v\|$.

A **normed vector space**, is a pair of a vector space V together with a norm function on V. This is often denoted as $(V, \|\cdot\|)$.

On any vector space $(V, \|\cdot\|)$ we have a natural notion of metric coming from the norm function. We present this in the next lemma.

Lemma 2.2. *Let V be a vector space, and $\|\cdot\| : V \to \mathbb{R}$ be a norm function on V. The function $\mathrm{d}_{\|\|} : V \times V \to \mathbb{R}$, defined as*

$$\mathrm{d}_{\|\|}(u, v) = \|u - v\|$$

is a metric on V.

Proof. Property M1 comes from the property N1 of the norm function, that is,

$$\mathrm{d}_{\|\|}(v, w) = \|v - w\| \geq 0.$$

Also,

$$\mathrm{d}_{\|\|}(v, w) = 0 \iff \|v - w\| = 0 \iff v - w = 0 \iff v = w.$$

Property M2 comes from the property N2 of the norm function, since

$$\mathrm{d}_{\|\|}(w, v) = \|w - v\| = \|(-1)(v - w)\| = |-1| \, \|v - w\| = \|v - w\| = \mathrm{d}_{\|\|}(v, w).$$

Property M3 comes from the property N3 for the norm. That is because

$$\mathrm{d}_{\|\|}(v, z) = \|v - z\| \leq \|v - w\| + \|w - z\| = \mathrm{d}_{\|\|}(v, w) + \mathrm{d}_{\|\|}(w, z).$$ □

Some of the examples we already seen are normed vector spaces. For example, the distance d_2 on $\mathbb{R}^n$ comes from the norm $\|\cdot\|$ on $\mathbb{R}^n$.

Example 2.11. Let $V = \mathbb{R}^n$, and consider the functions

$$\begin{aligned} \big\|(x^1, x^2, \dots, x^n)\big\|_1 &= |x^1| + |x^2| + \dots + |x^n|, \\ \big\|(x^1, x^2, \dots, x^n)\big\|_\infty &= \max\{|x^1|, |x^2|, \dots, |x^n|\}. \end{aligned}$$

One can easily see that these functions satisfy the three properties for the norm function. These norms induce the metrics d_1 and d_∞ on $\mathbb{R}^n$, respectively.

Assume that $a < b$ are real numbers, and let $C([a,b])$ denote the set of all continuous functions $f : [a,b] \to \mathbb{R}$. For f and g in $C([a,b])$, we define $f+g$ as the function $(f+g)(x) = f(x) + g(x)$, for all $x \in [a,b]$. Also, for $\lambda \in \mathbb{R}$, and $f \in C([a,b])$, we define $(\lambda f)(x) = \lambda f(x)$. These operations make $C([a,b])$ a vector space on $\mathbb{R}$. This vector space has infinite dimensions, since the functions $x \mapsto x$, $x \mapsto x^2$, $x \mapsto x^3$, $\dots$, are linearly independent.

Exercise 2.6. Assume that $a < b$ are real numbers. Show that each of the following functions is a norm on $C([a,b])$:

(i)
$$\|f\|_1 = \int_a^b |f(t)|\, dt$$

(ii)
$$\|f\|_\infty = \max_{t\in[a,b]} |f(t)|$$

(iii)
$$\|f\|_2 = \left(\int_a^b |f(t)|^2\, dt\right)^{1/2}$$

Remark 2.3. The norm $\|\cdot\|_1$ on $C([a,b])$ is called the l_1-norm, $\|\cdot\|_2$ on $C([a,b])$ is called the l_2-norm, and the norm $\|\cdot\|_\infty$ on $C([a,b])$ is called the l_∞-norm, or supremum norm. The metric induced from $\|\cdot\|_1$ on $C([a,b])$ is the d_1 metric we presented in Example 2.9 and the metric induced from $\|\cdot\|_\infty$ on $C([a,b])$ is the d_∞ metric we presented in Example 2.8.

You can learn more about these spaces in the modules Lebesgue Measure and Integration, and Functional Analysis.

It is not true that every metric on a vector space comes from a norm. You can show this by the following exercise.

Exercise 2.7. Show that if V is a vector space, and $\|\cdot\| : V \to \mathbb{R}$ is a norm function, then for any $v \in V$, we must have $\mathrm{d}_{\|\|}(0, 2v) = 2\,\mathrm{d}_{\|\|}(0, v)$. Conclude that there is no norm function on $\mathbb{R}^2$ which induced the discrete metric d_{disc} on $\mathbb{R}^2$.

As we shall see in later sections, the notion of metric allows us to develop analysis on general metric spaces. It is remarkable that such a simple notion can lead to a huge volume of mathematical theory. Of all the properties of a function which makes it a metric, the triangle inequality is the non-trivial one. It is worth taking a moment to build intuition about that property. The following exercise helps you to achieve that.

Exercise 2.8. Let (X, d) be a metric space.

(i) Show that for every x, y, and z in X, we have

$$|\,\mathrm{d}(x,z) - \mathrm{d}(y,z)| \leq \mathrm{d}(x,y).$$

(ii) Show that for all x, y, z and t in X, we have

$$|\,\mathrm{d}(x,y) - \mathrm{d}(z,t)| \leq \mathrm{d}(x,z) + \mathrm{d}(y,t).$$

(iii) Show that for all x_1, x_2, $\ldots$, x_n in X, we have

$$\mathrm{d}(x_1,x_n) \leq \mathrm{d}(x_1,x_2) + \mathrm{d}(x_2,x_3) + \cdots + \mathrm{d}(x_{n-1},x_n).$$

2.1.4 Open sets in metric spaces

The notion of a metric on a set allows us to describe some geometric properties of subsets of that set. We shall discuss some of these properties in Sections 2.1.4 and 2.1.6.

Definition 2.6. Consider a metric space (X, d), a point $x \in X$, and a real number $\epsilon > 0$. The **ball** of radius ϵ centred at x is the set of all points $x' \in X$ satisfying $\mathrm{d}(x,x') < \epsilon$. In other words,

$$B_\epsilon(x) = \{x' \in X \mid \mathrm{d}(x,x') < \epsilon\}.$$

This set is also referred to as **ϵ-ball** about x, or **ϵ-neighbourhood** of x. To emphasise the dependence of the ball on the metric d and the underlying space X, we may use the notation $B_\epsilon(x, X, \mathrm{d})$.

Example 2.12. We look at ϵ-balls in some of the metric spaces we introduced in the previous section.

(i) In $(\mathbb{R}, \mathrm{d}_1)$, for every $a \in \mathbb{R}$ and $\epsilon > 0$, we have

$$B_\epsilon(a) = \{x \in \mathbb{R} \mid \mathrm{d}_1(x,a) < \epsilon\} = \{x \in \mathbb{R} \mid |x-a| < \epsilon\} = (a-\epsilon, a+\epsilon).$$

(ii) In $(\mathbb{R}^n, \mathrm{d}_2)$, for every $a \in \mathbb{R}^n$ and $\epsilon > 0$, $B_\epsilon(a)$ consists of all the points inside a hypersphere.

(iii) In $(\mathbb{R}^2, \mathrm{d}_\infty)$, for every $a = (a^1, a^2) \in \mathbb{R}^2$ and $\epsilon > 0$,

$$\begin{aligned} B_\epsilon(a) &= \{(x^1,x^2) \in \mathbb{R}^2 \mid \mathrm{d}_\infty((a^1,a^2),(x^1,x^2)) < \epsilon\} \\ &= \{(x^1,x^2) \in \mathbb{R}^2 \mid \max\{|a^1 - x^1|, |a^2 - x^2|\} < \epsilon\}. \end{aligned}$$

This is a square with horizontal and vertical sides of lengths 2ϵ centre at a.

(iv) Let $I = [0, 1] \subset \mathbb{R}$, and d_I denote the induced metric on I from d_1 on $\mathbb{R}$. Then, in $(\mathbb{R}, \mathrm{d}_1)$, we have

$$B_1(1) = B_1(1, \mathbb{R}, \mathrm{d}_1) = \{x \in \mathbb{R} | |x - 1| < 1\} = (0, 2).$$

In (I, d_I), we have

$$\begin{aligned} B_1(1) = B_1(1, I, \mathrm{d}_I) &= \{x \in I \mid \mathrm{d}_I(x, 1) < 1\} \\ &= \{x \in [0, 1] \mid |x - 1| < 1\} \\ &= (0, 1]. \end{aligned}$$

In (I, d_I),

$$B_{1/2}(1/2) = B_{1/2}(1/2, I, \mathrm{d}_I) = \{x \in I \mid \mathrm{d}_I(x, 1/2) < 1/2\} = (0, 1).$$

(v) In $(X, \mathrm{d}_{\mathrm{disc}})$, where X is a non-empty set, and $\mathrm{d}_{\mathrm{disc}}$ is the discrete metric, for every $x \in X$ and $\epsilon > 0$ we have the following.

If $\epsilon \leq 1$, then

$$B_\epsilon(x) = \{x' \in X \mid \mathrm{d}_{\mathrm{disc}}(x, x') < \epsilon\} = \{x\}.$$

If $\epsilon > 1$,

$$B_\epsilon(x) = \{x' \in X \mid \mathrm{d}_{\mathrm{disc}}(x, x') < \epsilon\} = X.$$

(vi) In $(C([a, b]), \mathrm{d}_\infty)$, for $f \in C([a, b])$ and $\epsilon > 0$, we have

$$B_\epsilon(f) = \{g \in C([a, b]) \mid \mathrm{d}_\infty(f, g) < \epsilon\}$$
$$\{g \in C([a, b]) \mid \max_{t \in I} |f(t) - g(t)| < \epsilon\}.$$

This consists of all continuous functions $g : [a, b] \to \mathbb{R}$ such that the graph of g lies between the graphs of $f - \epsilon$ and $f + \epsilon$.

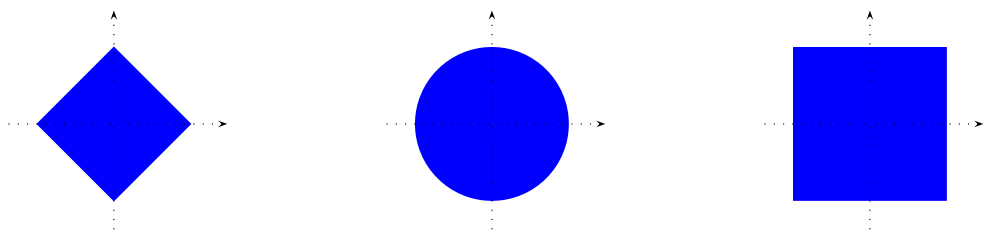

Figure 2.4: Figure on the left hand side shows $B_\epsilon(0, \mathbb{R}^2, \mathrm{d}_1)$, the figure in the middle shows $B_\epsilon(0, \mathbb{R}^2, \mathrm{d}_2)$, and the figure on the right hand side shows $B_\epsilon(0, \mathbb{R}^2, \mathrm{d}_\infty)$.

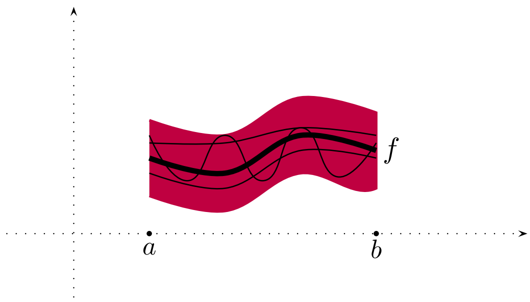

Figure 2.5: In $(C([a,b]), \mathrm{d}_\infty)$, $B_\epsilon(f)$ consists of all continuous functions on $[a,b]$ whose graphs lie in the red region. We have drawn the graphs of three functions in $B_\epsilon(f)$.

Exercise 2.9. Let (X, d) be a metric space.

(i) Show that if $\epsilon < \delta$, then $B_\epsilon(x) \subseteq B_\delta(x)$. By example, show that the equality may hold even if $\epsilon < \delta$.

(ii) Show that for every $x \in X$, we have

$$\bigcap_{n \in \mathbb{N}} B_{1/n}(x) = \{x\}.$$

Definition 2.7. Let (X, d) be a metric space, and $U \subseteq X$. We say that U is **open** in (X, d), if for every $u \in U$, there is $\delta > 0$ such that $B_\delta(u) \subseteq U$.

Lemma 2.3. *Let (X, d) be a metric space. For every $x \in X$ and $\epsilon > 0$, the ball $B_\epsilon(x)$ is open in X.*

Proof. Fix an arbitrary $y \in B_\epsilon(x)$. Let $\delta = \epsilon - \mathrm{d}(x,y)$. Since $y \in B_\epsilon(x)$, we have $\mathrm{d}(x,y) < \epsilon$, and hence $\delta > 0$.

Let $z \in B_\delta(y)$ be an arbitrary point. By the triangle inequality of the metric,

$$\mathrm{d}(z,x) \le \mathrm{d}(z,y) + \mathrm{d}(y,x) < \delta + (\epsilon - \delta) = \epsilon.$$

Hence, $z \in B_\epsilon(x)$. As $z \in B_\delta(y)$ was arbitrary, we conclude that $B_\delta(y) \subset B_\epsilon(x)$. As $y \in B_\epsilon(x)$ was arbitrary, we conclude that $B_\epsilon(x)$ is an open set. □

Due to the above lemma, $B_\epsilon(x)$ is also called an **open ball** of radius ϵ about x.

Lemma 2.4. *In any metric space (X, d), the empty set and the set X are open.*

Proof. To see that the empty set is open, we need to show that for every x in the empty set, there is $\delta >$ such that $B_\delta(x)$ is contained in the empty set. Since there is no such x in the empty set to begin with, for logical reasons this statement is true. So the empty set is open.

On the other hand, for every $x \in X$, we have $B_1(x) \subset X$. That is because of the definition of the ball. Thus we can take $\epsilon = 1$, in the criterion for open sets. $\square$

Note that the definition of open set in a metric space (X, d) depends on both the metric d and the underlying set X. We make this clear in the next example.

Example 2.13. Consider the discrete metric d_{disc} on $\mathbb{R}$, that is $(\mathbb{R}, \mathrm{d}_{\text{disc}})$. In this space, any subset of $\mathbb{R}$ is open. To see that let U be an arbitrary subset of $\mathbb{R}$, and let $u \in U$ be an arbitrary point. We let $\delta = 1/2$, and note that $B_{1/2}(u) = \{u\} \subset U$. This shows that U is open. But in the metric space $(\mathbb{R}, \mathrm{d}_1)$ it is not true that every subset of $\mathbb{R}$ is open. For example, the set with single element $\{1\}$ is open in $(\mathbb{R}, \mathrm{d}_{\text{disc}})$, but it is not open in $(\mathbb{R}, \mathrm{d}_1)$.

On the other hand, let $I = [0, 1] \subset \mathbb{R}$, and let d_I be the induced metric on $[0, 1]$ from d_1 on $\mathbb{R}$. The set $[0, 1/2)$ is not open in $(\mathbb{R}, \mathrm{d}_1)$ (the definition does not hold for the point $0 \in [0, 1/2)$). But $[0, 1/2)$ is open in $([0, 1], \mathrm{d}_I)$. To show the latter property, let $x \in [0, 1/2)$. If $x \in (0, 1/2)$, we define $\delta = \min\{x, 1/2 - x\}$, and see that $\delta > 0$ and

$$B_\delta(x, I, \mathrm{d}_I) = \{x' \in [0, 1/2) \mid \mathrm{d}_I(x, x') < \delta\} = (x - \delta, x + \delta) \subset [0, 1/2).$$

If $x = 0$, we let $\delta = 1/4$, and see that

$$B_\delta(0, I, \mathrm{d}_I) = \{x' \in [0, 1/2) \mid \mathrm{d}_I(0, x') < \delta\} = [0, 1/4) \subset [0, 1/2).$$

According to the definition of open sets, this shows that $[0, 1/2)$ is open in $([0, 1], \mathrm{d}_I)$.

Lemma 2.5. *Let $X = (X, \mathrm{d})$ be a metric space. The union of any number of (finite, countable, uncountable) open sets in X is an open set in X.*

Proof. Assume that $G_\alpha \subseteq X$ is open, for all α in a set I. Let $x \in \cup_{\alpha \in I} G_\alpha$. Then there exists some $\alpha_0 \in I$ such that $x \in G_{\alpha_0}$. Since G_{α_0} is an open set, there exists $\delta > 0$ such that $B_\delta(x) \subset G_{\alpha_0}$. This implies that $B_\delta(x) \subseteq \cup_{\alpha \in I} G_\alpha$. $\square$

Lemma 2.6. *Let $X = (X, \mathrm{d})$ be a metric space. The intersection of any finite number of open sets in X is an open set in X.*

Proof. Assume that $m \geq 1$ and $G_1, G_2, \dots, G_m$ are open sets in X. Fix an arbitrary $x \in \cap_{k=1}^m G_k$. For every $k \in \{1, 2, \dots, m\}$, $x \in G_k$. For every such k, since G_k is open, there exists $\epsilon_k > 0$ such that $B_{\epsilon_k}(x) \subset G_k$. Let $\epsilon = \min\{\epsilon_1, \dots, \epsilon_m\} > 0$.

By our choice of ϵ, for every $k \in \{1, 2, \dots, m\}$, $B_\epsilon(x) \subset B_{\epsilon_k}(x) \subset G_k$. Therefore, $B_\epsilon(x) \subset \cap_{k=1}^m G_k$. $\square$

The statement in the above lemma is not necessarily true if we drop the hypothesis of finiteness. For example, as we saw in Exercise 2.9, in the metric space $(\mathbb{R}^n, \mathrm{d}_2)$, we have $\cap_{n=1}^{\infty} B_{1/n}(x) = \{x\}$. And the set $\{x\}$ is not open in $(\mathbb{R}^2, \mathrm{d}_2)$.

We have already seen that there may be many metrics on a given set. For example, we have metrics d_1, d_2, and d_∞ on $\mathbb{R}^n$. The definition of open set in a metric space depends on the metric. So a priori, for each of these metrics on $\mathbb{R}^n$, we may have different open sets. This seems to be cumbersome, but can be alleviated by the following definition.

Definition 2.8. Let d_1 and d_2 be metrics on a set X. The metrics d_1 and d_2 are called **topologically equivalent**, if the following property holds. For every $U \subseteq X$, U is open in (X, d_1) if and only if U is open in (X, d_2).

Exercise 2.10. (i) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x, y) \leq \mathrm{d}_2(x, y) \leq \sqrt{n} \cdot \mathrm{d}_\infty(x, y).$$

(ii) Show that for all x and y in $\mathbb{R}^n$, we have

$$\mathrm{d}_\infty(x, y) \leq \mathrm{d}_1(x, y) \leq n \cdot \mathrm{d}_\infty(x, y).$$

(iii) Show/conclude that for all x and y in $\mathbb{R}^n$, we have

$$\frac{1}{\sqrt{n}} \mathrm{d}_2(x, y) \leq \mathrm{d}_1(x, y) \leq n\, \mathrm{d}_2(x, y).$$

(iv) Conclude that the metrics d_1, d_2 and d_∞ on $\mathbb{R}^n$ are topologically equivalent.

2.1.5 Convergence in metric spaces

Definition 2.9. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence of points in X. We say that the sequence $(x_n)_{n\geq 1}$ **converges** in (X, d), if there is $x \in X$ satisfying the following:

for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $\mathrm{d}(x_n, x) < \epsilon$.

In this case, we say that x is the **limit** of the sequence $(x_n)_{n\geq 1}$, or say that the sequence $(x_n)_{n\geq 1}$ converges to x in (X, d), and write $x_n \to x$ as $n \to \infty$, or $\lim_{n\to\infty} x_n = x$.

Notice the similarly between the above definition and the definition of convergence of sequences in Euclidean spaces.

Example 2.14. In the metric space $(\mathbb{R}, \mathrm{d}_1)$ the sequence $(1/n)_{n\geq 1}$ converges. That is because $0 \in \mathbb{R}$, and for every $\epsilon > 0$ we can choose an integer $N > 1/\epsilon$, so that for all $n \geq N$ we have $\mathrm{d}_1(1/n, 0) = 1/n < \epsilon$.

Now let $I = (0, 1)$, and d_I be the induced metric on I from d_1. In the metric space (I, d_I), the sequence $(1/n)_{n\geq 1}$ does not converge. That is because there is no $x \in (0, 1)$ satisfying the criterion for the convergence. Assume in the contrary that there is such an $x \in (0, 1)$. We choose $\epsilon = x/2 > 0$, and for every $N \in \mathbb{N}$, we choose $n \geq \max\{N, 2/x\}$. Then,

$$\mathrm{d}_I(1/n, x) = |1/n - x| = x - 1/n \geq x - x/2 = x/2 = \epsilon.$$

We say that a sequence $(x_n)_{n\geq 1}$ is **eventually constant**, if there is $n_1 \in \mathbb{N}$ such that for all $n \geq n_1$ we have $x_n = x_{n_1}$.

Exercise 2.11. Let $(X, \mathrm{d}_{\mathrm{disc}})$ be a discrete metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Then, $(x_n)_{n\geq 1}$ converges in $(X, \mathrm{d}_{\mathrm{disc}})$ if and only if the sequence $(x_n)_{n\geq 1}$ is eventually constant.

Lemma 2.7. *Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. If the sequence $(x_n)_{n\geq 1}$ converges in (X, d), then its limit is unique.*

Proof. Let us assume that there are two points x and y in X such that the sequence $(x_n)_{n\geq 1}$ converges to. Fix an arbitrary $\epsilon > 0$. Since the sequence converges to x, there is $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$, we have $\mathrm{d}(x_n, x) < \epsilon$. Similarly, since the sequence converges to y, there is $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$, we have $\mathrm{d}(x_n, y) < \epsilon$. Now, let $n = \max\{N_1, N_2\}$. We have

$$\mathrm{d}(x, y) \leq \mathrm{d}(x, x_n) + \mathrm{d}(x_n, y) < \epsilon + \epsilon = 2\epsilon.$$

By property M1 of metrics, $\mathrm{d}(x, y) \geq 0$, and since $\epsilon > 0$ was arbitrary, the above inequality shows that $\mathrm{d}(x, y) = 0$. Then, by property M1 of the metrics, we conclude that $x = y$. □

Exercise 2.12. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Prove that the sequence $(x_n)_{n\geq 1}$ converges to $x \in X$ if and only if, for every open set U in (X, d) with $x \in U$, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$.

As a corollary of the above exercise, we obtain the following result.

Corollary 2.8. *Let* d_1 *and* d_2 *be topologically equivalent metrics on* X*. Then, a sequence* $(x_n)_{n\geq 1}$ *in* X *converges in* (X, d_1) *if and only if it converges in* (X, d_2)*.*

Proof. Recall that by the definition of equivalent metrics, U is open in (X, d_1) if and only if U is open in (X, d_2). The result immediately follows from the previous exercise. □

2.1.6 Closed sets in metric spaces

Definition 2.10. Let (X, d) be a metric space, and $V \subseteq X$ be a set. We say that V is **closed** in (X, d), if for every sequence $(x_n)_{n\geq 1}$ in V which converges in (X, d), then the limit of $(x_n)_{n\geq 1}$ belongs to V.

When it is clear what metric is involved, we may simple say that V is closed in X. For example, when a metric is not specified on $\mathbb{R}^n$, it is assumed that it is the Euclidean metric d_2. Thus, when we say that E is closed in $\mathbb{R}^n$, we mean that E is closed in $(\mathbb{R}^n, \mathrm{d}_2)$.

Example 2.15. Consider real numbers $a < b$. The set $[a, b]$ is closed in $(\mathbb{R}^1, \mathrm{d}_1)$. That is because if $(x_n)_{n\geq}$ is a sequence in $[a, b]$ which converges to x in $\mathbb{R}$, then we have $a \leq x_n \leq b$, and hence $a \leq \lim_{n\to\infty} x_n \leq b$. This implies that $x \in [a, b]$.

The intervals (a, b) and $(a, b]$ are not closed in $(\mathbb{R}^1, \mathrm{d}_1)$. That is because

$$a + \frac{b-a}{n}, \quad n \geq 2$$

is a sequence in $(a, b]$ which converges to a in $(\mathbb{R}, \mathrm{d}_1)$, but a does not belong $(a, b]$.

On the other hand, let $I = (0, 1)$ and d_I be the induced metric on I from $(\mathbb{R}, \mathrm{d}_1)$. Then the set $V = (0, 1/2]$ is closed in $((0, 1), \mathrm{d}_I)$. To see this, assume that $(x_n)_{n\geq 1}$ is a sequence in $(0, 1/2]$ which converges in $((0, 1), \mathrm{d}_I)$. By the definition of convergence in $((0, 1), \mathrm{d}_I)$, the limit of the sequence must be in $(0, 1)$. However, since the sequence belongs to $(0, 1/2]$, its limit is at most $1/2$. Thus, the limit belongs to $(0, 1/2]$.

Exercise 2.13. Let $(X, \mathrm{d}_{\mathrm{disc}})$ be a discrete metric space. Then every set in X is closed.

Note that open is not the opposite of closed. If a set is not open, it does not mean that it is closed. For example, the set $(1, 2]$ is neither open or closed in $(\mathbb{R}, \mathrm{d}_1)$. There are sets that are both open and closed, as we shall see in a moment.

Theorem 2.9. *Let (X, d) be a metric space and $V \subseteq X$. Then, V is closed in (X, d) if and only if $X \setminus V$ is open in (X, d).*

Proof. First assume that V is closed. Assume in the contrary that $X \setminus V$ is not open. Then, there is $x \in X \setminus V$, such that for all $\delta > 0$, $B_\delta(x) \nsubseteq X \setminus V$. Equivalently, for all $\delta > 0$, $B_\delta(x) \cap V \neq \emptyset$. In particular, for each $n \in \mathbb{N}$, we let $\delta = 1/n$, and conclude that there is a point $x_n \in B_\delta(x) \cap V$. This process generates a sequence $(x_n)_{n \in \mathbb{N}}$ in V. The sequence $(x_n)_{n \in \mathbb{N}}$ converges to x in (X, d), because $x_n \in B_\delta(x)$ implies that $\mathrm{d}(x_n, x) < 1/n$. But the limit x does not belong to V, which contradicts V is closed.

Now assume that $X \setminus V$ is open. Let $(x_n)_{n \in \mathbb{N}}$ be an arbitrary sequence in V which converges to some $x \in X$. We need to show that $x \in V$. If $x \notin V$, then $x \in X \setminus V$. Then, since $X \setminus V$ is open, there is $\delta > 0$ such that $B_\delta(x) \subset X \setminus V$. On the other hand, since $(x_n)_{n \in \mathbb{N}}$ converges to x, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in B_\delta(x)$. Thus, for all $n \geq N$, $x_n \in X \setminus V$. This is a contradiction since $(x_n)_{n \in \mathbb{N}}$ is a sequence in V. □

Some authors define the notion of closed sets using the equivalence form in the above theorem. That is, a set is closed, if its complement is open. Then, they prove (as in the proof of the above theorem) that if a set is closed, it contains the limit of any convergent sequence in that set.

Lemma 2.10. *Let (X, d) be a metric space.*

(i) the intersection of any number (finite, countable or uncountable) of closed sets in (X, d) is a closed set in (X, d),

(ii) the union of any finite number of closed sets in (X, d) is a closed set in (X, d).

Proof. Let F_α, for $\alpha \in I$, be a collection of closed sets in X. By Theorem 2.9, for every $\alpha \in I$, $X \setminus F_\alpha$ is an open set. Then, by Lemma 2.5, $\cup_{\alpha \in I}(X \setminus F_\alpha)$ is open in X. Since

$$X \setminus (\cap_{\alpha \in I} F_\alpha) = \cup_{\alpha \in I}(X \setminus F_\alpha),$$

we conclude that $X \setminus (\cap_{\alpha \in I} F_\alpha)$ is open. Using Theorem 2.9 again, we conclude that $\cap_{\alpha \in I} F_\alpha$ is closed in X. This proves part (i) of the lemma.

The proof for part (ii) is similar, except that one uses Lemma 2.6 instead of Lemma 2.5. □

It is also possible to give a proof of the above lemma, directly using the definition of closed sets in Definition 2.10.

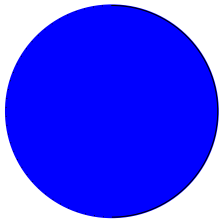

Figure 2.6: The solid black arc is part of V, but the dotted arc is not part of V.

2.1.7 Interior, isolated, limit, and boundary points in metric spaces

In a metric space (X, d), given a set $V \subset X$ and a point $x \in X$, the location of x in X relative the set V can be of several types. The simple case is if x belongs to V or not. But, one can also ask if all the balls around x meet V, or there is a ball about x which is contained in V, etc. We formalise these types in the next definition.

Definition 2.11. Let (X, d) be a metric space, $V \subset X$, and $x \in X$.

(i) The point x is called an **interior** point of V, or an **inner** point of V, if there is $\delta > 0$ such that $B_\delta(x) \subset V$.

(ii) The point x is called an **isolated** point of V, if there exists $\delta > 0$ such that $V \cap B_\delta(x) = \{x\}$. In other words, there is a δ-neighbourhood of x which does not contain any point of V except x.

(iii) The point x is called a **limit** point of V, or an **accumulation** point of V, if for every $\delta > 0$, $B_\delta(x) \cap V$ contains a point other than x. In other words, for every $\delta > 0$, $(B_\delta(x) \cap V) \setminus \{x\} \neq \emptyset$.

(iv) The point x is called a **boundary** point of V, if for every $\delta > 0$ we have $B_\delta(x) \cap V \neq \emptyset$ and $B_\delta(x) \setminus V \neq \emptyset$. In other words, x is a boundary point of V, if every δ-neighbourhood of x meets both V and the complement of V.

Note that in items (i) and (ii), any interior point and any isolated point of V is an element of V. But the limit point and the boundary point of a set V are not necessarily elements of V.

Example 2.16. Consider the Euclidean metric space $(\mathbb{R}^2, \mathrm{d}_2)$, and the set

$$V = \big\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| \leq 1, x \geq 0\big\} \bigcup \big\{(x, y) \in \mathbb{R}^2 \mid \|(x, y)\| < 1, x < 0\big\}.$$

You can see that (x, y) is an interior point of V if and only if $\|(x, y)\| < 1$. The set V has no isolated points. The point (x, y) is a limit point of V if and only if $\|(x, y)\| \leq 1$. The point (x, y) is a boundary point of V if and only if $\|(x, y)\| = 1$. Verify these statement.

Example 2.17. In the metric space $(\mathbb{R}, \mathrm{d}_1)$, consider the set

$$V = \{1/n \mid n \in \mathbb{N}\}.$$

Then, V has no interior point. Every point in V is an isolated point of V. The point 0 is the only limit point of V. Every point in V is a boundary point of V. But also the point 0, which is not in V, is a boundary point of V.

If $V = (0, 1] \cup \{2\}$ in $\mathbb{R}^1$, then a is an interior point of V if and only if $a \in (0, 1)$. The point 2 is the only isolated point of V. The point a is a limit point of V if and only if $a \in [0, 1]$. A point a is a boundary point of V if and only if $a \in \{0, 1, 2\}$.

Definition 2.12. Let (X, d) be a metric space, and $V \subset X$.

(i) The **interior** of V is defined as the set of all $v \in V$ such that v is an interior point of V. The interior of V is often denoted as V°.

(ii) The **closure** of V is the union of V and all the limit points of V. The closure of V is often denoted as $\overline{V}$.

(iii) The **boundary** of V is the set of all $v \in X$ such that v is a boundary point of V. The boundary of the set V is often denoted as ∂V.

Note that $\overline{V}$ consists of

(i) all elements of V,

(ii) all limit points of V which belong to V,

(iii) all limit points of V which do not belong to V.

Indeed, there is a simple equivalent definition of the closure of a set V in terms of balls. A point z belongs to the closure of V, if for every $\delta > 0$, $B_\delta(z) \cap V \neq \emptyset$.

Example 2.18. In the metric space $(\mathbb{R}^1, \mathrm{d}_1)$, we have $\mathbb{Q}^\circ = \emptyset$, $\overline{\mathbb{Q}} = \mathbb{R}$, and $\partial\mathbb{Q} = \mathbb{R}$. Also, $\mathbb{Z}^\circ = \emptyset$, $\overline{\mathbb{Z}} = \mathbb{Z}$, and $\partial\mathbb{Z} = \mathbb{Z}$.

Example 2.19. Let $V = (0, 1] \cup \{2\}$ in $(\mathbb{R}^1, \mathrm{d}_1)$. Then, $V^\circ = (0, 1)$, $\overline{V} = [0, 1] \cup \{2\}$, $\partial V = \{0, 1, 2\}$.

By the above definition, we note that a set V is open if and only if $V^\circ = V$.

Exercise 2.14. Let (X, d) be a metric space, and V be a subset of X. Show that the set V is closed if and only if $\overline{V} = V$.

Exercise 2.15. Let V and W be subsets of a metric space (X, d). The following properties hold:

(i) if $V \subset W$, then $V^\circ \subset W^\circ$,

(ii) if $V \subset W$, then $\overline{V} \subset \overline{W}$,

Exercise 2.16. Let V and W be subsets of a metric space (X, d). Prove that

$$\overline{V \cup W} = \overline{V} \cup \overline{W}.$$

Give an example of (X, d), V and W such that

$$(V \cup W)^\circ \neq V^\circ \cup W^\circ.$$

Lemma 2.11. *Let (X, d) be a metric space, and $V \subseteq X$. Then, $x \in X$ is a limit point of V if and only if there exists a sequence of points in $V \setminus \{x\}$ which converges to x.*

Proof. Assume that there is a sequence of points, say $(x_n)_{n \geq 1}$, in $V \setminus \{x\}$ which converges to x. We need to show that for every $\delta > 0$, $B_\delta(x) \cap V$ contains an element of V other than x. Fix an arbitrary $\delta > 0$. Because the sequence $(x_n)_{n \geq 1}$ converges to x, there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x_n \in B_\delta(x)$. As the sequence lies in $V \setminus \{x\}$, we conclude that x_N is distinct from x, and $x_N \in B_\delta(x)$. This completes the proof.

Now assume that x is a limit point of V. For each $n \in \mathbb{N}$, the number $\delta_n = 1/n$ is strictly positive. So, by the definition of limit points, $B_{1/n}(x) \cap V$ contains an element different from x. Let x_n be such an element. This process generates a sequence $(x_n)_{n \geq 1}$ in $V \setminus X$. We do not know that the points in the sequence x_1, x_2, x_3, ... are distinct points. But this does not matter for us. The sequence $(x_n)_{n \geq 1}$ converges to x since $\mathrm{d}(x_n, x) < 1/n$. □

Definition 2.13. Let (X, d) be a metric space.

- We say that a set $V \subseteq X$ is **dense** in X, if $\overline{V} = X$.
- We say that the metric space (X, d) is **separable**, if there is a countable set which is dense in X.

Example 2.20. In the metric space $(\mathbb{R}^1, \mathrm{d}_1)$, the set $\mathbb{Q}$ is countable and dense. So $(\mathbb{R}^1, \mathrm{d}_1)$ is separable.

In the metric space $(\mathbb{R}^n, \mathrm{d}_2)$ the set of all vectors with rational coordinates is countable and dense in $\mathbb{R}^n$.

Remark 2.4. By a classical theorem in analysis (Stone-Weierstrass theorem), any continuous function $f : [a, b] \to \mathbb{R}$ can be approximated by polynomials with real coefficients. In other words, the set of polynomials is dense in the metric space $(C([a, b]), \mathrm{d}_\infty)$. Since the set of polynomials with rational coefficients is countable and dense in the space of all polynomials with real coefficients, it follows that

$(C([a,b]), \mathrm{d}_\infty)$ is separable. You can see an elementary, but rather long, proof of this classic theorem in the Principles of Analysis by Rudin.

On the other hand, one can approximate any continuous function on $[0,1]$ by a (series) function of the form

$$\sum_{n=1}^{\infty} a_n \cos(2\pi n x) + b_n \sin(2\pi n x),$$

where a_n and b_n are real numbers. So they also form a dense subset of $(C([0,1]), \mathrm{d}_\infty)$. Functions of the above form are called Fourier series. There is an entire module called "Fourier Analysis and the Theory of Distributions" devoted to the properties of such functions.

Example 2.21.* Recall the space of all bounded sequences in $\mathbb{R}$ with the supremum metric d_∞. This metric space is not separable. To see that, Let E denote the set of all sequences of 0s and 1s (i.e. $00111010101010\dots$). You have already seen in Analysis I that E is uncountable.

Note that the d_∞ distance between any two distinct elements of E is equal to 1. Then, for distinct elements e_1 and e_2 in E, $B_{1/2}(e_1) \cap B_{1/2}(e_2) = \emptyset$. So any dense subset needs to have at least one element from each such ball, but there are an uncountable number of such balls. Hence, the dense subset can not be countable.

2.1.8 Continuous maps of metric spaces

Let us recall a terminology from basic set theory and maps.

Let $f : M \to N$. For any $m \in M$, $n = f(m) \in N$ is called the **image** of m under the map f. If A is a subset of M, the **image of** A under f is defined (and denoted) at

$$f(A) = \{f(m) \mid m \in A\}.$$

For a given $n \in N$, the set of elements $m \in M$ such that $f(m) = n$ is called the **pre-image** of n. This should be denoted as $f^{-1}(\{n\})$, but abusing the notation, it is often denoted as $f^{-1}(n)$. For any set $B \subseteq N$, the **pre-image** of B, is defined (and denoted) as

$$f^{-1}(B) = \{m \in M \mid f(m) \in B\}.$$

Of course it is possible that $f^{-1}(B) = \emptyset$, for some $B \subset N$.

Definition 2.14. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a map.

(i) We say that f is **continuous** at $x \in X$, if for every $\epsilon > 0$ there is $\delta > 0$ such that for every $x' \in X$ satisfying $\mathrm{d}_X(x', x) < \delta$ we have

$$\mathrm{d}_Y(f(x), f(x')) < \epsilon.$$

(ii) We say that $f : X \to Y$ is **continuous**, if f is continuous at every x in X.

(iii) We say that $f : X \to Y$ is **uniformly continuous**, if f is continuous at every $x \in X$, and $\delta = \delta(\epsilon)$ does not depend on x.

To emphasise the dependence of the notion of continuity on d_X and d_Y, we may say that f is continuous at x with respect to the metrics d_X and d_Y.

There is a remarkable equivalent criterion for continuity of maps between metric spaces. We state that as the next theorem.

Theorem 2.12. *Let (A_1, d_1) and (A_2, d_2) be metric spaces. A map $f : A_1 \to A_2$ is continuous if and only if the pre-image of any open set in A_2 is an open set in A_1.*

Proof. Let us first assume that f is continuous, and fix an arbitrary open set U in A_2. Take any $x \in f^{-1}(U)$, then $f(x) \in U$. As U is open in A_2, there is $\epsilon > 0$ such that $B_\epsilon(f(x)) \subset U$. As f is continuous $\exists \delta > 0$ such that $f(B_\delta(x)) \subset B_\epsilon(f(x)) \subset U$. Therefore $B_\delta(x) \subset f^{-1}(U)$. Since $x \in f^{-1}(U)$ was arbitrary, we deduce that $f^{-1}(U)$ is open.

Now assume that the pre-image of any open set is an open set. Let $x \in A_1$ and $\epsilon > 0$ be arbitrary elements. Consider the open set $B_\epsilon(f(x))$. By the assumption, $f^{-1}(B_\epsilon(f^{-1}(x)))$ is open. But $x \in f^{-1}(B_\epsilon(f(x)))$, so there is $\delta > 0$ such that $B_\delta(x) \subset f^{-1}(B_\epsilon(f(x)))$. That is, $f(B_\delta(x)) \subset B_\epsilon(f(x))$. Thus f is continuous at x. As x was arbitrary, we conclude that f is continuous on a_1. □

Exercise 2.17. Let (A_1, d_1) and (A_2, d_2) be metric spaces. A map $f : A_1 \to A_2$ is continuous if and only if the pre-image of any closed set in A_2 is a closed set in A_1.

Example 2.22. The function $f : \mathbb{R}^3 \to \mathbb{R}$ defined as

$$f(x, y, z) = x^2 + 10xy^3 + \sin(xy)$$

is continuous. Therefore, the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) \leq -1\},$$

is a closed set. The above set is the pre-image of the closed set $(-\infty, -1]$. Since f is continuous, by the above exercise, the pre-image of $(-\infty, -1]$ must be a closed set. For the same reason, the set

$$\{(x, y, z) \in \mathbb{R}^3 \mid f(x, y, z) \in (0, 1)\}$$

is an open set.

By exercise 2.17, we can easily verify the closed or openness of many sets in Euclidean spaces. For example,

$$\{x \in \mathbb{R}^n \mid \|x\| \in [1, 2]\}$$

is a closed set, and

$$\{x \in \mathbb{R}^n \mid \|x\| \in (5, \infty)\}$$

is an open set.

Theorem 2.13. *Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a map. The following statements are equivalent:*

(i) f is continuous at $x \in X$,

(ii) for any sequence $(x_n)_{n\geq 1}$ in X which converges to some $x \in X$, the sequence $(f(x_n))_{n\geq 1}$ converges to $f(x)$ in (Y, d_Y).

Proof. The proof is identical to the proof of this statement for higher dimensional Euclidean spaces. One only needs to replace the metric d_2 with the metrics d_X and d_Y is suitable places. □

Exercise 2.18. Recall that the set of all continuous functions from $[0, 1]$ to $\mathbb{R}$ is denoted by $C([0, 1])$. We also defined the metrics d_1 and d_∞. Consider the map

$$\Phi : C([0, 1]) \to \mathbb{R},$$

defined as

$$\Phi(f) = f(1/2).$$

(i) Is the map Φ from the metric space $(C([0, 1]), \mathrm{d}_\infty)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous?

(ii) Is the map Φ from the metric space $(C([0, 1]), \mathrm{d}_1)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous?

(iii) Is the map Φ from the metric space $(C([0, 1]), \mathrm{d}_2)$ to $(\mathbb{R}, \mathrm{d}_1)$ continuous?

Exercise 2.19. Consider the metric spaces $X = (\mathbb{R}, \mathrm{d}_1)$ and $Y = (\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$. Show that the map $f(x) = x$ from X to Y is not continuous. Show that the map $g(x) = x$ from Y to X is continuous.

Exercise 2.20. Consider the sequence of functions $f_n : [0, 1] \to \mathbb{R}$, for $n \geq 1$, defined as

$$f_n(x) = \begin{cases} 1 - nx & \text{if } x \in [0, 1/n] \\ 0 & \text{otherwise.} \end{cases}$$

Let $f : [0, 1] \to \mathbb{R}$ be the constant map $f \equiv 0$.

(i) show that the sequence $(f_n)_{n\geq 1}$ in $C([0, 1])$ converges to f in the metric space $(C([0, 1], \mathrm{d}_1)$.

(ii) show that the sequence $(f_n)_{n\geq 1}$ in $C([0, 1])$ does not converge to f in the metric space $(C([0, 1], \mathrm{d}_\infty)$.

(iii) conclude that the identity map

$$\mathrm{id} : (C([0,1]), \mathrm{d}_1) \to (C([0,1]), \mathrm{d}_\infty)$$

is not continuous.

Definition 2.15. Let (X_1, d_1) and (X_2, d_2) be metric spaces.

(i) A map $f : X_1 \to X_2$ is called a **homeomorphism**, if $f : X_1 \to X_2$ is a bijection and both of the maps $f : X_1 \to x_2$ and $f^{-1} : X_2 \to X_1$ are continuous.

(ii) Two metric spaces (X_1, d_1) and (X_2, d_2) are called **homeomorphic**, if there is a homeomorphism from X_1 to X_2.

Example 2.23. The sets $(-\infty, \infty)$ and $(-1, 1)$ with respect to the metric d_1 on $\mathbb{R}^1$ are homeomorphic. For example the map $f(x) = \arctan(x)$ is a homeomorphism between these two sets.

Definition 2.16. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$.

(i) We say that f is **Lipschitz**, if there is a constant $M > 0$ such that for all x_1 and x_2 in X, we have

$$\mathrm{d}_Y(f(x_1), f(x_2)) \leq M \cdot \mathrm{d}_X(x_1, x_2).$$

(ii) We say that f is **bi-Lipschitz**, if there are constant $M_1 > 0$ and $M_2 > 0$ such that for all x_1 and x_2 in X, we have

$$M_2 \cdot \mathrm{d}_X(x_1, x_2) \leq \mathrm{d}_Y(f(x_1), f(x_2)) \leq M_1 \cdot \mathrm{d}_X(x_1, x_2).$$

(iii) We say that f is an **isometry**, or **distance preserving**, if for every x_1 and x_2 in X, we have

$$\mathrm{d}_Y(f(x_1), f(x_2)) = \mathrm{d}_X(x_1, x_2).$$

Obviously, any isometry between metric spaces, is a Bi-lipschitz map (choose both constants 1). Also, any bi-Lipschitz map is injective.

Example 2.24. Let (S^1, d) be the metric space from Example 2.4, that is S^1 is the circle of radius 1 and d is the arc length between two points on S^1. Recall that every point on S^1 is equal to $(\cos(\theta), \sin(\theta))$, for a unique $\theta \in [0, 2\pi)$. For every $\alpha \in [0, 2\pi]$ we can consider the rotation by α on S^1, which may be defined as

$$R_\alpha (\cos(\theta), \sin(\theta)) = (\cos(\theta + \alpha), \sin(\theta + \alpha)).$$

For every $\alpha \in [0, 2\pi]$, the map $R_\alpha : S^1 \to S^1$ is an islometry.

Exercise 2.21. Let (X, d_X) and (Y, d_Y) be metric spaces, and $f : X \to Y$ be a surjective map. Show that if f is bi-Liptschitz, then it is a homeomorphisms.

2.2 Topological spaces

2.2.1 Motivation

In this section we are going to generalise the fundamental concepts of analysis, such as convergence of sequences and continuity of maps, to even wider settings. To understand what we are about to do, let us briefly look back at how the notion of a metric allowed us to define those fundamental concepts. We started with a set X, and a non-negative function on $X \times X$, called metric. We employed the metric to define balls around points in X, and then using those balls we defined open sets in X. So each metric on X gives rise to a collection of subsets of X which are called open sets. From there, we saw that the convergence of sequences, continuity of maps, etc, can be defined using open sets. See for instance, Exercise 2.12 and Theorem 2.12.

Isn't it easier to separate some subsets of X, call them open sets, and then use them to define the convergence and continuity in the same fashion. Through this approach, we avoid dealing with the notion of metric, which can be fairly complicated in general. This approach seems to be more natural, because it is based on the more basic objects; the subsets of X. Also, it is more direct, that is, we deal with things happening in X (such as convergence of sequences in X) using objects living in X.

There is also a practical side in making this generalisation. Although most of the spaces one comes across in mathematics are metric spaces, occasionally, one needs to work on some sets where there cannot be a natural notion of metric (for example, some function spaces). So this generalisation cannot be avoided.

Remark 2.5. As we will be using the word "set" and "subset" very often in this section, we will use the words "collection" and "class" to mean "set", and "subcollection" and "subclass" to mean "subset". So instead of saying "consider the set of all subsets of $\mathbb{R}$ such that .... ", we may prefer to say "consider the collection of all subsets of $\mathbb{R}$ such that ... ".

2.2.2 Topology on a set

Definition 2.17. Let A be an arbitrary set, and τ be a collection of subsets of A. We say that τ is a **topology** on A, if the following properties hold:

(T1) the empty set $\emptyset$, and the whole set A belong to τ,

(T2) if $G_\alpha \in \tau$, for α in a (finite or infinite) set I, then $\cup_{\alpha \in I} G_\alpha \in \tau$,

(T3) if $G_1, G_2, \dots, G_m$ belong to τ, then $\cap_{i=1}^m G_i \in \tau$.

A **topological space**, denoted as (A, τ), is a pair of a set A and a topology τ on A. Every element of A is called a **point**, and every element of τ is called an

open set in (A, τ). For a point $a \in A$, we say that U is a **neighbourhood** of a, if U is open (belongs to τ) and $a \in U$.

It is possible to define a topology on any set, as we see in the next examples.

Example 2.25. Let A be an arbitrary set, and $\tau = \{\emptyset, A\}$. It is easy to see that τ satisfies the three properties T1, T2, and T3 in the definition of topology. The collection τ is called the **coarse** topology on A.

Example 2.26. Let A be an arbitrary set, and let τ be the collection of all subsets of A. Evidently, τ satisfies the three properties T1, T2, and T3. In this topology, every subset of A is open. This topology on A is called the **discrete** topology.

Below we give some non-trivial examples of topologies.

Example 2.27. Let $A = \{a, b\}$, where a and b are the letters "a" and "b" (so they are distinct), and let

$$\tau = \{\emptyset, \{a, b\}, \{b\}\} .$$

It is easy to see that τ satisfies T1, T2, and T3, so it is a topology on A. The only open sets in this topology are the empty set, A and the set $\{b\}$. So A is the only open set containing a, and hence any open set containing a also contains b. The collection τ is called the **Sierpinski** topology, and the pair (A, τ) is called the Sierpinski topological space. Note that this topology is not equal to the coarse topology, and also it is not equal to the discrete topology.

Example 2.28. Let $A = \mathbb{R}$ and let τ be the collection of all subsets of $\mathbb{R}$ of the form $(a, +\infty)$ for some $a \in \mathbb{R} \cup \{+\infty, -\infty\}$. Here we assume that $(+\infty, +\infty)$ is the empty set. You can verify that this collection satisfies the properties T1, T2, and T3, so τ is a topology on $\mathbb{R}$. This is called the **order** topology on $\mathbb{R}$.

Example 2.29. Let X be an arbitrary set, and let

$$\tau = \{V \subset X \mid \mathrm{Card}(X \setminus V) < +\infty, \text{ or } V = \emptyset\}.$$

That is each set in τ is either empty, or its complement has a finite number of elements. You can see that this set satisfies the properties T1, T2, and T3. This topology on X is called the **co-finite** topology.

The following example shows that the topological spaces are, in a sense, generalisation of metric spaces.

Example 2.30. Let (X, d) be a metric space, and let τ be the collection of all open sets in (X, d). By Lemma 2.4, the empty set and the whole set X are open, so they belong to τ. This shows that T1 holds. By Lemma 2.5, the union of any arbitrary number of open sets in (X, d) is open, so property T2 holds. Similarly, by

Lemma 2.6, the intersection of any finite number of open sets in (X, d) is open in (X, d). Hence property T3 holds. Therefore, τ is a topology on X.

The topology τ on X is called the **induced** topology from the metric d.

The topology induced on $\mathbb{R}^n$ from the metric d_2 is called the **Euclidean topology** on $\mathbb{R}^n$. We note that since the metrics d_1, d_2 and d_∞ are equivalent, they all induce the same metric on $\mathbb{R}^n$.

As we explained in the above example, every metric on X naturally induces a topology on X (the induced topology). But, this is not a reversible process. First of all, distinct metrics on a set X may induce the same topology on X. For example, if d_1 and d_2 are topologically equivalent metrics on X, then they induce the same topology. Therefore, we cannot associate a unique metric to each topology. One might ask whether for every topology τ on X, there is a metric d on X which induces τ on X. For example, you can verify that the discrete topology on X is induced from the discrete metric on X. We say that a topological space (X, τ) is **metrisable**, if there is a metric on X which induces the topology τ.

Remark 2.6. In general, it is a difficult problem to find out if a given topology on a set is metrisable. There are important theorems in topology called metrisation theorems (such as Urysohn's metrisation theorem), which provide sufficient conditions for a topology to be metrisable. You can learn more about this topic if you take the module on Differential Topology, or Algebraic Topology.

Exercise 2.22. Consider a discrete metric space $(X, \mathrm{d}_{\mathrm{disc}})$, that is $\mathrm{d}_{\mathrm{disc}}$ is a discrete metric on X. Show that $\mathrm{d}_{\mathrm{disc}}$ induces the discrete topology on X.

There are standard approaches to define new topologies using old ones. We explain two of these approaches below.

Example 2.31. Let (X, τ) be a topological space, and let Y be a subset of X. Consider the collection of sets

$$\tau_Y = \{U \cap Y \mid U \in \tau\}.$$

This is a collection of subsets of Y, and one can verify that τ_Y is a topology on Y. In other words, τ_Y satisfies properties T1, T2, and T3. The topology τ_Y is called the **induced** topology on Y from (X, τ). We may also say that (Y, τ_Y) has the **subspace topology** induced from (X, τ).

Exercise 2.23. Let (X, τ), Y, and τ_Y be as in Example 2.31. Show that τ_Y is a topology on Y.

Example 2.32. Assume that (X, τ) and (Y, μ) are two topological spaces. Consider the product set

$$X \times Y = \{(x, y) \mid x \in X, y \in Y\}.$$

Let $\tau * \mu$ be the collection of all sets $\Omega \subseteq X \times Y$ such that for every $(x, y) \in \Omega$, there are $U_x \in \tau$ and $V_y \in \mu$ such that $x \in U_x$, $y \in V_y$ and $U_x \times V_y \subseteq \Omega$.

One can see that the collection $\tau * \mu$ is a topology on $X \times Y$. This is called the **product** topology on $X \times Y$.

To define a topology on $X \times Y$, one might wish to simply take the sets of the form $U \times V$, such that $U \in \tau$ and $V \in \mu$. By the next exercise, you can see that this does not work in general.

Exercise 2.24. Let τ_{Eucl} be the Euclidean topology on $\mathbb{R}$, that is τ_{Eucl} is the collection of all open sets in $(\mathbb{R}, \mathrm{d}_1)$. Show that the collection

$$\{U \times V \mid U \in \tau_{\text{Eucl}}, V \in \tau_{\text{Eucl}}\}.$$

is not a topology on $\mathbb{R} \times \mathbb{R}$. Is condition T2 satisfied? How about condition T3?

Definition 2.18. Let A be a set, and τ_1 and τ_2 be two topologies on A. We say that the topology τ_1 is **stronger** (or **finer**) than τ_2, if $\tau_2 \subset \tau_1$.

Example 2.33. For every set A, the coarse topology on A is the weakest (the least strong) topology on A, and the discrete topology on A is the strongest topology on A.

Note that it is not always possible to compare two topologies on a given set A in the sense of Definition 2.18. That is, there may be topologies τ_1 and τ_2 on a set A such that neither τ_1 is stronger than τ_2, nor τ_2 is stronger than τ_1. For example, let

$$A = \{a, b\}, \quad \tau_1 = \{\emptyset, \{a, b\}, \{a\}\}, \quad \tau_2 = \{\emptyset, \{a, b\}, \{b\}\}.$$

Recall that in a topological space (X, τ), members of τ are called open sets. This is in analogy with the way we defined open sets in metric spaces using balls (see Definition 2.7).

Lemma 2.14. *Let (A, τ) be a topological space. A set $G \subseteq A$ is open in A if and only if for all $x \in G$ there is a neighbourhood of x contained in G.*

Proof. Let us first assume that G is open. Since G is an open set in A, we have $G \in \tau$. Thus, for every $x \in G$, G is a neighbourhood of x, and G is a subset of G.

On the other hand, assume that there is a set $G \subset X$ such that for every $x \in G$ there exists a neighbourhood G_x contained in G. By property T2, $\cup_{x \in G} G_x$ belongs to τ, and hence it is an open set. Since $G = \cup_{x \in G} G_x$, we conclude that G is an open set. $\square$

Definition 2.19. Let (A, τ) be a topological space, and Ω be a subset of A. A point $z \in \Omega$ is called an **interior point** of Ω, if there is $U \in \tau$ such that $z \in U$ and $U \subset \Omega$. The **interior** of the set Ω is defined as the set of all $z \in \Omega$ such that z is an interior point of Ω. The interior of Ω is denoted by Ω°.

It follows from the above definition that the interior of any set is a subset of that set. That is, if $S \subseteq A$, then $S^\circ \subseteq S$.

Exercise 2.25. Let (A, τ) be a topological space, and let S and T be subsets of A. The following properties hold:

(i) if $S \subset T$ then $S^\circ \subset T^\circ$,

(ii) S is open in A if and only if $S = S^\circ$,

(iii)* S° is the largest open set contained in S.

2.2.3 Convergence, and Hausdorff property

Definition 2.20. Let (A, τ) be a topological space, and $(x_n)_{n=1}^\infty$ be a sequence in A. We say that $(x_n)_{n=1}^\infty$ **converges** in (A, τ), if there is $x \in A$ satisfying the following property: for any $G \in \tau$ with $x \in G$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in G$.

When this occurs, we say that x_n converges to x as n tends to ∞, or write $\lim_{n\to\infty} x_n = x$.

Example 2.34. Let (A, τ) be a topological space, with τ the coarse topology on A. Then any sequence in A is convergent, and converges to any element in A.

On the other hand, if τ is the discrete topology on A, then a sequence $(x_n)_{n\in\mathbb{N}}$ is convergent if and only if, the sequence is eventually constant.

The above example shows that behaviour of sequences in a topological space may be strange, and counter intuitive. For example, it shows that the limit of a convergent sequence may not be unique.

Definition 2.21. A topological space (A, τ) is called **Hausdorff**, if the following property holds: For every x and y in A with $x \neq y$, there are open sets U and V such that $x \in U$, $y \in V$, and $U \cap V = \emptyset$. In this case we say that U and V **separate** x and y.

Example 2.35. Consider the set $A = \{a, b, c\}$, and

$$\tau = \big\{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}\big\}.$$

You can shows that τ is a topology on A. The space (A, τ) is not Hausdorff, since b and c cannot be separated. The only open set in A which contains c is $\{a, b, c\}$, and that set also contains b.

Exercise 2.26. Let (X, d) be a metric space, and let τ be the topology on X induced from the metric d. Show that (X, τ) is a Hausdorff topological space.

The important property of Hausdorff spaces is stated in the next theorem.

Theorem 2.15. *Let (A, τ) be a Hausdorff topological space, and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A. If the sequence $(x_n)_{n\in\mathbb{N}}$ converges in (A, τ), then its limit is unique.*

Proof. Assume in the contrary that there are distinct points x and y in A such that

$$\lim_{n\to\infty} x_n = x, \text{ and } \qquad \lim_{n\to\infty} x_n = y.$$

Because (A, τ) is a Hausdorff space, there are open sets G_x and G_y such that $x \in G_x$, $y \in G_y$, and $G_x \cap G_y = \emptyset$. Since the sequence x_n converges to x, there is $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, we have $x \in G_x$. Similarly, there is $N_y \in \mathbb{N}$ such that for all $n \geq N_y$ we have $x_n \in G_y$. Now, for $n = \max\{N_x, N_y\}$, we have $x_n \in G_x$ and $x_n \in g_y$. This contradicts $G_x \cap G_y = \emptyset$. □

2.2.4 Closed sets in topological spaces

It is possible to give a definition of closed sets in a topological space in the same fashion as we defined closed sets in a metric space (refer to Definition 2.10). However, for technical reasons, in a topological space, one has to consider the limit points of the set itself rather than the limit points of sequences in the set. It is convenient to use the criterion in Theorem 2.9 for the definition of closed sets for topological spaces, while we show in a moment that a set in a topological space is closed if and only if it contains its limit points (see Lemma 2.19-(ii) and Remark 2.7).

Definition 2.22. Let (A, τ) be a topological space, and let $V \subseteq A$. We say that V is **closed** in (A, τ), if $A \setminus V$ is open in (A, τ). That is, V is closed in (A, τ) if and only if $A \setminus V \in \tau$.

Theorem 2.16. *This is not a theorem, this is only inserted to make the numberings of theorem, lemmas, etc, in the typed notes consistent with the ones in the hand written notes.*

Lemma 2.17. *Let (A, τ) be a topological space. Then, the empty set and the set A are closed in (A, τ). Moreover, we have*

(i) the intersection of any number of (finite, countable, uncountable) closed sets in (A, τ) is a closed set in (A, τ),

(ii) the union of any finite number of closed sets in (A, τ) is a closed set in (A, τ).

Proof. This follows from Definition 2.22, and the properties T1, T2, and T3 of topology, by taking complements. See the proof of Lemma 2.10 for a similar argument. □

Lemma 2.18. *Let (A, τ) be a Hausdorff topological space, and $a \in A$. Prove that the set $\{a\}$ is a closed set.*

Proof. For any $b \in A$ with $b \neq a$, there are open sets G_b and G_a such that $b \in G_b$, $a \in G_a$, and $G_a \cap G_b = \emptyset$. Then, by Definition 2.22, $A \setminus G_b$ is a closed set. By Lemmas 2.17, the intersection

$$\bigcap_{b \in A \setminus \{a\}} (A \setminus G_b)$$

is a closed set. Since for every $b \in A \setminus \{a\}$, $A \setminus G_b$ contains a and does nor contain b, the above intersection is equal to $\{a\}$. This completes the proof. $\square$

Definition 2.23. Let (A, τ) be a topological space, and S be a subset of A. A point $x \in A$ is called a **limit** point of S, or an **accumulation** point of S, if the following property holds: for any neighbourhood U of x, U contains a point in S different from x. In other words, for any neighbourhood U of x, we have $(S \cap U) \setminus \{x\} \neq \emptyset$. Note that the point x may not be in S.

The **closure** of S is defined as the set of all points in S and all limit points of S. The closure of S is denoted by $\overline{S}$. Obviously, for any set $S \subset A$, $S \subset \overline{S}$.

Example 2.36. Let τ be the Sierpinski topology on $A = \{a, b\}$. The constant sequence $b, b, b, b, \dots$ converges to the point a (and also to b) in this topology. That is because, the only open set in (A, τ) which contains a is A. Obviously, all points in the sequence belongs to $\{b\} \subset A$. This implies that the closure of the set $\{b\}$ is A.

Lemma 2.19. *Let (A, τ) be a topological space, and assume that S and T are subsets of A. The following properties hold:*

(i) if $S \subset T$, then $\overline{S} \subset \overline{T}$,

(ii) S is closed in (A, τ) if and only if $S = \overline{S}$,

Remark 2.7. One can take the statement in part (ii) of the above lemma as the definition of closed sets in a topological space. In other words, V is closed, if it contains all the limit points of V. This is in the spirit of how we defined closed sets in metric spaces, but it is not identical to that. If a set V is closed in a topological space (A, τ), in particular, for any sequence in V which converges to some point in A, the limit of the sequence must belong to V. That is because the limit of the sequence in V belongs to the limit set of V. However, one has to note that limits of sequences are not necessarily unique in topological spaces. By considering the limit points of the set we avoid discussing the notion of Hausdorff property when defining closed sets.

Proof of Lemma 2.19. Part (i): Let $x \in \overline{S}$ be an arbitrary point. Any neighbourhood of x contains a point in S and hence a point in T. This implies that $x \in \overline{T}$.

Part (ii): First assume that S is closed. Let us suppose in the contrary that $S \neq \overline{S}$. Then there is $x \in \overline{S}$ such that $x \notin S$. This implies that $x \in A \setminus S$. Since S is closed, then $A \setminus S$ is open. This open set contains x, but does not contain any element of S. So x cannot be a limit point of S, which is a contradiction.

Now assume that $S = \overline{S}$. Assume in the contrary that S is not closed. Then $A \setminus S$ is not open. This implies that there exists $x \in A \setminus S$ such that any neighbourhood G_x of x is not contained in $A \setminus S$. Thus, G_x contains an element of S. This implies that $x \in \overline{S}$, and hence $x \in S$. This is a contradiction. □

2.2.5 Continuous maps on topological spaces

Definition 2.24. Let (X, τ_X) and (Y, τ_Y) be two topological spaces, and $f : X \to Y$ be a map. We say that f is **continuous** on X, if for any open set U in Y, $f^{-1}(U)$ is open in X.

Note that the continuity of f in the above definition does not just depend on f but also on the topologies on X and Y. This is illustrated in the next example.

Example 2.37. Let (X, τ_X) and (Y, τ_Y) be topological spaces.

(i) If τ_X is the discrete topology on X, then any $f : X \to Y$ is continuous.

(ii) If τ_Y is the coarse topology on Y, then any $f : X \to Y$ is continuous.

We have the following equivalent criterion for the continuity.

Theorem 2.20. *Let (X, τ_X) and (Y, τ_Y) be two topological spaces. Then, $f : X \to Y$ is continuous if and only if the pre-image of any closed set in Y is closed in X.*

Proof. First note that for any set V in Y, we have $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. Now, the theorem follows from theorem 2.9. □

Theorem 2.21. *Let (X, τ_X), (Y, τ_Y) and (Z, τ_Z) be topological spaces, and assume that $f : X \to Y$ and $g : Y \to Z$ are continuous. Then, $g \circ f : X \to Z$ is continuous.*

Proof. This easily follows from the definition of continuity. □

Lemma 2.22. *Let (X, τ_X) and (Y, τ_Y) be topological spaces, and $y \in Y$. The constant map $f : X \to Y$ defined as $f(x) = y$, for all $x \in X$, is continuous.*

Proof. Let $U \subseteq Y$ be an arbitrary open set. Then

$$f^{-1}(U) = \begin{cases} \emptyset & \text{if } y \notin U \\ X & \text{if } y \in U. \end{cases}$$

Since the empty set and the whole set are open in any topology, we conclude that $f^{-1}(U)$ is open in X. Because U was arbitrary, we conclude that f is continuous. $\square$

Definition 2.25. Let (X, τ_X) and (Y, τ_Y) be topological spaces, and $f : X \to Y$.

(i) We say that $f : X \to Y$ is a **homeomorphism**, if $f : X \to Y$ is a bijection, and both maps $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous.

(ii) the topological spaces (X, τ_X) and (Y, τ_Y) are called **topologically equivalent**, or **homeomorphic**), if there is a homeomorphism from X to Y.

Note that topological equivalence gives an equivalence relation on the set of topological spaces.

Example 2.38. In the Euclidean space $\mathbb{R}$, for every $a < b$,

(i) the sets $[a, b]$ and $[0, 1]$ are homeomorphic, by the map $x \mapsto (x - a)/(b - a)$ from $[a, b]$ to $[0, 1]$,

(ii) the sets (a, b) and $(0, 1)$ are homeomorphic, by the map $x \mapsto (x - a)/(b - a)$,

(iii) the sets $(-\infty, +\infty) = \mathbb{R}$ and $(-1, 1)$ are homeomorphic, by the map $x \mapsto \tan(\pi x/2)$,

(iv) the sets $(0, +\infty)$ and $(0, 1)$ are homeomorphic by the map $x \mapsto x/(x + 1)$.

(v) the sets $(-\infty, +\infty)$ and $(0, +\infty)$ are homeomorphic, by the map $x \mapsto e^x$.

(vi) the sets $[0, 1)$ and $(0, 1]$ are homeomorphic by the map $x \mapsto -x + 1$.

Exercise 2.27. Assume that the topological spaces (X, τ_X) and (Y, τ_Y) are topologically equivalent. Then, (X, τ_X) is Hausdorff if and only if (Y, τ_Y) is Hausdorff.

From here onward, we will only study metric spaces, as they are fairly general and capture almost all settings you will come across in mathematics. However, we will present most of the definitions, statements and proofs using open sets in the metric space. Thus, most definitions, statements, and proofs can be readily presented for topological spaces, replacing open sets in the metric with elements of the topology.

2.3 Connectedness

The intermediate value theorem is one of the main features in real analysis with many applications. If $f : [a, b] \to \mathbb{R}$ is a continuous function, and there are α and β in $[a, b]$ such that $f(\alpha) < 0$ and $f(\beta) > 0$, then there must be γ between α and β such that $f(\gamma) = 0$. What is it about the domain $[a, b]$, or the range $\mathbb{R}$, or the continuity of the map f which makes this theorem work? Is there any way to extend this useful statement to more general settings. This is the purpose of this section, and we will see that there is indeed a natural way to extend this property to more general settings.

2.3.1 Connected sets

Definition 2.26. Let (X, d) be a metric space, and consider a subset $T \subseteq X$. We say that T is **disconnected**, if there are open sets U and V in X satisfying the following properties:

(i) $U \cap V = \emptyset$,

(ii) $T \subseteq U \cup V$,

(iii) $T \cap U \neq \emptyset$ and $T \cap V \neq \emptyset$.

In particular, X is disconnected, if there are two open sets in X which are non-empty, disjoint, and their union is equal to X.

Intuitively, the above definition suggests that T is disconnected, if it can be separated into more than one piece using open sets. The separate pieces are $T \cap U$ and $T \cap V$.

Example 2.39. Consider the set $\mathbb{R}^2$ with the Euclidean metric d_2. Let

$$T = \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y = -1\} \cup \{(x, y) \in \mathbb{R}^2 \mid x \in [-1, 1], y = 1\}.$$

That is, T consists of two horizontal line segments in the plane. Intuitively, we see T as having more than one piece. Indeed, T is disconnected. For example, let

$$U = \{(x, y) \in \mathbb{R}^2 \mid x \in (-2, 2), y \in (-5/4, -3/4)\},$$

$$V = \{(x, y) \in \mathbb{R}^2 \mid x \in (-2, 2), y \in (3/4, 5/4)\}.$$

The sets U and V are open in $\mathbb{R}^2$, $U \cap V = \emptyset$,

$$U \cap T = [-1, 1] \times \{-1\} \neq \emptyset, V \cap T = [-1, 1] \times \{1\} \neq \emptyset.$$

We also have $T \subseteq U \cup V$.

Note that being disconnected does not only depend on the set T, but it crucially depends on the metric d on X. We illustrate this by the following example.

Example 2.40. Let $(X, \mathrm{d}_{\mathrm{disc}})$ be a discrete metric space, and assume that X has at least two elements. Then, X is disconnected. To see this, recall that in the discrete topology, any subset of X is open. Let $x \in X$ be an arbitrary elements. Define $U = \{x\}$ and $V = X \setminus \{x\}$. Then, since X has at least two points, V must be non-empty. Then, U and V satisfy the three properties in the definition of disconnectedness.

Definition 2.27. Let (X, d) be a metric space, and let $T \subset X$ be an arbitrary subset. We say that T is **connected**, if T is not disconnected. Equivalently, T is connected, if for every pair of open sets U and V in X satisfying $U \cap V = \emptyset$ and $T \subseteq U \cup V$, we must have either $U \cap T = \emptyset$ or $T \cap V = \emptyset$.

In particular, the whole set X is connected, if for every pair of open sets U and V satisfying $U \cup V = X$ and $U \cap V = \emptyset$, we must have either $U = \emptyset$ or $V = \emptyset$.

Exercise 2.28. Let (X, d) be a metric space. Show that X is connected if and only if the only subsets of X which are both open and closed are X and $\emptyset$.

Example 2.41. Consider the set of real numbers with the Euclidean metric, and let $a \in \mathbb{R}$. Then the set $\mathbb{R} \setminus \{a\}$ is not connected (disconnected).

Let $U = (-\infty, a)$ and $V = (a, +\infty)$. Clearly, U and V are open, non-empty, disjoint, and their union covers $\mathbb{R} \setminus \{a\}$.

Exercise 2.29. Show that in the Euclidean metric space $(\mathbb{R}^1, \mathrm{d}_1)$, the set of rational numbers $\mathbb{Q}$ is disconnected.

Lemma 2.23. *Let (X, d) be a metric space, and $T \subseteq X$. Then, T is disconnected if and only if there exists a continuous map $f : T \to \mathbb{R}$ satisfying $f(T) = \{0, 1\}$.*

Proof. First assume that such a map f exists. Let $U = f^{-1}(0)$ and $V = f^{-1}(1)$. Since $f(T) = \{0, 1\}$, $U \neq \emptyset$ and $V \neq \emptyset$. Also, since f is continuous, $U = f^{-1}(0) = f^{-1}(-1/2, 1/2)$ and $V = f^{-1}(1) = f^{-1}(1/2, 3/2)$ are open sets. Moreover, as $f(T) = \{0, 1\}$, $T \subseteq U \cup V$. Obviously, $U \cap V = \emptyset$. These imply that T is disconnect.

Now assume that T is disconnected. By definition, there are non-empty, disjoint, open sets U and V in X such that $T \subseteq U \cup V$, $U \cap T \neq \emptyset$ and $V \cap T \neq \emptyset$. Let us define the map $f : T \to \mathbb{R}$ as

$$f(x) = \begin{cases} 0 & \text{if } x \in U \cap T, \\ 1 & \text{if } x \in V \cap T. \end{cases}$$

Since $(U \cap T) \cap (V \cap T) = \emptyset$, the above conditions make sense, and since $T \subset U \cup V$, the map f is defined on T. We need to show that f is continuous on T.

Let x be an arbitrary point in T and let $(x_n)_{n\geq 1}$ be a sequence in T which converges to x. Since $T \subset U \cup V$ and $U \cap V = \emptyset$, x belongs to one of U and V. Without loss of generality, assume that $x \in U$. Since U us open, by the definition of converges of sequences, there is $N \in \mathbb{N}$ such that for all $n \geq N$, we have $x_n \in U$. Thus, for all $n \geq N$, $f(x_n) = 0$. This implies that the sequence $(f(x_n))_{n\in\mathbb{N}}$ converges to $0 = f(x)$. Therefore, f is continuous at x. Since x was arbitrary in T, we conclude that f is continuous on T. $\square$

It is easier to show that a set is disconnected than to show that it is connected. In the former case, it is enough to find examples of two open sets with those properties. But in the latter case, one needs to show that such pairs do not exist. Of course that becomes a difficult task if there are two many open sets in the metric. You can see this below, as we try to prove that the interval $[a, b]$ is connected.

By an interval in $\mathbb{R}$ we mean any of the sets (a, b), $(a, b]$, $[a, b)$, $[a, b]$, $(-\infty, +\infty)$, $(-\infty, b)$, $(-\infty, b]$, $(a, +\infty)$, or $[a, +\infty)$, for some a and b in $\mathbb{R}$.

Lemma 2.24. *Let $S \subseteq \mathbb{R}$ be a non-empty set. Then, S is an interval if and only if for all x and y in S and all $z \in \mathbb{R}$ satisfying $x < z < y$ we have $z \in S$.*

Proof. If S is an interval, then by the definition of an interval, the latter side of the lemma holds.

Now assume that the latter side of the theorem holds. If S is not bounded from above, we define $b = +\infty$, and if S is bounded from above, we define $b = \sup S$. Similarly, if S is not bounded from below, we define $a = -\infty$, and if S is bounded from below, we let $a = \inf S$.

Let us first show that the open interval $(a, b) \subseteq S$. To see this, fix an arbitrary $z \in (a, b)$. Since $z < b$, z cannot be an upper bound for S (otherwise, $\sup S \leq z$). Therefore, there is $b' \in S$ such that $b' > z$. Similarly, since $z > a$, z cannot be a lower bound for S (otherwise $\inf S \geq z$). Therefore, there is $a' \in S$ such that $a' < z$. Combining these together, we have $a' < z < b'$, $a' \in S$, and $b' \in S$. By the assumption in the latter side of the theorem, we must have $z \in S$. Because $z \in (a, b)$ was arbitrary, we conclude that $(a, b) \subseteq S$.

Note that the supremum and infimum of a set do not have to be in the set itself. There are several possibilities for the set S depending on whether each of a and b belongs to S or not. (of course if $a = -\infty$ or $b = +\infty$, they cannot be in S). Then,

$$S = \begin{cases} [a, b] & \text{if } a \in S \text{ and } b \in S, \\ [a, b) & \text{if } a \in S \text{ and } b \notin S, \\ (a, b] & \text{if } a \notin S \text{ and } b \in S, \\ (a, b) & \text{if } a \notin S \text{ and } b \notin S. \end{cases}$$

$\square$

Theorem 2.25. *Consider the Euclidean metric space* $(\mathbb{R}, \mathrm{d}_1)$ *and let* $S \subseteq \mathbb{R}$*. If* S *is connected, then* S *is an interval.*

Proof. Suppose S is connected, but it is not an interval. By Lemma 2.24, there exist x and y in S and $z \in \mathbb{R}$ such that $x < z < y$ and $z \notin S$.

Consider the sets $U = (-\infty, z)$ and $V = (z, +\infty)$. Then, the sets U and V are open in $\mathbb{R}^1$, $U \cap V = \emptyset$, $S \subseteq U \cup V$, and $U \cap S \neq \emptyset$ (since it contains x) and $V \cap S \neq \emptyset$ (since it contains y). These show that U and V disconnect S, which is a contradiction. □

Theorem 2.26. *For every* a *and* b *in* $\mathbb{R}$ *with* $a < b$*, the interval* $[a, b]$ *is connected in the metric space* $(\mathbb{R}, \mathrm{d}_1)$*.*

Proof. Let us assume that $[a, b]$ is disconnected. Then, there must be open sets U and V in $\mathbb{R}$ such that

$$U \cap [a, b] \neq \emptyset, \quad V \cap [a, b] \neq \emptyset, \quad [a, b] \subset U \cup V, \quad U \cap V = \emptyset.$$

Since $a \in U \cup V$, we must have either $a \in U$ or $a \in V$. By relabelling U and V if necessary, we may assume that $a \in U$. Consider the set

$$I = \{s \in [a, b] \mid [a, s] \subseteq U\}.$$

As $a \in I$, the set I is not empty, and since $I \subseteq [a, b]$, I is bounded from above. Therefore, I has a supremum, which we denote by t. Note that $t \in [a, b]$, and t may or may not be in I. We consider three cases below.

(I) Assume that $t \in I$ and $t = b$. These imply that $[a, b] \subset U$, which is a contradiction, since $[a, b] \cap V \neq \emptyset$ and $U \cap V = \emptyset$.

(II) Assume that $t \notin I$. This implies that $t \notin U$, $t \neq a$ and $[a, t) \subset U$. As $t \in [a, b]$ and $[a, b] \subset U \cup V$, we must have $t \in V$. Now, since V is an open set in $\mathbb{R}$, there is $\delta > 0$ such that $(t - \delta, t + \delta) \subset V$. As $U \cap V = \emptyset$, we must have $(t - \delta, t + \delta) \cap U = \emptyset$. This contradicts $[a, t) \subset U$.

(III) Assume that $t \neq b$. We either have $t \in U$ or $t \in V$. If $t \in U$, by the openness of U, there is $\delta' > 0$ such that $(t - \delta', t + \delta') \subset U$. This contradicts $t = \sup I$. If $t \in V$, by the openness of V, there is $\delta'' > 0$ such that $(t - \delta'', t + \delta'') \subset V$. Thus $(t - \delta'', t + \delta'') \cap U = \emptyset$. This contradicts $t = \sup I$. □

Exercise 2.30.* Consider the Euclidean metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that a and b are real numbers with $a < b$.

(i) Show that the interval $[a, b)$ is connected.

(ii) Show that the interval $(a, b]$ is connected.

(iii) Show that the interval (a, b) is connected.

2.3.2 Continuous maps and connected sets

Theorem 2.27. *Let (A_1, d_1) and (A_2, d_2) be metric spaces, and $f : A_1 \to A_2$ be a continuous map. If $S \subset A_1$ is connected, then $f(S)$ is connected.*

Proof. Let us assume in the contrary that $f(S)$ is not connected. Then, there are open sets U and V in A_2 such that

$$U \cap V = \emptyset, \quad f(S) \subset U \cup V, \quad f(S) \cap U \neq \emptyset, \quad f(S) \cap V \neq \emptyset.$$

Since f is continuous, the sets $U' = f^{-1}(U)$ and $V' = f^{-1}(V)$ are open in A_1. Moreover, we have

$$U' \cap V' = \emptyset, \quad S \subset U' \cup V', \quad S \cap U' \neq \emptyset, \quad S \cap V' \neq \emptyset.$$

These show that S is not connected in (A_1, d_1), which is a contradiction. □

Corollary 2.28. *Assume that $f : (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is a homeomorphism. Then X is connected if and only if Y is connected.*

Theorem 2.29. *Let (X, d) be a connected metric space, and let $f : X \to \mathbb{R}$ be a continuous map. Assume that there are a and b in X satisfying $f(a) < 0$ and $f(b) > 0$. Then, there is $c \in X$ such that $f(c) = 0$.*

Proof. Assume in the contrary that there is no $c \in X$ satisfying $f(c) = 0$. Consider the sets

$$U = f^{-1}((-\infty, 0)), V = f^{-1}((0, +\infty)).$$

These are subsets of X. As f is continuous, and the sets $(-\infty, 0)$ and $(0, +\infty)$ are open in $\mathbb{R}$, the sets U and V are open in (X, d). Obviously, $U \cap V = \emptyset$. Moreover, $U \neq \emptyset$ since $a \in U$, and $V \neq \emptyset$ since $b \in V$. Also, since there is no $c \in X$ satisfying $f(c) = 0$, $U \cup V = X$. These show that X is disconnected, contradicting the hypothesis in the theorem. □

The connectedness of the domain X is a necessary condition for the intermediate value theorem for arbitrary metric spaces. To see that, assume that X is a disconnected topological space. By Lemma 2.23 there is a continuous and surjective map $f : X \to \{0, 1\}$. Consider the map $f - 1/2$, which takes both values $+1/2$ and $-1/2$, but does not take the value 0 at any point in X.

Corollary 2.30. *Let $f : [a, b] \to \mathbb{R}$ be a continuous map, and assume that there are x and y in $[a, b]$ satisfying $f(x) < 0$ and $f(y) > 0$. Then, there is $z \in [a, b]$ such that $f(z) = 0$.*

Proof. This immediately follows from Theorem 2.29 and Theorem 2.26. □

Example 2.42. The intervals $(0, 1)$ and $(0, 1]$ are not homeomorphic. Assume in the contrary that there is a homeomorphism $f : (0, 1) \to (0, 1]$. Let $x = f^{-1}(1) \in (0, 1)$. Then, $f : (0, 1) \setminus \{x\} \to (0, 1)$ is a homeomorphism. This contradicts 2.28, since $(0, 1)$ is connected but $(0, 1) \setminus \{x\}$ is nor connected.

By a similar argument, one can show that the pair of intervals $(0, 1)$ and $[0, 1]$, as well as the pair of interval $(0, 1]$ and $[0, 1]$ are not homeomorphic.

2.3.3 Path connected sets

We already mentioned that in general it is easier to show that a set is disconnected than to show that it is connected. In this section we aim to provide a constructive criterion to show that a set is connected.

Definition 2.28. Consider a metric space (X, d). Given a pair of points a and b in X, a **path** from a to b in X is a continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$. This is also called a **path joining** a and b.

Remark 2.8. In the above definition, the closed interval $[0, 1]$ can be replaced by any closed interval $[\alpha, \beta]$.

Definition 2.29. A metric space (X, d) is called **path-connected**, if for any pair of points a and b in X there is a path from a to b in X.

Exercise 2.31. Show that the following metric spaces are path connected.

(i) the Euclidean space $\mathbb{R}^n$, for any $n \geq 1$,

(ii) the open ball $B_1(0)$ in $(\mathbb{R}^n, \mathrm{d}_2)$, for any $n \geq 2$,

(iii) the annulus $\{(x, y) \in \mathbb{R}^2 \mid 1 \leq \|(x, y)\| \leq 2\}$.

Theorem 2.31. *If a metric space (X, d) is path connected, then it is connected.*

Proof. Let us assume that there is a metric space (X, d) which is path connected, but not connected. By Lemma 2.23, there is a continuous map $f : X \to \mathbb{R}$ satisfying $f(X) = \{0, 1\}$. Then, there exist x and y in X such that $f(x) = 0$ and $f(y) = 1$.

Because X is path connected, there is a continuous map $g : [0, 1] \to X$ satisfying $g(0) = x$ and $g(1) = y$. Then, $f \circ g : [0, 1] \to \mathbb{R}$ is continuous, and its image is equal to $\{0, 1\}$.

Let us consider the map $(f \circ g) - 1/2$ on the interval $[0, 1]$. It take both values $-1/2$ and $+1/2$, but it does not take the value 0. However, by Corollary 2.30, this map must take the value 0 at some point in $[0, 1]$. □

By the above theorem, the sets in Exercise 2.31 are connected. In the same fashion, one can show that the cube $[0, 1]^n$ is connected in $\mathbb{R}^n$. Compare this argument with how difficult it is to show that the interval $[0, 1]$ is connected.

Exercise 2.32. Consider the set of all continuous functions $f : [0,1] \to \mathbb{R}$, that is $C([0,1])$, with the metric d_1.

(i) Show that the space $(C([0,1]), \mathrm{d}_1)$ is path connected.

(ii) Conclude that the space $(C([0,1]), \mathrm{d}_1)$ is connected.

Exercise 2.33.* In this exercise, we aim to show that the converse of Theorem 2.31 is not true.

Consider the following subset of $\mathbb{R}^2$:

$$A = \{(x, \sin(1/x)) \in \mathbb{R}^2 \mid x > 0\} \cup \{(x, y) \in \mathbb{R}^2 \mid x = 0, y \in [-1, +1]\}.$$

That is, A is the union of the oscillating curve which is the graph of $\sin(1/x)$, and the vertical line segment $\{0\} \times [-1, +1]$.

(i) show that the set A is connected.

(ii) show that the set A is not path connected.

2.4 Compactness

2.4.1 Compactness by covers

Definition 2.30. Let (X, d) be a metric space, and $Y \subseteq X$.

(i) A collection $\mathcal{R}$ of open subsets of X is called an **open cover** for Y, if

$$Y \subseteq \bigcup_{U \in \mathcal{R}} U.$$

(ii) Given an open cover $\mathcal{R}$ for Y, we say that $\mathcal{C}$ is a **sub-cover** of $\mathcal{R}$ for Y, if

$$\mathcal{C} \subseteq \mathcal{R} \quad \text{and} \quad Y \subseteq \bigcup_{U \in \mathcal{C}} U.$$

(iii) An open cover $\mathcal{R}$ for Y is called a **finite cover**, if the number of elements in $\mathcal{R}$ is finite.

Example 2.43. In the metric space $(\mathbb{R}, \mathrm{d}_1)$, the collection

$$\mathcal{R}_1 = \{(-n, n) \mid n \in \mathbb{N}\}$$

is an open cover for $\mathbb{R}$. This cover is not finite. The collection

$$\{(-2n, 2n) \mid n \in \mathbb{N}\}$$

is a sub-cover of $\mathcal{R}_1$ for $\mathbb{R}$. The collection of open sets

$$\{(n - 1/4, n + 1/4) \mid n \in \mathbb{Z}\}$$

is an open cover for $\mathbb{Z}$.

The key concept we aim to study in this section is the following definition.

Definition 2.31. Let (X, d) be a metric space, and $Y \subseteq X$. We say that Y is **compact** in (X, d), if every open cover for Y has a finite sub-cover.

This definition may appear strange at this point, but by the end of this section, it will be clear how important it is.

Example 2.44. In the metric space $(\mathbb{R}, \mathrm{d}_1)$, the set $\mathbb{R}$, and the open interval $(0, 1)$ are not compact.

In order to show this, we need to present an open cover which does not have a finite sub-cover. The collection $\mathcal{R}_1$ for $\mathbb{R}$ introduced in the above example does not have a finite sub-cover for $\mathbb{R}$. That is because, for any finite sub-cover, say

$$\{(-n_1, n_1), (-n_2, n_2), \dots, (-n_k, n_k)\},$$

the point $\max\{n_1, n_2, \dots, n_k\}$ does not belong to the above cover, but it belongs to $\mathbb{R}$. In fact, if $n = \max\{n_1, n_2, \dots, n_k\}$, then the union of the sets in the above collection is equal to $(-n, n)$, which clearly does not cover $\mathbb{R}$.

Here is another open cover for $\mathbb{R}$, which has no finite sub-cover

$$\{(n-1, n+1) \mid n \in \mathbb{Z}\}.$$

For the interval $(0, 1)$, we can consider the open cover

$$\{(1/n, 1) \mid n \in \mathbb{N}, n \geq 2\}.$$

This cover does not have a finite sub-cover. Suppose in the contrary that there is a finite sub-cover of the above collection, say

$$\{(1/n_1, 1), (1/n_2, 1), \dots, (1/n_k, 1)\}.$$

Define $m = \max\{n_1, n_2, \dots, n_k\}$. We note that $1/m \in (0, 1)$ but $1/m$ does not belong to any of the sets in the above collection. Thus, the above finite collection does not cover $(0, 1)$.

Another example is given by

$$\left\{ \left(\frac{1}{n+1}, \frac{1}{n-1} \right) \Bigm| n \in \mathbb{N}, n \geq 2 \right\}.$$

To see that the above collection is an open cover, we note that for every $r \in (0, 1)$, $1/r > 1$. Thus, we can choose an integer $n \geq 2$ such that $1/r \in (n-1, n+1)$. This implies that $r \in (1/(n+1), 1/(n-1))$. By a similar argument, you can show that this cover does not have a finite sub-cover.

Exercise 2.34. Consider the metric space $(\mathbb{R}, \mathrm{d}_1)$, and assume that a and b are real numbers with $a < b$. Show that all of the intervals $(a, b]$, $[a, b)$, $[a, +\infty)$, and $(-\infty, b]$ are not compact.

Example 2.45. Let (X, d) be a metric space, and assume that Y is a subset of X with a finite number of elements. Then Y is compact.

To see this, let $\mathcal{R}$ be an arbitrary open cover of Y. Since Y only has finite number of elements, and each of those elements belongs to one set in $\mathcal{R}$, those finite number of elements in $\mathcal{R}$ cover Y. Thus, $\mathcal{R}$ has a finite sub-cover.

Example 2.46. In the metric space $(\mathbb{R}, \mathrm{d}_1)$ the set $\mathbb{Q} \cap [0, 1]$ is not compact.

To see this, let α be an irrational number in $[0, 1]$ (for example, $\alpha = \sqrt{2}/2$). We can consider the open cover

$$\{(-\infty, \alpha - 1/n) \cup (\alpha + 1/n, +\infty) \mid n \in \mathbb{N}\}.$$

Obviously, each set $(-\infty, \alpha - 1/n) \cup (\alpha + 1/n, +\infty)$ is open in $\mathbb{R}$, and the above collection covers $\mathbb{Q} \cap [0, 1]$. But there is no finite sub-cover of the above cover for $\mathbb{Q} \cap [0, 1]$.

Exercise 2.35. Show that if A and B are compact subsets of a metric space (X, d), then $A \cup B$ is a compact set.

Exercise 2.36. Show that the ball

$$\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

in the metric space $(\mathbb{R}^2, \mathrm{d}_2)$ is not compact.

As you may have noted from the definition of compactness, and the above examples, it is easier to show that a non-compact set is not compact than to show that a compact set is compact. To show that a set is not compact, it suffices to give one open cover which has no finite sub-cover. But to show that a set is compact, we need to show that any open cover has a finite sub-cover. To deal with any open cover brings a level of sophistication.

Proposition 2.32. *Let a and b be real numbers with $a \leq b$. In the metric space $(\mathbb{R}, \mathrm{d}_1)$, the closed interval $[a, b]$ is compact.*

Proof. Let $\mathcal{R}$ be an arbitrary open cover for $[a, b]$. Let us consider the set

$$I = \big\{s \in [a, b] \;\big|\; \text{there is a finite sub-cover of } \mathcal{R} \text{ for } [a, s]\big\}.$$

The set I is not empty, as it contains a. That is because, there is one element in $\mathcal{R}$ which covers the interval $[a, a] = \{a\}$. Also, the set I is bounded from above, since $I \subseteq [a, b]$. Thus, by the completeness of the set of real numbers, I has a supremum. Let $t = \sup(I)$. Note that since $I \subseteq [a, b]$, we have $t \in [a, b]$. First we show that $t = b$.

Assume that $t = a$. Since $\mathcal{R}$ is an open cover for $[a, b]$, there is an open set $U \in \mathcal{R}$ such that $a \in U$. As U is an open set in $\mathbb{R}$, there is $\delta > 0$ such that $(-\delta, +\delta) = B_\delta(a) \subseteq U$. By choosing δ smaller, if necessary, we may assume that $\delta < b - a$. Now, the collection $\{U\}$ is a finite sub-cover of $\mathcal{R}$ for $[a, \delta/2]$. Thus, $\delta/2 \in I$, contradicting $\sup(I) = a$.

Assume that $t \in (a, b)$. Since $\mathcal{R}$ is an open cover for $[a, b]$ there is $U \in \mathcal{R}$ such that $t \in U$. As $U \cap (a, b)$ is an open set, and $t \in U \cap (a, b)$, there is $\delta > 0$ such that $(t - \delta, t + \delta) \subseteq U \cap (a, b)$. By the definition of supremum, there must be $s \in I$ such that $s \in (t - \delta, t]$. As $s \in I$, $[a, s]$ can be covered by a finite sub-cover of $\mathcal{R}$, say $\mathcal{C} \subseteq \mathcal{R}$. Now the collection $\mathcal{C} \cup \{U\}$ is a finite sub-cover of $\mathcal{R}$, and it covers the set

$$[a, t + \delta/2] \subseteq [a, s] \cup (t - \delta, t + \delta).$$

This shows that $t + \delta/2 \in I$, contradicting $\sup(I) = t$.

By the above two paragraphs, we must have $t = b$. This does not immediately mean that $[a, b]$ can be covered by a finite sub-cover of $\mathcal{R}$ (supremum may not belong to the set). Again, since $\mathcal{R}$ is an open cover for $[a, b]$ there is an open set $U \in \mathcal{R}$ such that $b \in U$. As U is an open set, there is $\delta > 0$ such that $(b - \delta, b + \delta) \subseteq U$. By the definition of supremum, there must be $s \in I$ such that $s \in (b - \delta, b]$. As $s \in I$, $[a, s]$ can be covered by a finite sub-cover of $\mathcal{R}$, say $\mathcal{C} \subseteq \mathcal{R}$. Now the collection $\mathcal{C} \cup \{U\}$ is a finite sub-cover of $\mathcal{R}$, and it covers the set $[a, b] \subseteq [a, s] \cup (b - \delta, b + \delta)$. This shows that there is a sub-cover of $\mathcal{R}$ for $[a, b]$. □

Let us look at some basic properties of compact sets.

Proposition 2.33. *Let (X, d) be a metric space, and let $Y \subseteq X$. If X is compact and Y is closed, then Y is compact.*

Proof. Let $\mathcal{R}$ be a cover for Y. Because Y is closed, $X \setminus Y$ is open. Therefore, $\mathcal{R} \cup \{X \setminus Y\}$ is an open cover for X. Since X is compact, there is a finite sub-cover of $\mathcal{R} \cup \{X \setminus Y\}$, which covers X. This sub-cover also covers Y. However, we do not need $X \setminus Y$ to cover Y. Hence we may remove $X \setminus Y$ from that sub-cover, and still cover Y. Thus, there is a finite sub-cover of $\mathcal{R}$ which covers Y. □

Theorem 2.34. *Let (X, d) be a metric space, and $Y \subseteq X$. If Y is compact, then Y is closed.*

Proof. Let $Y \subseteq X$ be a compact set. We aim to show that $X \setminus Y$ is open (by Theorem 2.9 this implies that Y is closed). Fix an arbitrary point $z \in X \setminus Y$.

For each $y \in Y$, we define $r_y = d(z, y)/2 > 0$. Consider the ball $B_{r_y}(y)$. The collection

$$\{B_{r_y}(y) \mid y \in Y\}$$

is an open cover for Y. Since Y is compact, there is a finite sub-cover of this cover for Y. Thus, there are a finite number of points $y_1, y_2, \dots, y_k$ in Y and positive real numbers $r_{y_1}, r_{y_2}, \dots, r_{y_k}$ such that

$$Y \subseteq \bigcup_{i=1}^{k} B_{r_{y_i}}(y_i).$$

Let $r = \min\{r_{y_i} \mid 1 \leq i \leq k\}$, and note that

$$B_r(z) = \bigcap_{i=1}^{k} B_{r_{y_i}}(z).$$

By our choice of r_y, we have

$$B_r(z) \cap B_{r_{y_i}}(y_i) \subseteq B_{r_{y_i}}(z) \cap B_{r_{y_i}}(y_i) = \emptyset.$$

Therefore,

$$B_r(z) \cap Y \subset B_r(z) \bigcap \left(\bigcup_{i=1}^{k} B_{r_i}(y_i) \right) = \emptyset.$$

Thus, $B_r(z) \subseteq X \setminus Y$. As $z \in X \setminus Y$ was arbitrary, we conclude that $X \setminus Y$ is open. $\square$

Theorem 2.35. *Let (X, d_X) and (Y, d_Y) be metric spaces, and consider the product space $X \times Y$ with any of the metrics d from Definition 2.4. If X and Y are compact, then $(X \times Y, \mathrm{d})$ is compact.*

Proof. Let $\mathcal{R}$ be an arbitrary open cover for $X \times Y$.

Let us first assume that every set in $\mathcal{R}$ is of the from $U \times V$, where U is an open set in X and V is an open set in Y. Thus, for every $(x, y) \in X \times Y$, there is W_{xy} in $\mathcal{R}$ such that $(x, y) \in W_{xy}$, and $W_{xy} = U_{xy} \times V_{xy}$ for some open sets U_{xy} in X and $V_{x,y}$ in Y.

Fix an arbitrary $x \in X$. For any $y \in Y$, there is $W_{xy} \in \mathcal{R}$ such that $(x, y) \in W_{xy}$. Let us consider the collection

$$\mathcal{R}_x = \{V_{xy} \mid W_{xy} \in \mathcal{R}, (x, y) \in W_{xy}, W_{xy} = U_{xy} \times V_{xy}\}.$$

Since $\mathcal{R}$ is an open cover for $X \times Y$, $\mathcal{R}_x$ is an open cover for Y. As Y is compact, there is a finite sub-collection of $\mathcal{R}_x$ for Y, say $\{V_{xy_1}, V_{xy_2}, \dots, V_{xy_n}\}$. Consider the set

$$U_x = \bigcap_{i=1}^{n} U_{xy_i}.$$

Since each U_{xy_i} is an open set in X, and the above intersection is finite, the set U_x is open in X. In particular, we have

$$U_x \times Y \subseteq \bigcup_{i=1}^{n} \big(U_{xy_i} \times V_{xy_i}\big) = \bigcup_{i=1}^{n} W_{xy_i}.$$

As $x \in X$ was arbitrary, by the above argument, for each $x \in X$ we obtain an open set U_x in X. Let us consider the collection of open sets $\{U_x \mid x \in X\}$. This is an open cover for X. Because X is compact, there is a finite sub-cover of this cover for X, say $\{U_{x_1}, U_{x_2}, \dots, U_{x_m}\}$. Combining with the above equation, we note that

$$X \times Y \subseteq \bigcup_{i=1,j=1}^{i=m,j=n} W_{x_i y_j}.$$

Thus, there is a finite sub-cover of $\mathcal{R}$ for $X \times Y$. This completes the proof in this case.

Now assume that $\mathcal{R}$ is an arbitrary open cover for $X \times Y$. For each $(x, y) \in X \times Y$, there is an open set W_{xy} in $\mathcal{R}$ such that $(x, y) \in W_{x,y}$. Let us choose an open set U_{xy} in X and an open set V_{xy} in Y such that $x \in U_{xy}$, $y \in V_{xy}$, and $U_{xy} \times V_{xy} \subseteq W_{xy}$. The collection of all such open sets $U_{xy} \times V_{xy}$, for all $x \in X$ and $y \in Y$, is an open cover of $X \times Y$. By the above proof, there is a finite sub-cover of this cover, say $U_{x_i y_j} \times V_{x_i y_j}$ for $(i, j) \in I$, which covers $X \times Y$.

For each $(i, j) \in I$, there is $W_{x_i y_j} \in \mathcal{R}$ such that $U_{x_i y_j} \times V_{x_i y_j} \subseteq W_{x_i y_j}$. Therefore, $\{W_{x_i y_j} \mid (i, j) \in I\}$, is a finite sub-cover of $\mathcal{R}$ for $X \times Y$. This completes the proof. □

By Proposition 2.32 and Theorem 2.35, we obtain the following corollary.

Corollary 2.36. *The set* $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ *in the Euclidean space* $\mathbb{R}^n$ *is compact.*

The above results show us how difficult it is to prove that a given set compact set is compact. One may imagine how difficult it can be to deal with an unusual set in $\mathbb{R}^n$. Thus, it is important to have some criteria which can be verified easily, and imply compactness. In the remaining of this section we aim to introduce few such criteria.

Definition 2.32. Let (X, d) be a non-empty metric space, and $Z \subseteq X$. We say that the set Z is **bounded** in (X, d), if there exists $M \in \mathbb{R}$ such that for all x and y in Z we have $\mathrm{d}(x, y) \leq M$.

Let S be an arbitrary set, and $f : S \to X$. We say that f is **bounded**, if the set $f(S)$ is bounded in X.

Exercise 2.37. Let (X, d) be a metric space, and A_1, A_2, ..., A_n be a finite number of bounded sets in X. Then $\cup_{i=1}^n A_i$ is a bounded set in X.

Exercise 2.38. Let (X, d) be a non-empty metric space, and let $Z \subseteq X$. Show that Z is bounded if and only if there is $x \in X$ and $r \in \mathbb{R}$ such that $Z \subseteq B_r(x)$.

Lemma 2.37. *If* (X, d) *is a compact metric space, then* X *is bounded.*

Proof. Fix an arbitrary $x \in X$ and consider the open cover $\mathcal{R} = \{B_n(x) \mid n \in \mathbb{N}\}$. As X is compact, there is a finite sub-cover of $\mathcal{R}$ which covers X. Let $B_{n_i}(x)$, for $i = 1, 2, \ldots, k$, be those finite sets. Define $m = \max_{1 \leq i \leq k} n_i$. We have

$$X \subset \cup_{i=1}^k B_{n_i}(x) = B_m(x).$$ □

The main criterion for compactness of subsets of $\mathbb{R}^n$ is presented in the next theorem.

Theorem 2.38 (Heine Borel)**.** *Consider the Euclidean metric space* $(\mathbb{R}^n, \mathrm{d}_2)$*, and let* $X \subseteq \mathbb{R}^n$*. Then,* X *is compact if and only if* X *is closed and bounded.*

Proof. Let us first assume that X is compact. By Lemma 2.37, X is bounded, and by Theorem 2.34, X is closed.

Now assume that X is closed and bounded. Since X is bounded, there is $N \in \mathbb{N}$ such that $C \subseteq [-N, N]^n$. By Corollary 2.36, the set $[-N, N]^n$ is compact. Thus, X is a closed set in a compact set. By Proposition 2.33, that implies that X is compact. □

In the above theorem it is important that the set X is contained in $\mathbb{R}^n$. The statement of the theorem is not true for general metric spaces, as you show in the next exercise.

Exercise 2.39. Consider the set $\mathbb{R}$ with the discrete metric $\mathrm{d}_{\mathrm{disc}}$. The set $(0,1)$ is closed and bounded in $(\mathbb{R}, \mathrm{d}_{\mathrm{disc}})$, but it is not compact.

We say that a sequence of sets V_n, for $n \geq 1$, is a nest, if for all $i \geq 1$ we have $V_{i+1} \subseteq V_i$. That is,

$$V_1 \supseteq V_2 \supseteq V_3 \supseteq V_4 \supseteq \dots .$$

Exercise 2.40. Let (X, d) be a metric space, and assume that V_n, for $n \geq 1$, be a nest of non-empty closed sets in X.

(i) Show that if X is compact, then $\cap_{n\geq 1} V_n$ is not empty.

(ii) Give an example of a nest of non-empty closed sets V_n, for $n \geq 1$, in a metric space such that $\cap_{n\geq 1} V_n$ is empty.

2.4.2 Sequential compactness

In this section we aim to find a simpler criterion which implies the compactness property.

Definition 2.33. We say that a metric space (X, d) is **sequentially compact**, if every sequence in X has a sub-sequence which converges in (X, d). That is, for every sequence $(x_n)_{n\geq 1}$ in X, there is a sub-sequence $(x_{n_k})_{k\geq 1}$ and a point $x \in X$ such that $x_{n_k} \to x$, as $k \to +\infty$.

Example 2.47. The metric space $(\mathbb{R}, \mathrm{d}_1)$ is not sequentially compact. For example, the sequence $(n)_{n\geq 1}$ does not have any sub-sequence which converges in $(\mathbb{R}, \mathrm{d}_1)$.

Consider $(0,1) \subseteq \mathbb{R}$, and let d be the induced metric from d_1 on $(0,1)$. The metric space $((0,1), \mathrm{d})$ is not sequentially compact. To see this, consider the sequence $(1/n)_{n\geq 1}$. This sequence belongs to $(0,1)$, and converges to 0 in the metric space $(\mathbb{R}, \mathrm{d}_1)$. So any subsequence of this sequence also converges to 0 in $(\mathbb{R}, \mathrm{d}_1)$. But, since $0 \notin (0,1)$, this sequence has no sub-sequence which converges in $((0,1), \mathrm{d})$.

Lemma 2.39. *Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. Then, $(x_n)_{n\geq 1}$ has a sub-sequence which converges to an element in X if and only if there is $x \in X$ such that for every $\epsilon > 0$, there are infinitely many i satisfying $x_i \in B_\epsilon(x)$.*

Proof. First assume that $(x_n)_{n\geq 1}$ has a sub-sequence which converges to $x \in X$. Let $(x_{n_i})_{i\geq 1}$ be a sub-sequence which converges to x. Fix an arbitrary $\epsilon > 0$. By the definition of convergence, there is $N \in \mathbb{N}$ such that for all $i \geq N$, we have $x_{n_i} \in B_\epsilon(x)$. This shows that there are infinitely many n such that $x_n \in B_\epsilon(x)$.

For the other direction, we aim to find a subsequence of x_n which converges to x. We shall do this inductively. Let $n_1 = 1$. Suppose we have defined x_{n_1}, x_{n_2}, $\dots$, $x_{n_{i-1}}$. Then by the assumption, for infinitely many n we have $x_n \in B_{1/i}(x)$. So take n_i to be the smallest such n such that $n_i \geq n_{i-1}$ and $x_{n_i} \in B_{1/i}(x)$. With this process, we define a sub-sequence of $(x_n)_{n\geq 1}$. We note that for every $i \geq 1$, we have $\mathrm{d}(x_{n_i}, x) < 1/i$. This shows that the sub-sequence x_{n_i} converges to x as $i \to \infty$. □

Exercise 2.41. Show that if a metric space is sequentially compact, then it is bounded.

Theorem 2.40. *If a metric space* (X, d) *is compact, then it is sequentially compact.*

Proof. Suppose in the contrary that X is not sequentially compact. Then, there is a sequence $(x_n)_{n\geq 1}$ in X which has no convergent sub-sequence. Therefore, for every $x \in X$, there is no subsequence of this sequence which converges x. Thus, using Lemma 2.39, for every $x \in X$, there is $\epsilon_x > 0$ such that only for finitely many n we have $x_n \in B_{\epsilon_x}(x)$.

Let $U_x = B_{\epsilon_x}(x)$. Then, the collection

$$\{U_x \mid x \in X\}$$

is an open cover for X. By the compactness of X, there is a finite sub-cover $\{U_{x_1}, U_{x_2}, \dots, U_{x_m}\}$ such that $X = \cup_{i=1}^m U_{x_i}$. But, for each i, $x_n \in U_{x_i}$ for only finitely many n. Thus, $x_n \in X$, for only finitely many n, which is a contradiction, since the whole sequence $(x_n)_{n\geq 1}$ belongs to X. □

Note that in the above proof we did not say that there are finitely many x_n in each U_{x_i}, but we say that $x_n \in U_{n_i}$ for finitely many n. This is important since, the sequence x_n may be constant, or there may be infinitely many entries in the sequence which are the same.

Theorem 2.41 (Bolzano-Weierstrass)**.** *Any bounded sequence in* $\mathbb{R}^m$ *has a convergent subsequence.*

Proof. Let $(x_n)_{n\geq 1}$ be a bounded sequence in $\mathbb{R}^m$. Then, there is $M > 0$ such that for all $n \geq 1$, we have $\|x_n\| \leq M$. Since $[-M, M]^m$ is compact in the Euclidean metric, by Theorem 2.40, $([-M, M]^n, \mathrm{d}_2)$ is sequentially compact. Therefore, $(x_n)_{n\geq 1}$ has a convergent subsequence. □

The opposite direction of the statement in Theorem 2.40 is also true. But the proof requires some technical steps, which we break into few optional exercises.

Exercise 2.42.* Let (X, d) be a sequentially compact metric space. Show that X is separable, that is, there is a countable dense set in X.

Exercise 2.43.* Let (X, d) be a sequentially compact metric space, and $\mathcal{R}$ be an open cover for X. Show that there is a countable sub-cover of $\mathcal{R}$ for X.

Theorem 2.42. *Assume that* (X, d) *is a metric space. If* X *is sequentially compact, then* X *is compact.*

The statement of the above theorem is not optional, but its proof is optional.

*Proof.** Let $\mathcal{R}$ be an arbitrary open cover for X. By Exercise 2.43, we can extract a countable sub-cover of $\mathcal{R}$, say $\{V_1, V_2, V_3, \dots\}$

Suppose that there is no finite sub-cover of $\{V_1, V_2, \dots\}$ for X. Then for every $n \geq 1$, $\{V_1, ..., V_n\}$ does not cover X. Hence, for each n we can choose $x_n \in X$ such that $x_n \notin \cup_{i=1}^n V_i$. In particular, $x_n \notin V_i$ for every $i \leq n$. This implies that only finitely many entries of the sequence lie in each V_i. This generates a sequence $(x_n)_{n\geq 1}$ in X.

Since X is sequentially compact, we can find a convergent sub-sequence, say $(x_{n_j})_{j\geq 1}$. Suppose this converges to $x \in X$. Then, since $\{V_1, V_2, \dots\}$ is a cover for X, there is $m \geq 1$ such that $x \in V_m$. Since V_m is open, by the definition of convergence, there is $N \in \mathbb{N}$ such that for all $j \geq N$, we have $x_{n_j} \in V_m$. Hence, infinitely many entries in the sequence $(x_n)_{n\geq 1}$ lie in V_m, which is a contradiction. □

2.4.3 Continuous maps and compact sets

Theorem 2.43. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces, and* $f : X \to Y$ *be a continuous map. If* Z *is a compact set in* X*, then* $f(Z)$ *is a compact set in* Y*.*

Proof. Let $\mathcal{R} = \{V_\alpha \mid \alpha \in I\}$ be an open cover for $f(Z)$. Define $U_\alpha = f^{-1}(V_\alpha)$. Note that each U_α is an open set in X, since f is continuous. Moreover, $\cup_{\alpha\in I} U_\alpha$ covers Z. Since Z is compact, there exists a finite sub-cover $U_1, U_2, \dots, U_n$ for Z. Then, $V_1 = f(U_1)$, $V_2 = f(U_2)$, $\dots$, $V_n = f(U_n)$ is a finite sub-cover of $\mathcal{R}$ for $f(Z)$. □

Corollary 2.44. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces, and* $f : X \to Y$ *be a homeomorphism. Then,* X *is compact if and only* Y *is compact.*

The above corollary allows us to immediately conclude that some pairs of sets are not homeomorphic. For example, the intervals $(0, 1)$ and $[0, 1]$ are not homeomorphic, since one of them is compact and the other one is not.

Compactness is an extremely useful property in analysis. We shall study some of the conveniences that come with it.

Recall that a map $f : (X, \mathrm{d}_X) \to (Y, \mathrm{d}_Y)$ is uniformly continuous, if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all x_1 and x_2 in X satisfying $\mathrm{d}_X(x_1, x_2) < \delta$ we have $\mathrm{d}_Y(f(x_1), f(x_2)) < \epsilon$. Note that δ in the above definition is independent of x_1

and x_2. In general it is fairly difficult to verify that a map is uniformly continuous. You have already seen this for maps from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 2.45. *Every continuous map from a compact metric space to a metric space is uniformly continuous.*

Proof. To prove this, fix arbitrary metric spaces (A_1, d_1) and (A_2, d_2), and assume that A_1 is compact and $f : A_1 \to A_2$ is continuous.

Suppose in the contrary that f is not uniformly continuous. Then, for some $\epsilon > 0$ and any $n \in \mathbb{N}$ there exists x_n and x'_n in A_1 such that

$$\mathrm{d}_1(x_n, x'_n) < 1/n \quad \text{and} \quad \mathrm{d}_2(f(x_n), f(x'_n)) \geq \epsilon.$$

Because A_1 is compact, by Theorem 2.42, A_1 is sequentially compact. Thus, there exists a sub-sequence $(x_{n_k})_{k\geq 1}$ which converges to some $x \in A_1$. We note that the sub-sequence $(x'_{n_k})_{k\geq 1}$ also converges to x. That is because, by the triangle inequality,

$$\mathrm{d}_1(x'_{n_k}, x) \leq \mathrm{d}_1(x'_{n_k}, x_{n_k}) + \mathrm{d}_1(x_{n_k}, x) \leq 1/n + \mathrm{d}_1(x_{n_k}, x).$$

On the other hand, since f is continuous, and the sequences $(x_{n_k})_{k\geq 1}$ and $(x'_{n_k})_{k\geq 1}$ converge to x, the sequences $(f(x_{n_k}))_{k\geq 1}$ and $(f(x'_{n_k}))_{k\geq 1}$ converge to $f(x)$. But, by our choice of these sub-sequences, we have

$$\epsilon \leq \mathrm{d}_2(f(x_{n_k}), f(x'_{n_k})) \leq \mathrm{d}_2(f(x_{n_k}), f(x)) + \mathrm{d}_2(f(x), f(x'_{n_k}))$$

This is a contradiction. □

Corollary 2.46. *Assume that a and b are real numbers with $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous, then it is uniformly continuous.*

Theorem 2.47. *Let (X, d) be a compact metric space, and $f : X \to \mathbb{R}$ be a continuous map. Then f is bounded from above and below on X, and attains its upper and lower bounds.*

Proof. By Theorem 2.43, $f(X) \subseteq \mathbb{R}$ is compact. Then, by Theorem 2.38, $f(X)$ is closed and bounded in $\mathbb{R}$. In particular, $f(X)$ is bounded from above and below.

Let $M = \sup f(X)$. Since M is the least upper bound for $f(X)$, for every $n \geq 1$, $M - 1/n$ is not an upper bound for $f(X)$. Thus, for every $n \geq 1$, there is $x_n \in X$ such that $f(x_n) \geq M - 1/n$.

Consider the sequence $(x_n)_{n\geq 1}$. Since X is compact, it is sequentially compact. Thus, there is a sub-sequence of $(x_n)_{n\geq 1}$, say $(x_{n_k})_{k\geq 1}$, which converges to some x in X. As f is continuous, $f(x_{n_k}) \to f(x)$ as $k \to \infty$. Taking limits in the inequality

$$f(x_{n_k}) \geq M - 1/n_k,$$

as $k \to \infty$, we obtain $f(x) \geq M$. On the other hand, since $f(x) \in f(X)$, and $\sup f(X) = M$, we must have $f(x) \leq M$. Therefore, $f(x) = M$.

Similarly, we may show that there is $x' \in X$ such that $f(x') = \inf f(X)$. $\square$

Exercise 2.44. Let (X, d) be a compact metric space, and assume that $f : X \to X$ is a continuous map such that for all $x \in X$, we have $f(x) \neq x$. Show that there is $\delta > 0$ such that for all $x \in X$, we have $\mathrm{d}(x, f(x)) \geq \delta$.

Theorem 2.48. *Assume that $f : \mathbb{R} \to \mathbb{R}$ is a continuous map with respect to the Euclidean metrics on the domain and the range. For any interval $[a, b]$, $f([a, b])$ is an interval of the form $[m, M]$, for some real numbers m and M.*

Proof. By Theorem 2.26, the interval $[a, b]$ is connected in $\mathbb{R}$. Since the image of any connected set by a continuous map is connected (see Theorem 2.27), $f([a, b])$ is connected. Then, by Theorem 2.25, $f([a, b])$ must be an interval. By the definition of interval, $f([a, b])$ is equal to one of the sets (m, M), $(m, M]$, $[m, M)$, or (m, M), for some $m \in \mathbb{R} \cup \{-\infty\}$ and $M \in \mathbb{R} \cup \{+\infty\}$ with $m \leq M$.

By Theorem 2.47, $f([a, b])$ is compact. Thus, m and M are finite numbers and $f([a, b]) = [m, M]$. $\square$

2.5 Completeness

2.5.1 Complete metric spaces and Banach space

The completeness of the set of real numbers is a fundamental property widely used in analysis. It allows us to solve equations such as $x^2 = 2$ in $\mathbb{R}$, which have no solutions in $\mathbb{Q}$. Evidently, it is useful to have such a property in more general settings. However, the completeness of $\mathbb{R}$ in terms of least upper bounds, uses the order on the set of real numbers. This cannot be generalised to arbitrary sets in a meaningful fashion. But, the completeness of $\mathbb{R}$ in terms of Cauchy sequences can be generalised to arbitrary metric spaces. In this section, we aim to develop this theory. You will see many applications of the completeness results of this section in the second year module, Differential Equations.

Definition 2.34. Let (X, d) be a metric space, and $(x_n)_{n\geq 1}$ be a sequence in X. We say that $(x_n)_{n\geq 1}$ is a **Cauchy** sequence in (X, d), if for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all n and m bigger than N_ϵ we have

$$\mathrm{d}(x_n, x_m) < \epsilon.$$

Exercise 2.45. Show that any convergent sequence in a metric space, is a Cauchy sequence.

Exercise 2.46. Let (X, d) be a metric space, and assume that $(x_n)_{n\geq 1}$ is a Cauchy sequence in X. If there is a subsequence of $(x_n)_{n\geq 1}$ which converges to some $x \in X$, then the sequence $(x_n)_{n\geq 1}$ converges to x.

Definition 2.35. (i) A metric space (X, d) is called **complete**, if every Cauchy sequence in X converges to a limit in X.

(ii) A normed vector space $(V, \|\cdot\|)$ is called a **Banach** space, if V with the induced metric space $\mathrm{d}_{\|\|}$ is a complete metric space.

Example 2.48. You have already seen in Analysis I that any Cauchy sequence in $\mathbb{R}$ is convergent. You can also prove this using Exercise 2.46 and the Bolzano-Weierstrass Theorem 2.41. Thus, the metric space $(\mathbb{R}, \mathrm{d}_1)$ is complete.

The metric space $(\mathbb{Q}, \mathrm{d})$ is not complete (here d_1 is the induced metric on $\mathbb{Q}$). For example, any sequence in $\mathbb{Q}$ which converges to $\sqrt{2}$, is Cauchy but does not converge in $(\mathbb{Q}, \mathrm{d}_1)$.

In the same fashion, the metric space $((0, 1], \mathrm{d}_1)$ is not complete. For example, the sequence $(1/n)_{n\geq 1}$ in $(0, 1]$ is Cauchy, but not convergent (the limit does not belong to $(0, 1]$).

However, the metric space $([0, 1], \mathrm{d}_1)$ is complete.

Lemma 2.49. *For every $m \geq 1$, the metric space $(\mathbb{R}^m, \mathrm{d}_2)$ is complete.*

Proof. Assume that $(x_n)_{n\geq 1}$ is a Cauchy sequence in $\mathbb{R}^m$. For each $n \geq 1$, let us write $x_n = (x_n^1, x_n^2, \dots, x_n^m)$. Recall that for every $z = (z^1, z^2, \dots, z^m)$ and $y = (y^1, y^2, \dots, y^m)$ in $\mathbb{R}^m$, and every $k \in \{1, 2, \dots, m\}$, we have

$$|z^k - y^k| \leq \|z - y\| .$$

This implies that for every $k \in \{1, 2, \dots, m\}$, the sequence $(x_n^k)_{n\geq 1}$ is a Cauchy sequence in $(\mathbb{R}, \mathrm{d}_1)$. To see this, fix an arbitrary $\epsilon > 0$. Since $(x_n)_{n\geq 1}$ is Cauchy in $(\mathbb{R}^m, \mathrm{d}_2)$, there is $N_\epsilon \in \mathbb{N}$, such that for all i and j bigger than N_ϵ we have $\mathrm{d}_2(x_i, x_j) < \epsilon$. Then, by the above inequality, for all i and j bigger than N_ϵ, we have

$$\mathrm{d}_1(x_i^k, x_j^k) = |x_i^k - x_j^k| \leq \|x_i - x_j\| = \mathrm{d}_2(x_i, x_j) < \epsilon.$$

Now, since every Cauchy sequence in $\mathbb{R}$ is convergent, the sequence $(x_n^k)_{n\geq 1}$ converges to some point in $\mathbb{R}$, say x^k. This implies that the sequence $(x_n)_{n\geq 1}$ converges to $x = (x^1, x^2, \dots, x^m)$ in $\mathbb{R}^m$. $\square$

Alternatively, by the the above lemma, we can say that the normed vector space $(\mathbb{R}^m, \|\cdot\|_2)$ is a Banach space.

Example 2.49. In any discrete metric space, only eventually constant sequences are Cauchy. Obviously, any eventually constant sequence is convergent. Therefore, any set with the discrete metric is complete.

Recall that for real numbers a and b with $a \leq b$, $C([a, b])$ denotes the set of all continuous functions $f : [a, b] \to \mathbb{R}$. We defined two norms on $C([a, b])$ denoted by $\|\cdot\|_2$ and $\|\cdot\|_\infty$. These induce the metrics d_2 and d_∞, respectively. In these metrics, for f and g in $C([a, b])$, we have

$$\mathrm{d}_\infty(f, g) = \sup_{t\in[a,b]} |f(t) - g(t)|,$$

and

$$\mathrm{d}_2(f, g) = \Big(\int_a^b |f(t) - g(t)|^2 \Big)^{1/2}$$

Proposition 2.50. *The metric space* $(C([a, b], \mathrm{d}_2)$ *is not complete. Equivalently, the normed vector space* $(C([a, b]), \|\cdot\|_2)$ *is not a Banach space.*

Proof. To simplify the argument, let us assume that $a = -1$ and $b = 1$ (one can adapt the following example to the general case). For $n \geq 1$, consider the functions

$$\phi_n(t) = \begin{cases} -1 & \text{if } -1 \leq t \leq -1/n, \\ nt & \text{if } -1/n \leq t \leq 1/n, \\ 1 & \text{if } 1/n \leq t \leq 1. \end{cases}$$

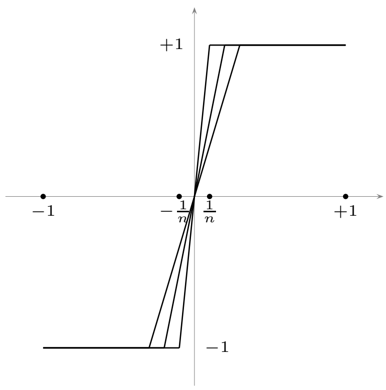

Figure 2.7: The graphs of three functions in the sequence $(\phi_n)_{n\geq 1}$.

See Figure 2.7 for the graphs of these functions.

Each ϕ_n is a continuous and hence belongs to $C([-1,+1])$. We note that for every m and n in $\mathbb{N}$, we have

$$\int_{-1}^{1} |\phi_n(t) - \phi_m(t)|^2 \, dt \leq \frac{2}{\min(n,m)}.$$

This implies that the sequence ϕ_n is Cauchy.

We claim that the sequence $(\phi_n)_{n\geq 1}$ does not converge in $(C([-1,+1]), \mathrm{d}_2)$. To see this let us consider the function

$$\psi(t) = \begin{cases} -1 & \text{if } t \in [-1, 0), \\ 1 & \text{if } t \in [0, 1]. \end{cases}$$

For every $n \geq 1$, we have

$$\int_{-1}^{+1} |\phi_n(t) - \psi(t)|^2 \, dt \leq 2 \cdot \frac{1}{n}$$

Now, assume in the contrary that the sequence $(\phi_n)_{n\geq 1}$, converges to some f in $C([-1,+1])$. By the triangle inequality for the metric d_2, we have

$$\Big(\int_{-1}^{1} |f(t) - \psi(t)|^2 dt\Big)^{1/2} \leq \Big(\int_{-1}^{1} |f(t) - \phi_n(t)|^2 \, dt\Big)^{1/2} + \Big(\int_{-1}^{1} |\phi_n(t) - \psi(t)|^2 \, dt\Big)^{1/2}.$$

By the above properties, the right hand side of the above equation tends to 0 as $n \to \infty$. As the left hand side is a non-negative number, we must have

$$\int_{-1}^{1} |f(t) - \psi(t)|^2 = 0.$$

This implies that

$$\int_{-1}^{0} |f(t) - \psi(t)|^2 = 0 \quad \text{and} \quad \int_{0}^{1} |f(t) - \psi(t)|^2 = 0.$$

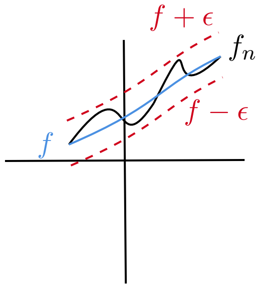

Figure 2.8: In the uniform convergence, for all $n \geq N_\epsilon$, the graph of the function f_n lies between $f-\epsilon$ and $f+\epsilon$.

Then, since f and ϕ are continuous on the intervals $(-1,0)$ and $(0,1)$, the above equations imply that $f = \psi$ on $(-1,0)$ and on $(0,1)$. Such a map f cannot be continuous. $\square$

Remark 2.9. Just like building the completion of $\mathbb{Q}$ to get the set of real numbers, one can build the completion of the metric space $(C([a,b]), \mathrm{d}_2)$. This results in a complete metric space of functions, where one can look for solutions to functional equations. You can learn about this and similar results by taking the optional module *Lebesgue Measure and Integration*.

Recall that a sequences of functions $f_n : [a,b] \to \mathbb{R}$ converges *point-wise* to $f : [a,b] \to \mathbb{R}$, if for every $x \in [a,b]$, the sequence of real numbers $f_n(x)$ converges to $f(x)$. That is, for every $x \in [a,b]$ and every $\epsilon > 0$ there exists $N_{x,\epsilon} \in \mathbb{N}$ such that for all $n \geq N_{x,\epsilon}$ we have $|f_n(x) - f(x) < \epsilon$.

Recall that the sequence $f_n : [a,b] \to \mathbb{R}$ converges *uniformly* to $f : [a,b] \to \mathbb{R}$, if for all $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $n \geq N_\epsilon$ and for all $x \in [a,b]$ we have $|f_n(x) - f(x)| < \epsilon$. This is equivalent to

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0, \text{ as } n \to \infty.$$

Example 2.50. (i): Consider the functions $f_n : [0,1] \to \mathbb{R}$ defined as $f_n(x) = x^n$, for $n \geq 1$. The sequence f_n converges point-wise to the function

$$f = \begin{cases} 0 & \text{if } x \in [0,1), \\ 1 & \text{if } x = 1. \end{cases}$$

But, for every $n \geq 1$, we have

$$\sup_{x\in[0,1]} |f_n(x) - f(x)| = 1.$$

That is because, as x tends to 1 from the left, we have $f_n(x) - f(x) = x^n \to 1$. Therefore, f_n does not converge uniformly to f.

(ii) Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$, defined as

$$f_n(x) = n^2 x(1-x)^n, \quad \forall n \geq 1.$$

This sequence converges point-wise to $f \equiv 0$.

To see whether f_n converges uniformly to f, we examine the functions f_n. Each f_n takes non-negative values, with $f_n(0) = 0$ and $f_n(1) = 0$. Also, each f_n is differentiable on $(0,1)$, so it takes its maximum where the derivative of f_n becomes 0. By calculation, we see that $f_n'(1/(n+1)) = 0$, and

$$f_n\left(\frac{1}{n+1}\right) = \frac{n^2}{n+1}\left(\frac{n}{n+1}\right)^n.$$

Therefore,

$$\sup_{t\in[0,1]} |f_n(t) - f(t)| = \frac{n^2}{n+1}\left(\frac{n}{n+1}\right)^n \to \infty, \text{ as } n \to \infty.$$

This implies that f_n does not converge uniformly to f.

(iii): Consider the sequence of functions $f_n : [0,1] \to \mathbb{R}$ defined as $f_n = xe^{-nx^2}$. The sequence $(f_n)_{n\geq 1}$ converges uniformly (and hence point-wise) to $f \equiv 0$. That is because,

$$\sup_{x\in[0,1]} xe^{-nx^2} \to 0, \text{ as } n \to \infty.$$

It is likely that you have seen the following theorem in Analysis I.

Theorem 2.51. *Assume that $(f_n : [a,b] \to \mathbb{R})_{n\geq 1}$ is a sequence of continuous functions which converges uniformly to $f : [a,b] \to \mathbb{R}$. Then, $f : [a,b] \to \mathbb{R}$ is continuous.*

Proof. Fix an arbitrary $c \in [a,b]$. In order to prove that f is continuous at c, let us also fix an arbitrary $\epsilon > 0$.

Because the sequence $(f_n)_{n\geq 1}$ converges uniformly to f, there is $N_\epsilon \in \mathbb{N}$, such that for all $n \geq N_\epsilon$, and all $x \in [a,b]$ we have $|f_n(x) - f(x)| < \epsilon/3$.

Now, fix an arbitrary $n \geq N_\epsilon$. Since f_n is continuous at c, there is $\delta > 0$ such that for all $x \in B_\delta(c) \cap [a,b]$, we have $|f_n(x) - f_n(c)| \leq \epsilon/3$.

By the above inequalities, and the triangle inequality for the modulus function, for all $x \in B_\delta(c) \cap [a,b]$, we have

$$\begin{aligned}|f(x) - f(c)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(c)| + |f_n(c) - f(c)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, this shows that f is continuous at c. As $c \in [a,b]$ was arbitrary, we conclude that f is continuous on $[a,b]$. □

Theorem 2.52. *The metric space $(C([a,b]), \mathrm{d}_\infty)$ is complete. Equivalently, the normed vector space $(C([a,b]), \|\cdot\|_\infty)$ is a Banach space.*

Proof. Let $(\phi_n)_{n\geq 1}$ be a Cauchy sequence in $(C([a,b]), \mathrm{d}_\infty)$. By definition, for every $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that for all $x \in [a,b]$ and all m and n bigger than N_ϵ we have $|\phi_n(x) - \phi_m(x)| < \epsilon$.

Now, fix an arbitrary $x \in [a,b]$. By the above paragraph, the sequence of real numbers $(\phi_n(x))_{n\geq 1}$ is a Cauchy sequence in $(\mathbb{R}, \mathrm{d}_1)$. Then, by the completeness of the set of real numbers, the sequence of real numbers $(\phi_n(x))_{n\geq 1}$ converges to a (unique) real number, which we denote by l_x. As x in $[a,b]$ was arbitrary, for each $x \in [a,b]$, we obtain a real number l_x.

Let us define the function $\phi : [a,b] \to \mathbb{R}$ as $\phi(x) = l_x$. We claim that ϕ_n converges uniformly to ϕ on $[a,b]$. To see this, fix an arbitrary $\epsilon > 0$. Since $(\phi_n)_{n\geq 1}$ is a Cauchy sequence in $(C([a,b]), \mathrm{d}_\infty)$, (for $\epsilon/2 > 0$) there exists $M_\epsilon \in \mathbb{N}$ such that for all $x \in [a,b]$ and all m and n bigger than M_ϵ we have

$$|\phi_n(x) - \phi_m(x)| < \epsilon/2.$$

Taking limit as $m \to \infty$, the above inequality implies that

$$|\phi_n(x) - \phi(x)| \leq \epsilon/2 < \epsilon.$$

Thus, for all $x \in [a,b]$ and all $n \geq M_\epsilon$, we have

$$|\phi_n(x) - \phi(x)| < \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $(\phi_n)_{n\geq 1}$ converges uniformly to ϕ. By Theorem 2.51, $\phi : [a,b] \to \mathbb{R}$ is continuous. Therefore, any Cauchy sequence in $(C([a,b]), \mathrm{d}_\infty)$ converges to an element of $C([a,b])$. $\square$

Theorem 2.53. *If (X, d) is a compact metric space, then (X, d) is complete.*

Proof. Let $(x_n)_{n\geq 1}$ be a Cauchy sequence in (X, d). By theorem 2.40, (X, d) is sequentially compact. Thus, there exists a subsequence $(x_{n_k})_{k\geq 1}$ which converges to some $x \in X$. By Exercise 2.46, x_n converges to x in (X, d). $\square$

2.5.2 Arzelà-Ascoli

There is an important corollary of the completeness of $(C([a,b]), \mathrm{d}_\infty)$, which we present in this section.

Definition 2.36. Let $\mathcal{C}$ be a collection of functions $f : [a,b] \to \mathbb{R}$.

(i) We say that the collection $\mathcal{C}$ is **uniformly bounded**, if there exists M such that for all $f \in \mathcal{C}$ and all $x \in [a,b]$ we have $|f(x)| < M$.

(ii) We say that the collection $\mathcal{C}$ is uniformly **equi-continuous**, if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $f \in \mathcal{C}$, and all x_1 and x_2 in $[a, b]$ satisfying $|x_1 - x_2| < \delta$, we have $|f(x_1) - f(x_2)| < \epsilon$.

Note that in the second part of the above definition, the number δ does not depend on the function f, but only on the collection $\mathcal{C}$.

Exercise 2.47. Let $\mathcal{C}$ be a collection of functions $f : [a, b] \to \mathbb{R}$. Assume that there is $K > 0$ such that for all $f \in \mathcal{C}$ and all x and y in $[a, b]$, we have

$$|f(x) - f(y)| \leq K|x - y|.$$

Show that the family $\mathcal{C}$ is uniformly equi-continuous.

Theorem 2.54 (Arzelà-Ascoli). *Assume that $\mathcal{C}$ is a collection of continuous functions $f : [a, b] \to \mathbb{R}$. If $\mathcal{C}$ is uniformly bounded and uniformly equi-continuous, then every sequence in $\mathcal{C}$ has a sub-sequence which converges in $(C([a, b]), \mathrm{d}_\infty)$.*

Proof. Let us fix an arbitrary sequence $(f_n)_{n\geq 1}$ in $\mathcal{C}$. We need to show that there is a sub-sequence of this sequence which converges to some continuous function $f : [a, b] \to \mathbb{R}$ with respect to the metric d_∞. We break the proof into several steps:

Step 1. The sequence $(f_i)_{i=0}^\infty$ has a sub-sequence $(g_i)_{i=0}^\infty$ which converges pointwise on $[a, b] \cap \mathbb{Q}$.

Proof of Step 1: Note that the set $[a, b] \cap \mathbb{Q}$ is countable. This means that we may write $[a, b] \cap \mathbb{Q} = \{x_1, x_2, \dots\}$.

Let us denote the function f_i by the notation $f_{0,i}$, that is, for all $i \in \mathbb{N}$ and for all $x \in [a, b]$, we have $f_{0,i}(x) = f_i(x)$.

Now consider the sequence of numbers $(f_{0,i}(x_1))_{i=0}^\infty$. This is a bounded sequence of real numbers. By Bolzano–Weierstrass, this sequence has a convergent subsequence, say $(f_{1,i}(x))_{i=0}^\infty$. Now let us consider $(f_{1,i}(x_2))_{i=0}^\infty$, which again is a bounded sequence of real numbers, with a convergent subsequence $f_{2,i}(x_2)$. This is a subsequence of $f_{1,i}$ such that $f_{2,i}(x_1)$ and $f_{2,i}(x_2)$ both converge. We can repeat this process of extracting subsequences to obtain functions $f_{k,i}$ for $k, i \in \mathbb{N}$ with the property that $(f_{k+1,i})_{i=0}^\infty$ is a subsequence of $(f_{k,i})_{i=0}^\infty$, and moreover for all $l \leq k$, the sequence $f_{k,i}(x_l)$ converges.

Let us define the sequence of functions $g_i = f_{i,i}$, for $i \in \mathbb{N}$. Each g_i is defined on $[a, b]$. To illustrate the above process, one may think of $f_{i,i}$ as the diagonal of the array

$$\begin{matrix} f_{0,0} & f_{0,1} & f_{0,2} & f_{0,3} & \cdots \\ f_{1,0} & f_{1,1} & f_{1,2} & f_{1,3} & \cdots \\ f_{2,0} & f_{2,1} & f_{2,2} & f_{2,3} & \cdots \\ f_{3,0} & f_{3,1} & f_{3,2} & f_{3,3} & \cdots \end{matrix}$$

Clearly, $(g_i)_{i=0}^{\infty}$ is a subsequence of F, and moreover for every $l \in \mathbb{N}$ the sequence $g_i(x_l)$ converges. Let us define $g(x_l) = \lim_{i\to\infty} g_i(x_l)$.

Step 2. The sequence of functions $g_i : [a, b] \to \mathbb{R}$, for $i \geq 0$, is Cauchy with respect to the metric d_∞.

Proof of Step 2: Let us fix an arbitrary $\epsilon > 0$. Since $\mathcal{C}$ is uniformly equi-continuous, we may find $\delta > 0$ such that for all x and y in $[a, b]$ and all $i \in \mathbb{N}$ we have

$$|x - y| < \delta \implies |g_i(x) - g_i(y)| < \epsilon/3.$$

Since $[a, b]$ is bounded, there are rational numbers $x_1, \ldots, x_k$ in $[a, b]$ such that

$$[a, b] \subset \cup_{m=1}^{k}(x_m - \delta, x_m + \delta).$$

Since g_i converges at each rational point, for each $m = 1, \ldots k$ there exists N_m such that for all $i, j \geq N_m$ we have

$$|g_i(x_m) - g_j(x_m)| < \epsilon/3.$$

Let $N = \max\{N_1, \ldots N_k\}$, and suppose $i, j \geq N$. Fix $x \in [a, b]$. By construction, there is $m \in \{1, \ldots, k\}$ such that $|x - x_m| < \delta$. We have

$$\begin{aligned}
|g_i(x) - g_j(x)| &= |g_i(x) - g_i(x_m) + g_i(x_m) - g_j(x_m) + g_j(x_m) - g_j(x)| \\
&\leq |g_i(x) - g_i(x_m)| + |g_i(x_m) - g_j(x_m)| + |g_j(x_m) - g_j(x)| \\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon.
\end{aligned}$$

Step 3. The sequence $(g_i)_{i=0}^{\infty}$ converges in $(C([a, b]), \mathrm{d}_\infty)$.

Proof of Step 3: By Step 2, g_i is a Cauchy sequence in $(C([a, b]), \mathrm{d}_\infty)$. Then, by Theorem 2.52, $(g_i)_{i\geq 1}$ converges to some g in the metric space $(C([a, b], \mathrm{d}_\infty)$. □

2.5.3 Fixed point Theorem

Definition 2.37. Let (X_1, d_1) and (X_2, d_2) be metric spaces, and $f : X_1 \to X_2$. We say that f is **contracting**, if there exists $K \in (0, 1)$ such that for all a and b in X_1 we have

$$\mathrm{d}_2(f(a), f(b)) \leq K \cdot \mathrm{d}_1(a, b).$$

It is easy to see that every contracting map is continuous.

For a map $f : X \to X$, we say that $x \in X$ is a fixed point of f, if $f(x) = x$.

Theorem 2.55 (Banach fixed point Theorem)**.** *Let (X, d) be a non-empty complete metric space, and $f : X \to X$ be a contracting map. Then, f has a unique fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point. Let us define the sequence of points $(x_n)_{n\geq 0}$ according to $x_{n+1} = f(x_n)$, for $n \geq 0$.

Since f is contracting, there is $K \in (0,1)$ such that for all a and b in X we have $\mathrm{d}(f(a), f(b)) \leq K \cdot \mathrm{d}(a,b)$. Then, for every $j \in \mathbb{N}$, we have

$$\mathrm{d}(x_{j+1}, x_j) = \mathrm{d}(f(x_j), f(x_{j-1})) \leq K\,\mathrm{d}(x_j, x_{j-1}) \leq \cdots \leq K^j d(x_1, x_0).$$

Therefore, for integers $m > n$, we have

$$\begin{aligned}\mathrm{d}(x_m, x_n) &\leq \mathrm{d}(x_m, x_{m-1}) + \cdots + \mathrm{d}(x_{n+1}, x_n)\\ &\leq (K^{m-1} + K^{m-2} + \cdots + K^n)\,\mathrm{d}(x_1, x_0)\\ &\leq K^n \frac{1}{1-K}\,\mathrm{d}(x_1, x_0).\end{aligned}$$

Because $K \in (0,1)$, the last expression in the above equation converges to 0 as $n \to \infty$. This implies that the sequence $(x_n)_{n\geq 1}$ is Cauchy in (X, d).

Since (X, d) is complete, the sequence $(x_n)_{n\geq 1}$ converges to some x in X. As f is continuous, $f(x_n) \to f(x)$, as $n \to \infty$. But $f(x_n) = x_{n+1} \to x$, as $n \to \infty$. By the uniqueness of the limits of convergent sequences in metric spaces, we must have $x = f(x)$.

The above argument shows that f has a fixed point. To show the uniqueness of the fixed point, assume that there is $y \in X$ such that $f(y) = y$. By the contraction property of f, we have

$$\mathrm{d}(x, y) = \mathrm{d}(f(x), f(y)) < K\,\mathrm{d}(x, y).$$

Since $K < 1$, we must have $\mathrm{d}(x, y) = 0$, and hence $x = y$. □

Exercise 2.48. Let $x_1 = \sqrt{2}$, and define the sequence $(x_n)_{n\geq 1}$ according to

$$x_{n+1} = \sqrt{2 + \sqrt{x_n}}.$$

Show that the sequence $(x_n)_{n\geq 1}$ converges to a root of the equation

$$x^4 - 4x^2 - x + 4 = 0$$

which lies in the interval $[\sqrt{3}, 2]$.

Exercise 2.49. Consider the map $f : (0, 1/3) \to (0, 1/3)$, defined as $f(x) = x^2$. Show that the map f is a contraction with respect to the Euclidean metric d_1. But, f has no fixed point in $(0, 1/3)$.

Exercise 2.50. Consider the map $f : [1, \infty) \to [1, \infty)$ defined as $f(x) = x + 1/x$. Show that $([1, +\infty), \mathrm{d}_1)$ is a complete metric space, and for all x and y in $[1, \infty)$ we have

$$\mathrm{d}_1(f(x), f(y)) \leq \mathrm{d}(x, y).$$

But, f has no fixed point.