Multivariable Calculus + Differential Equations Concise Notes

MATH50004

Year 2 Content

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Colour Code - **Definitions** are **green** in these notes, **Consequences** are **red** and **Causes** are **blue**

Content from MATH40002 assumed to be known.

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Contents

Part I

Term 1

1 Vector Calculus

1.1 Prelim

Definition 1.1.1 - Einstein Summation Convention

$$a_i x_i = \sum_{i=1}^{3} x_i$$

Definition 1.1.2 - The Kronecker delta

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Definition 1.1.3 - The Permutation Symbol

$$\epsilon_{ijk} = \begin{cases} 0, & \text{if any 2 elements } i, j, k \text{ equal} \\ 1, & \text{if } i, j, k \text{ a cyclic permutation of } 1, 2, 3 \\ -1, & \text{if } i, j, k \text{ an acyclic permutation } 1, 3, 2 \end{cases}$$

Formula - Relation between Kroenecker Delta and Permutation Symbol

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta kl$$

Definition 1.1.4 - Vector Products
Here are some identities:

- $\mathbf{a} \cdot \mathbf{b} = a_i b_i$
- $[\mathbf{a} \times \mathbf{b}]_i = \epsilon_{ijk} a_j b_k$
- $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \Rightarrow [a \times b]_i = \epsilon_{ijk} a_j b_k$
- $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \epsilon_{ijk} a_i b_j c_k$
- $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \Rightarrow [\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i$

1.2 Gradient, Div, and Curl

Definition 1.2 - Gradient, Directional Derivatives

$\phi =$ constant, defines a surface in $3D$, varying the constant yields a family of surfaces.

$$\hat{\mathbf{n}}\frac{\partial \phi}{\partial n} = \nabla = (\frac{\delta}{\delta x}, \frac{\delta}{\delta y}, \frac{\delta}{\delta z}) \Rightarrow \nabla\phi = \frac{\delta\phi}{\delta x} + \frac{\delta\phi}{\delta y} + \frac{\delta\phi}{\delta z}$$

Thus, directional derivative towards s $= \frac{\delta\phi}{\delta s} = \nabla\phi \cdot \hat{\mathbf{s}}$

In **cylindrical coordinates** r, θ, z parametrized by $x = r\cos\theta$, $y = r\sin\theta$ yields $\nabla\phi = \hat{\mathbf{r}}\frac{\delta\phi}{\delta r} + \frac{\hat{\theta}}{r}\frac{\delta\phi}{\delta\theta} + \mathbf{k}\frac{\delta\phi}{\delta z}$

Definition 1.2.3 - Tangent Plane to $\phi(P)$

$$(\mathbf{r} - \mathbf{r}_p) \cdot (\nabla\phi)_P = 0$$

$$\left(\frac{\delta\phi}{\delta x}\right)_P (x - x_P) + \left(\frac{\delta\phi}{\delta y}\right)_P (y - y_P) + \left(\frac{\delta\phi}{\delta z}\right)_P (z - z_P) = 0$$

1.3 Divergence & Curl

Definition 1.3.1 - Divergence and Curl

A a vector function of position

$$\textbf{Div A} = \nabla \cdot \mathbf{A} = \frac{\delta A_1}{\delta x} + \frac{\delta A_2}{\delta y} + \frac{\delta A_3}{\delta z} \text{ where } A = A_1\hat{\mathbf{i}} + A_2\hat{\mathbf{j}} + A_3\hat{\mathbf{k}}$$

$$\textbf{Curl A} = \nabla \times \mathbf{A} = \hat{\mathbf{i}}\left(\frac{\delta A_3}{\delta y} - \frac{\delta A_2}{\delta z}\right) - \hat{\mathbf{j}}\left(\frac{\delta A_3}{\delta x} - \frac{\delta A_1}{\delta z}\right) + \hat{\mathbf{k}}\left(\frac{\delta A_2}{\delta x} - \frac{\delta A_1}{\delta y}\right)$$

Definition - Laplacian Operator

$$\nabla^2\phi = \text{div}(\nabla\phi) = \frac{\delta^2\phi}{\delta x^2} + \frac{\delta^2\phi}{\delta y^2} + \frac{\delta^2\phi}{\delta z^2}$$

1.4 Operations with Grad operator

Resulting Equalities

(i) $\nabla(\phi_1 + \phi_2) = \nabla\phi_1 + \nabla\phi_2$

(ii) $\text{div }(\mathbf{A} + \mathbf{B}) = \text{div }\mathbf{A} + \text{div }\mathbf{B}$

(iii) $\text{curl }(\mathbf{A} + \mathbf{B}) = \text{curl }\mathbf{A} + \text{curl }\mathbf{B}$

(iv) $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$

(v) $\text{div}(\phi\mathbf{A}) = \phi \text{ div }\mathbf{A} + \nabla\phi \cdot \mathbf{A}$

(vi) $\text{curl}(\phi\mathbf{A}) = \phi \text{ curl }\mathbf{A} + \nabla\phi \times \mathbf{A}$

(vii) $\text{div}(\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl }\mathbf{A} - \mathbf{A} \cdot \text{curl }\mathbf{B}$

(viii) $\text{curl}(\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\ \mathbf{A} - \mathbf{B} \text{ div }\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} \text{ div }\mathbf{B}$

(ix) $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{B} \times \text{curl }\mathbf{A} + \mathbf{A} \times \text{curl }\mathbf{B}$

(x) $\text{curl }(\nabla\phi) = 0$

(xi) $\text{curl }(\text{curl }\mathbf{A}) = \nabla(\text{div }\mathbf{A}) - \nabla^2\mathbf{A}$

(xii) $\text{div }(\text{curl }\mathbf{A}) = 0$

1 Integration

Definition 1.4.6 - Scalar and Vector Fields
If at each point of region V, scalar function ϕ defined - ϕ **a scalar field over** V
Similarly if vector function A defined $\forall v \in V$, A **a vector field.**
If curl $A = 0$, A is an **irrotational vector field**. If div $A = 0$, A a **solenoidal vector field**

1.5 Path Integrals

Definition 1.5.1 - Definition of a Path Integral

$$\lim_{n\to\infty}\sum_{n=1}^{N} f_n \delta s_n = \int_\gamma f ds \Rightarrow \int_\gamma \mathbf{F}\cdot d\mathbf{r} \int_\gamma \mathbf{F}\cdot\hat{\mathbf{t}}\ ds \text{ where } \hat{\mathbf{t}} \text{ is the normalized vector tangent to the path}$$

Definition 1.5.3 - Conservative forces

If $F = \nabla\phi$ for a **differentiable scalar function** ϕ, F **is said to be a conservative field**, which has the following properties:

$$\int_\gamma \mathbf{F}\cdot d\mathbf{r} = \phi(B) - \phi(A)$$

Result independent of path joining **A** and **B**, in particular for γ a closed curve ($B \equiv A$) We have:

$$\oint_\gamma \mathbf{F}\cdot d\mathbf{r} = 0$$

Call this a **circulation of F** around γ
If a vector field **F** s.t $\oint_\gamma F\cdot dr = 0$, for any closed curve γ say **F a conservative field**, if $\mathbf{F} = \nabla\phi \implies \mathbf{F}$ conservative.
If **F** conservative $\implies$ can always find differentiable scalar function ϕ s.t $\mathbf{F} = \nabla\phi$, call ϕ the **potential of field F**

Definition 1.5.4 - Calculation of Path Integrals

When $\mathbf{F} = \mathbf{F}(x, y, z)$ and the path γ can be parametrized by $(x(t), y(t), z(t))$, then:

$$\mathbf{r} = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{k}} \Rightarrow d\mathbf{r} = \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{k}}$$

$$\implies \int_\gamma \mathbf{F}\cdot d\mathbf{r} = \int_{t_0}^{t_1}\left(\mathbf{F}_1\frac{dx}{dt} + \mathbf{F}_2\frac{dy}{dt} + \mathbf{F}_3\frac{dz}{dt}\right)dt$$

1.6 Surface Integrals

Definition 1.6.1 - Surface Integral

Consider a surface S,where we find the surface integral of $f = f(P)$ over S.
Dividing S into small elements of area δS_i, with f_i the values of f at typical points P_i of δS_i
The **surface integral of f over S** is

$$\int_S f dS = \lim_{\substack{N\to\infty \\ max(\delta S_n)\to 0}} \sum_{n=1}^{N} f_n \delta S_n$$

f may be a vector or a scalar.

1.6.2 Types of Surfaces

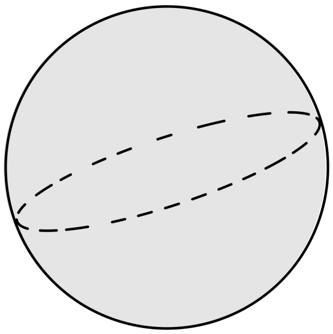


Figure 1: Closed Surface

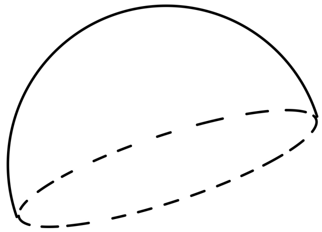


Figure 2: Open Surface

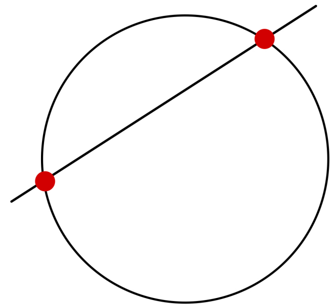


Figure 3: Convex Surface

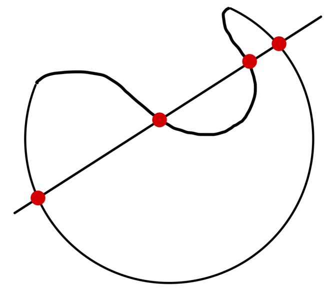
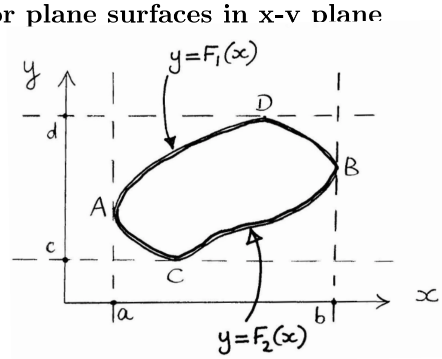


Figure 4: Non-Convex Surface

Definitions

1. **Closed Surface** - Divides $3D$ space into 2 non-connected regions; interior and exterior.
2. **Open Surface** - Does not divide $3D$ space into 2 non-connected regions - has a rim which can be represented by closed curve.
   Can think of closed surfaces as sum of 2 open surfaces.
3. **Convex Surface** - A surface which is crossed by a straight line at most twice

1.6.3 Evaluating surface integrals for plane surfaces in x-y plane



dS infinitesimal area $\implies$ think of as approx. plane.
Vector areal element $d\mathbf{S}$ is the vector $\hat{\mathbf{n}}dS$ for $\hat{\mathbf{n}}$ the unit normal vector to dS.
For a plane lying in $z = 0$, we can say $dS = dxdy$

For a rectangle, $x = a, b$ and $y = c, d$ circumscribing convex S. We let

$$y = \begin{cases} F_1(x) & \text{upper half ADB} \\ F_2(x) & \text{lower half ACB} \end{cases}$$

$$\textbf{Area of } S = \int_S dS = \int_{x=a}^{x=b} \int_{y=F_2(x)}^{y=F_1(x)} dydx = \int_a^b [F_1(x) - F_2(x)]dx$$

For $f(x, y)$ a function of position

$$\int_S f dS = \int_{x=a}^{x=b} \int_{y=F_2(x)}^{y=F_1(x)} f(x, y)dydx$$

Equivalently;

$$x = \begin{cases} G_1(x) & \text{right half CBD} \\ G_2(x) & \text{left half CAD} \end{cases}$$

$$\textbf{Area of } S = \int_S dS = \int_c^d G_1(y) - G_2(y)dy$$

$$\int_S f dS = \int_{y=c}^{y=d} \int_{x=G_2(x)}^{x=G_1(x)} f(x, y)dxdy$$

1.6.5 Projection of an area onto a plane

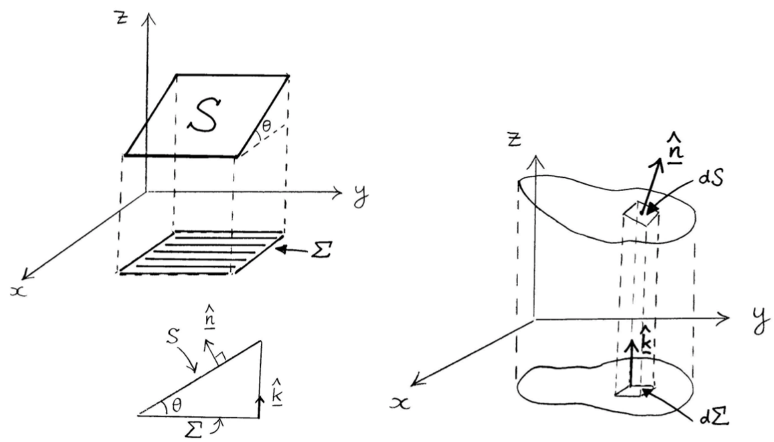


Figure 9: Left; Projection of plane area S onto $x-y$ plane
Figure 9: Right; Projection of curved surface S onto $x-y$ plane

$$dS = \frac{d\Sigma}{|\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|}$$

1.6.6 The Projection Theorem

P a point on surface S, which at no point is orthogonal to $\mathbf{k}$

$$\int_S f(P)dS = \int_\Sigma f(P)\frac{dx\ dy}{|\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|}$$

For a projection of S onto $z=0$, with $\hat{\mathbf{n}}$ normal to S
For S given by $z=\phi(x,y)$

$$\int_S f(x,y,z)dS = \int_{\Sigma_z} f(x,y,\phi(x,y))\frac{dx\ dy}{|\hat{\mathbf{n}} \cdot \hat{\mathbf{k}}|}$$

Projecting onto $x=0$ or $y=0$

$$\int_S f(P)dS = \int_{\Sigma_x} f(x,y,\phi(x,y))\frac{dy\ dz}{|\hat{\mathbf{n}} \cdot \hat{\mathbf{i}}|} = \int_{\Sigma_y} f(x,y,\phi(x,y))\frac{dx\ dz}{|\hat{\mathbf{n}} \cdot \hat{\mathbf{j}}|}$$

Σ_x, projection onto $x=0$, Σ_y, projection onto $y=0$

1.7 Volume Integrals

Definition 1.7.1 - Volume Integral
Considering a volume τ, split into N subregions, $\{\delta\tau_i\}$, with $\{P_i\}$ typical points of $\{\delta\tau_i\}$.

$$\int_\tau f d\tau = \lim_{\substack{N\to\infty \\ max(\delta\tau_i)\to 0}} \sum_{i=1}^{N} f(P_i)\delta\tau_i$$

In Cartesian coordinates, the volume element $d\tau = dxdydz$

1.8 Results relating line,surface and volume integrals

1.8.1 Green's Theorem in the plane

R a closed plane region bounded by a simple plane closed convex curve in $x-y$ plane.
L, M continuous functions of x, y with continuous derivatives throughout R. Then:

$$\oint_C (L\ dx + M\ dy) = \int_R (\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y})dxdy,$$

For C the boundary of R described in the counter-clockwise sense.

1.8.2 Vector forms of Green's Theorem

(i) 2D Stokes Theorem
Let $F = L\mathbf{i} + M\mathbf{j}$ and $d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$. Then

$$\text{curl } \mathbf{F} = \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)\mathbf{k}$$

Over region R write $dxdy = dS$.

$$\begin{aligned}\oint_C F \cdot dr &= \int_R k \cdot \text{curl } F dS \\ &= \int_R \text{curl } F \cdot d\mathbf{S}, \qquad d\mathbf{S} = \hat{\mathbf{k}} dS\end{aligned} \tag{1}$$

(ii) Divergence Theorem in 2D
Let $\mathbf{F} = M\mathbf{i} - L\mathbf{j}$. Then

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}$$

So we can rewrite Green's Theorem as

$$\int_R \text{div } \mathbf{F} dxdy = \oint_C F \cdot \hat{\mathbf{n}} ds$$

Green's Theorem holds for more complicated geometries too, if C not convex we can see it as the composition of 2 or more simple convex closed curves.
Joining A, A' form C_1, C_2 enclosing R_1, R_2 s.t $R_1 + R_2 = R$

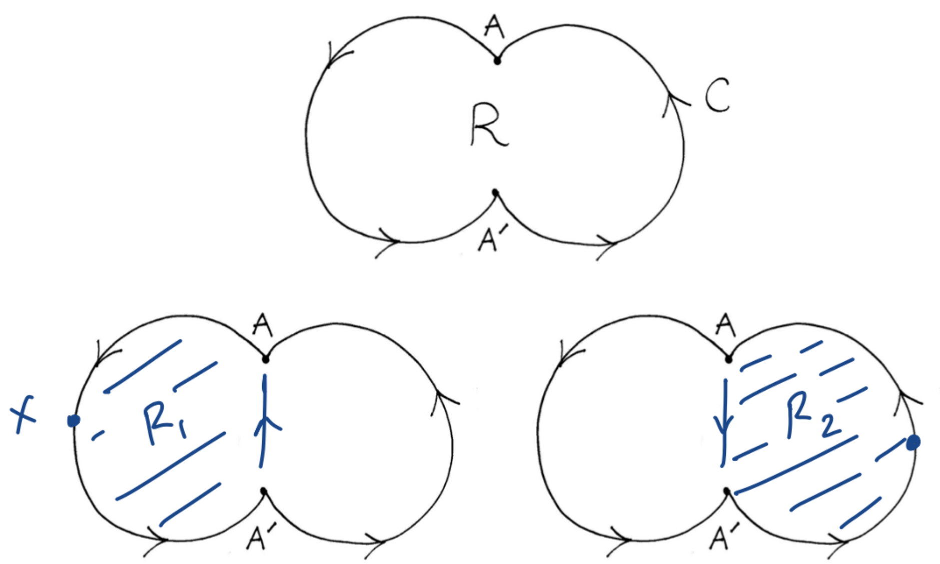


Figure 13: A non-convex boundary

$$\oint_C \mathbf{F} \cdot dr = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_R \text{curl } \mathbf{F} \cdot d\mathbf{S}$$

$$\begin{aligned}\oint_{C_1} &= \int_{AXA'} + \int_{A'}^{A} \\ \oint_{C_2} &= \int_{A'YA} + \int_{A}^{A'}\end{aligned} \tag{2}$$

1.8.4 Green's Theorem in multiply-connected regions

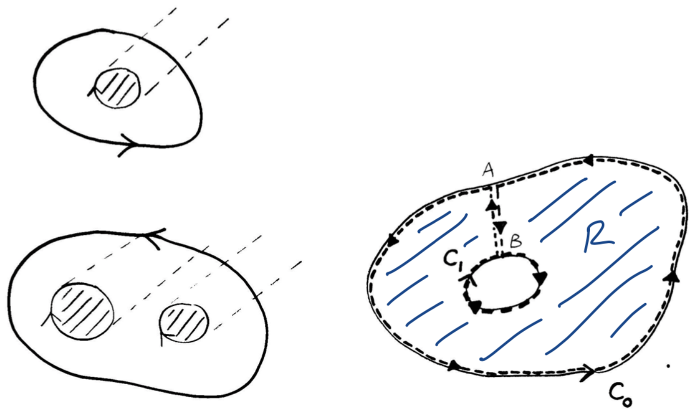


Figure 14: Left; Doubly- and triply- connected regions

Figure 14: Right; Green's Theorem in multiply-connected regions

R **simply-connected** if any closed curve in R can be shrunk to a point without leaving R.
For $2D$ any region with a hole in it; **not simply connected**, we say it is **multiply-connected**
Green's theorem still holds in multiply-connected regions. C interpreted as the entire inner and outer boundary.

For doubly-connected region, describe outer C_0 anti-clockwise, C_1 clockwise, and join them via A on C_0 and B on C_1
R now a simply connected region bounded by $(C_0 + AB + C_1 + BA)$

$$\int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} = \left(\oint_{C_0} + \int_A^B + \oint_{C_0} + \int_B^A \right) (\mathbf{F} \cdot d\mathbf{r})$$

$$\int_R \text{curl } \mathbf{F} \cdot d\mathbf{S} = \left(\oint_{C_0} + \oint_{C_1} \right) (\mathbf{F} \cdot d\mathbf{r}) = \left(\oint_C \mathbf{F} \cdot d\mathbf{r} \right)$$

Where $C = C_0 + C_1$

1.8.5 Flux

If S is a surface then the flux of A across S is defined as

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS$$

If S a closed surface then by convention draw unit normal $\hat{\mathbf{n}}$ **out** of S.

1.8.6 The divergence theorem

If τ the volume enclosed by a closed surface S with unit outward normal $\hat{\mathbf{n}}$ and $\mathbf{A}$ is a vector field with continuous derivatives throughout τ, then:

$$\int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_\tau div \mathbf{A} d\tau$$

1.8.7 The Divergence theorem in more complicated geometries

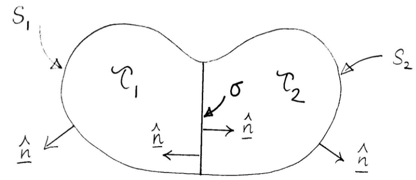


Figure 17: The divergence theorem for a non-convex surface

(i) **Non-convex surfaces** *non-convex surface S can be divided by surfaces(s) σ into 2 (or more) parts S_1 and S_2 which together with σ form convex surfaces $S_1 + \sigma, S_2 + \sigma$/*
Applying divergence theorem to the convex parts, upon addition yields the same result as before.

(ii) **A region with internal boundaries**

(a) *Simply-connected regions* - e.g space between concentric spheres..

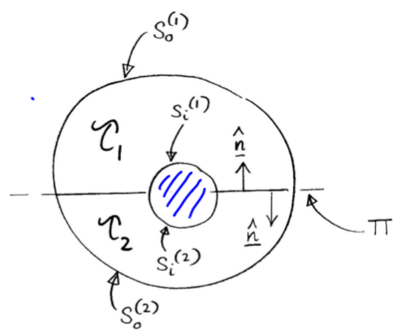


Figure 18: Simply-connected regions

Given interior surface S_i and outer surface S_o. A plane Π cutting both S_o, S_i, divides S_o, S_i into open $S_o^{(1)}, S_o^{(2)}$ and $S_i^{(1)}, S_i^{(2)}$ respectively.
Apply divergence theorem to τ_1, τ_2 bounded by closed $S_o^{(1)} + S_i^{(1)} + \Pi$ and $S_o^{(2)} + S_i^{(2)} + \Pi$. Upon addition contribution from Π cancels.

$$\int_{S_o+S_i} \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_S \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_{\tau_1} div \mathbf{A} d\tau + \int_{\tau_2} div \mathbf{A} d\tau = \int_\tau div \mathbf{A} d\tau$$

(b) *Multiply-connected regions*
e.g. region between 2 cyclinders.

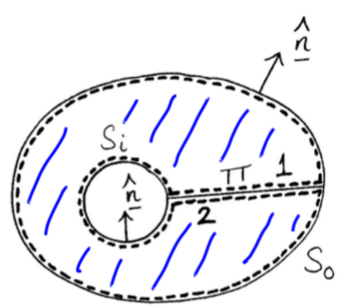


Figure 18: Multiply-connected regions

Given interior surface S_i and outer surface S_o, linked by plane Π.
Consider the closed surface, enclosing simply connected region τ

$$S_i + \text{ side 1 of } \Pi + S_o + \text{ side 2 of } \Pi$$

Applying divergence theorem to τ. Once again gives

$$\int_{S_0+S_i} \mathbf{A} \cdot \hat{\mathbf{n}} dS = \int_\tau div \mathbf{A} d\tau$$

1.8.8 Green's identity in 3D

For ϕ and ψ 2 scalar fields with continuous derivatives. We consider $\mathbf{A} = \phi\nabla\psi$, for which we have

$$div\mathbf{A} = \phi\nabla^2\psi + (\underline{\nabla}\phi)\cdot(\underline{\nabla}\psi)$$
$$\hat{\mathbf{n}}\cdot\mathbf{A} = \phi(\underline{\nabla}\psi)\cdot\hat{\mathbf{n}} = \phi\frac{\partial\psi}{\partial n}$$

Green's first identity

$$\int_S \left\{\phi\frac{\partial\psi}{\partial n}\right\} dS = \int_\tau \phi\nabla^2\psi + (\underline{\nabla}\phi)\cdot(\underline{\nabla}\psi)d\tau$$

Green's Second identity

$$\int_S \left\{\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right\} dS = \int_\tau \phi\nabla^2\psi - \psi\nabla^2\phi d\tau$$

1.8.9 Green's identities in 2D

Divergence theorem in 2D: $\int_F div\mathbf{F}dxdy = \oint_C \mathbf{F}\cdot\hat{\mathbf{n}}ds$
Giving the following Green's identities:

$$\oint_C \phi\frac{\partial\psi}{\partial n}ds = \int_R [\phi\nabla^2\psi + (\nabla\psi)\cdot(\nabla\phi)dxdy$$

and

$$\oint_C \left[\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right] ds = \int_R \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right] dxdy$$

$\int_R \phi\nabla^2\psi\ dxdy = \oint_C \phi\frac{\partial\psi}{\partial n}ds - \int_R(\nabla\psi)\cdot(\nabla\phi)dxdy$ - **Looks like Integration by parts**

1.8.10 Gauss' Flux Theorem

Let S a closed surface with outward unit normal $\hat{\mathbf{n}}$ and let O the origin of the coordinate system.
$\mathbf{A} = \frac{\mathbf{r}}{r^3}$ Then:

$$\int_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{r^3} = \begin{cases} 0, & \text{if } O \text{ is exterior to } S \\ 4\pi, & \text{if } O \text{ interior to } S \end{cases}$$

1.8.11 Stokes Theorem

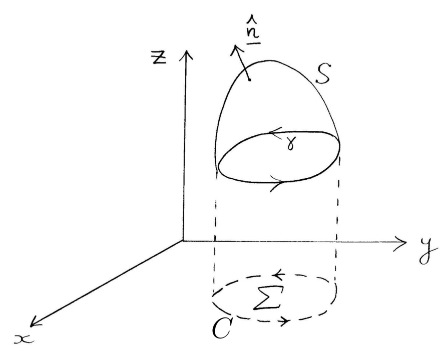


Figure 20: Diagram for proof of Stokes' Theorem

Suppose S is **open** surface with simple closed curve γ forming its boundary.
A a vector field with continuous partial derivatives, Then:

$$\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = \int_S curl\mathbf{A}\cdot\hat{\mathbf{n}}dS$$

This holds for **any** open surface with γ as a boundary.

Theorem
For **A** continuously differentiable and simply connected region:

$$\underbrace{\oint_\gamma \mathbf{A}\cdot d\mathbf{r} = 0}_{\mathbf{A}\text{ conservative}} \iff curl\mathbf{A} = 0,\ \text{throughout region for which } \gamma \text{ is drawn}$$

1.9 Curvilinear Coordinates

1.9.1 Intro + Definition

Consider generally cartesian coordinates: (x_1, x_2, x_3) with each expressible as single-valued differentiable functions of the new coorinates (u_1, u_2, u_3)

$$x_i = x_i(u_1, u_2, u_3)$$
$$\frac{\partial x_i}{\partial x_j} = \delta_{ij} = \frac{\partial x_i}{\partial u_1}\frac{\partial u_1}{\partial x_j} + \frac{\partial x_i}{\partial u_2}\frac{\partial u_2}{\partial x_j} + \frac{\partial x_i}{\partial u_3}\frac{\partial u_3}{\partial x_j}$$

With the following matrix equation

$$\begin{pmatrix} \partial x_1/\partial u_1 & \partial x_1/\partial u_2 & \partial x_1/\partial u_3 \\ \partial x_2/\partial u_1 & \partial x_2/\partial u_2 & \partial x_2/\partial u_3 \\ \partial x_3/\partial u_1 & \partial x_3/\partial u_2 & \partial x_3/\partial u_3 \end{pmatrix}\begin{pmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3 \\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3 \end{pmatrix} = I$$

Or more succinctly

$$J(x_u)\cdot J(u_x) = I$$

We say $J(x_u)$ the **Jacobian matrix** for the (x_1, x_2, x_3) system.

$$det\left(J(x_u)\right) \neq 0 \implies J(u_x) \text{ exists}$$
$$det(J(x_u)) = \tfrac{1}{det(J(u_x))}$$

We say (u_1, u_2, u_3) define a curvilinear coordinate system.
With each u_i = constant, defining a family of surfaces, with a member of each family passing through each $P(x, y, z)$
Let $(\hat{\mathbf{a_1}}, \hat{\mathbf{a_2}}, \hat{\mathbf{a_3}})$ unit vectors at P in the direction normal to $u_i = u_i(P)$, s.t u_i increasing in the direction $\hat{\mathbf{a_i}}$

$$\hat{\mathbf{a_i}} = \frac{\nabla \mathbf{u_i}}{|\nabla \mathbf{u_i}|}$$

if we have that $(\hat{\mathbf{a_1}}, \hat{\mathbf{a_2}}, \hat{\mathbf{a_3}})$ mutually orthogonal $\implies$ **orthogonal curvilinear coordinate system.**

$$\frac{\partial \mathbf{r}}{\partial u_i} = \hat{\mathbf{e_i}} h_i$$

For which we define $h_i = |\partial\mathbf{r}/\partial u_i|$. We call these the **length scales**

1.9.2 Path element

$\mathbf{r} = \mathbf{r}(u_1, u_2, u_3)$ **path element** $d\mathbf{r}$ given by

$$\begin{aligned} d\mathbf{r} &= \frac{\partial \mathbf{r}}{\partial u_1}du_1 + \frac{\partial \mathbf{r}}{\partial u_2}du_2 + \frac{\partial \mathbf{r}}{\partial u_3}du_3 \\ &= h_1 du_1 \hat{e_1} + h_2 du_2 \hat{e_2} + h_3 du_3 \hat{e_3} \end{aligned}$$

For an orthongal system

$$(ds)^2 = (d\mathbf{r})\cdot(d\mathbf{r}) = h_1(du_1)^2 + h_2(du_2)^2 + h_3(du_3)^2$$
$$\hat{e}_i = \hat{a}_i = \frac{\nabla \mathbf{u_i}}{|\nabla \mathbf{u_i}|}$$

1.9.3 Volume Element

$$\begin{aligned} d\tau &= (h_1 du_1)(h_2 du_2)(h_3 du_3) \\ &= h_1 h_2 h_3 du_1 du_2 du_3 \end{aligned}$$

1.9.4 Surface element

For u_1 constant.

$$dS = h_2 h_3 du_2 du_3$$

similarly for u_2, u_3

1.9.5 Properties of various orthogonal coordinates

(i) **Cartesisan coordinates** (x, y, z)

$$d\tau = dxdydz \qquad d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$$
$$(ds)^2 = (d\mathbf{r}) \cdot (d\mathbf{r}) = (dx)^2 + (dy)^2 + (dz)^2$$

We have $h_1 = h_2 = h_3$

(ii) **Cylindrical polar coordinates** (r, ϕ, z)
Related to cartesian by

$$x = r\cos\theta \quad y = r\sin\phi \quad z = z$$

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r} &= (\frac{\partial x}{\partial r})\hat{\mathbf{i}} + (\frac{\partial y}{\partial r})\hat{\mathbf{j}} + (\frac{\partial z}{\partial r})\hat{\mathbf{k}} = (\cos\phi)\hat{\mathbf{i}} + (\sin\phi)\hat{\mathbf{j}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial \mathbf{r}}{\partial \phi}) &= 0 & h_1 &= |\frac{\partial \mathbf{r}}{\partial r}| = 1 \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (\frac{\partial x}{\partial \phi})\hat{\mathbf{i}} + (\frac{\partial y}{\partial \phi})\hat{\mathbf{j}} + (\frac{\partial z}{\partial \phi})\hat{\mathbf{k}} = -(r\sin\phi)\hat{\mathbf{i}} + (r\cos\phi)\hat{\mathbf{j}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial z}{\partial \phi}) &= 0 & h_2 &= |\frac{\partial \mathbf{r}}{\partial \phi}| = r \\
\frac{\partial \mathbf{r}}{\partial z} &= \hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial \phi}) \cdot (\frac{\partial \mathbf{r}}{\partial z}) &= 0 & h_3 &= |\frac{\partial \mathbf{r}}{\partial z}| = 1
\end{aligned}$$

Yielding length and volume elements:

$$(ds)^2 = (dr)^2 + r^2(d\phi)^2 + (dz)^2 \qquad d\tau = rdrd\phi dz$$

(iii) **Spherical polar coordinates** (r, θ, ϕ)
Related to cartesian by:

$$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$$

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r} &= (\sin\theta\cos\phi)\hat{\mathbf{i}} + (\sin\theta\sin\phi)\hat{\mathbf{j}} + (\cos\theta)\hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial \mathbf{r}}{\partial \theta}) &= 0 & h_1 &= |\frac{\partial \mathbf{r}}{\partial r}| = 1 \\
\frac{\partial \mathbf{r}}{\partial \theta} &= (r\cos\theta\cos\phi)\hat{\mathbf{i}} + (r\cos\theta\sin\phi)\hat{\mathbf{j}} + (-r\sin\theta)\hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial r}) \cdot (\frac{\partial \mathbf{r}}{\partial \phi}) &= 0 & h_2 &= |\frac{\partial \mathbf{r}}{\partial \theta}| = r \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (-r\sin\theta\sin\phi)\hat{\mathbf{i}} + (r\sin\theta\cos\phi)\hat{\mathbf{j}} + (0)\hat{\mathbf{k}} & (\frac{\partial \mathbf{r}}{\partial \phi}) \cdot (\frac{\partial \mathbf{r}}{\partial \theta}) &= 0 & h_3 &= |\frac{\partial \mathbf{r}}{\partial \phi}| = r\sin\theta
\end{aligned}$$

Volume element:

$$d\tau = r^2 \sin\theta dr d\theta d\phi$$

1.9.6 Gradient in orthogonal curvilinear coordinates

Let $\nabla\Phi = \lambda_1\hat{\mathbf{e}}_\mathbf{1} + \lambda_2\hat{\mathbf{e}}_\mathbf{2} + \lambda_3\hat{\mathbf{e}}_\mathbf{3}$.
In a general coordinate system for λ_is to be found.

$$d\mathbf{r} = h_1 du_1 \hat{e}_1 + h_2 du_2 \hat{e}_2 + h_3 du_3 \hat{e}_3$$

$$\begin{aligned}
d\mathbf{\Phi} &= (\frac{\partial \phi}{\partial u_1})du_1 + (\frac{\partial \phi}{\partial u_2})du_2 + (\frac{\partial \phi}{\partial u_3})du_3 \\
&= (\frac{\partial \phi}{\partial x})dx + (\frac{\partial \phi}{\partial y})dy + (\frac{\partial \phi}{\partial z})dz \\
&= \boxed{(\nabla\Phi) \cdot d\mathbf{r} = \lambda_1 h_1 du_1 + \lambda_2 h_2 du_2 + \lambda_3 h_3 du_3}
\end{aligned}$$

$$h_i\lambda_i = \frac{\partial\Phi}{\partial u_i}$$

$$\implies \nabla\Phi = \frac{\hat{\mathbf{e}}_1}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\hat{\mathbf{e}}_2}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\hat{\mathbf{e}}_3}{h_3}\frac{\partial\Phi}{\partial u_3}$$

(i) **Cylindrical polars** (r, ϕ, z)

We have: $h_1 = 1$, $h_2 = r$, $h_3 = 1$ $\implies \nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\phi}}{r}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}$

(ii) **Spherical polars** (r, θ, ϕ)

We have: $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$ $\implies \nabla = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial\phi}$

1.9.7 Expressions for unit vectors

$$\hat{\mathbf{e}}_i = h_i\nabla u_i$$

Alternatively, unit vectors orthogonal $\implies$ if we know 2 already then

$$\hat{\mathbf{e}}_1 = (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) = h_2h_3(\nabla u_2 \times \nabla u_3)$$

1.9.8 Divergence in orthogonal curvilinear coordinates

Suppose we have vector field

$$\mathbf{A} = A_1\hat{\mathbf{e}}_1 + A_2\hat{\mathbf{e}}_2 + A_3\hat{\mathbf{e}}_3$$

$$\implies \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_3h_1) + \frac{\partial}{\partial u_3}(A_3h_1h_2)\right\}$$

So we have divergence in other coordinate systems as follows:

(i) **Cylindrical polars** (r, ϕ, z)

We have: $h_1 = 1$, $h_2 = r$, $h_3 = 1$ $\implies \nabla\cdot A = \frac{\partial A_1}{\partial r} + \frac{A_1}{r} + \frac{1}{r}\frac{\partial A_2}{\partial\phi} + \frac{\partial A_3}{\partial z}$

(ii) **Spherical polars** (r, θ, ϕ)

We have: $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$ $\implies \nabla\cdot A = \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}(r^2\sin\theta A_1) + \frac{\partial}{\partial\theta}(r\sin\theta A_2) + \frac{\partial}{\partial\phi}(rA_3)\right\}$

1.9.9 Curl in orthogonal curvilinear coordinates

$$curl\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$$

(i) **Cylindrical polars**

$$curl\mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & \hat{\mathbf{k}} \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_1 & A2 & A_3 \end{vmatrix}$$

(ii) **Spherical polars**

$$curl\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\mathbf{r}} & r\hat{\phi} & r\sin\theta\hat{\phi} \\ \partial/\partial r & \partial/\partial\phi & \partial/\partial z \\ A_1 & rA2 & r\sin\theta A_3 \end{vmatrix}$$

1.9.10 The Laplacian in orthogonal curvilinear coordinates

From the above grad and div;

$$\nabla^2\Phi = \nabla\cdot(\nabla\Phi)$$
$$= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3}\right)\right\}$$

(i) **Cylindrical polars** (r,ϕ,z)

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\phi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\Phi}{\partial z}\right)\right\}\\ &= \frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\phi^2}+\frac{\partial^2\Phi}{\partial z^2}\end{aligned}$$

(ii) **Spherical polars** (r,θ,ϕ)

$$\begin{aligned}\nabla^2\Phi &= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\Phi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{\partial}{\partial\phi}\left(\frac{1}{\sin\theta}\frac{\partial\Phi}{\partial\phi}\right)\right\}\\ &= \frac{\partial^2\Phi}{\partial r^2}+\frac{2}{r}\frac{\partial\Phi}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial\Phi}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\end{aligned}$$

1.10 Changes of variables in surface integration

Suppose we have surface S, parametrized by quantities u_1, u_2. We can write:

$$x = x(u_1,u_2),\quad y = y(u_1,u_2),\quad z = z(u_1,u_2)$$

Consider surface to be comprised of arbitrarily small parallelograms, its sides given by keeping either u_1 or u_2

$$\begin{aligned}dS &= \text{ Area of parallelogram with sides } \frac{\partial\mathbf{r}}{\partial u_1}du_1 \text{ and } \frac{\partial\mathbf{r}}{\partial u_2}du_2\\ &= |\mathbf{J}|du_1du_2\end{aligned}$$

Vector Jacobian J given by $\mathbf{J} = \frac{d\mathbf{r}}{du_1}\times\frac{d\mathbf{r}}{du_2}$.
Useful in substitution of surface integrals:

$$\int_S f(x,y,z)dS = \int_S F(u_1,u_2)|\mathbf{J}|du_1du_2]$$

$F(u_1,u_2) = f(x(u_1,u_2), y(u_1,u_2), z(u_1,u_2))$

For S a region R in the $x-y$ plane we can write:

$$\int_R f(x,y)dxdy = \int_R F(u_1,u_2)|det(J(x_u))|du_1du_2$$

$$|\mathbf{J}| = |\frac{d\mathbf{r}}{du_1}\times\frac{d\mathbf{r}}{du_2}| = det(J(x_u)) = \begin{vmatrix}\partial x/\partial u_1 & \partial x/\partial u_2\\ \partial y/\partial u_1 & \partial y/\partial u_2\end{vmatrix}$$

For a surface described by $z = f(x,y)$. We have $x = u_1, y = u_2$ and $\mathbf{r} = (x,y,f(x,y))$
We have:

$$\begin{aligned}\frac{\partial\mathbf{r}}{\partial u_1} &= \frac{\partial\mathbf{r}}{\partial x} &= \hat{\mathbf{i}}+\frac{\partial f}{\partial x}\hat{\mathbf{k}}\\ \frac{\partial\mathbf{r}}{\partial u_2} &= \frac{\partial\mathbf{r}}{\partial y} &= \hat{\mathbf{j}}+\frac{\partial f}{\partial y}\hat{\mathbf{k}}\end{aligned}$$

$$\begin{aligned}|\frac{\partial\mathbf{r}}{\partial u_1}\times\frac{\partial\mathbf{r}}{\partial u_2}| &= \left\|\begin{matrix}\hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}}\\ 1 & 0 & \partial f/\partial x\\ 0 & 1 & \partial f/\partial y\end{matrix}\right\|\\ &= \sqrt{1+|\nabla f|^2}\end{aligned}$$

So we have area of surface given by

$$\int_\Sigma \sqrt{1+|\nabla f|^2}dxdy$$

for the projection of S onto the $x-y$ plane.

Part II

Term 2

1 Introduction

1.1 ODEs and initial value problems

Definition 1.2. Ordinary differential equation

Consider $d \in \mathbb{N}$ an open set $D \subset \mathbb{R} \times \mathbb{R}^d$ and function $f : D \to \mathbb{R}^d$ Call

$$\dot{x} = f(t, x)$$

a **d-dimensional (first-order) ordinary differential equation**

Differentiable function $\lambda : I \to \mathbb{R}^d$ on interval $I \subset \mathbb{R}$ a **solution** to a differential equation if $(t, \lambda(t)) \in D$ and

$$\dot{\lambda}(t) = f(t, \lambda(t)) \quad \forall t \in I$$

Say ODE **autonomous** if of form

$$\dot{x} = f(x)$$

for $f : D \to \mathbb{R}^d, D \subset \mathbb{R}^d$

Proposition 1.3.

$D \subset \mathbb{R}^d$ open. $f : D \to \mathbb{R}^d$ with autonomous ODE

$$\dot{x} = f(x)$$

$\implies \exists$ constant solution $\lambda : \mathbb{R} \to \mathbb{R}^d$ with $a \in \mathbb{R}^d$ at $\lambda(t) = a \iff f(a) = 0 \forall t$

Definition 1.4. Initial value problem

$d \in \mathbb{N}$ open $D \subset \mathbb{R} \times \mathbb{R}^d,\ f : D \to \mathbb{R}^d$.
Call the following pair a **initial value problem**

$$\underbrace{\dot{x} = f(t, x)}_{\text{ODE}} \text{ and } \underbrace{x(t_0) = x_0}_{\text{Initial condition}}$$

Solutions s.t $\lambda : I \to \mathbb{R}^d$ with t_0 in interior of I and $\lambda(t_0) = x_0$

1.3 Visualisations

1.3.1 Solution portrait

$f : D \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ with $\dot{x} = f(t, x)$
Graph of solutions given by

$$\textbf{Solution Curve}: G(\lambda) = \{(t, \lambda(t)) : t \in I\} \subset \mathbb{R} \times \mathbb{R}^d$$

derivative of curve at point $t_0 \in I$ is

$$\frac{d}{dt}(t, \lambda(t))|_{t=t_0} = (t, \dot{\lambda}(t_0)) = (1, f(t_0, \lambda(t_0))$$

Vector field a map $(t, x) \mapsto (1, f(t, x))$, defined on D

Solution Curves are tangential to vector field.
Solution portrait given by visualisations of several solution curves in both

$$\underbrace{(t, x) - \text{space}}_{\text{extended phase space}} \text{ and } \underbrace{x - \text{space}}_{\text{phase space}}$$

1.3.2 Phase Portraits

Autonomous differential equations not dependent on time. Visualisations in phase-space alone suffice.

Proposition 1.9. ***(Translation invariance)***

$\lambda : I \to \mathbb{R}^d$ a solution to $\dot{x} = f(x)$
$\implies \forall \tau \in \mathbb{R}, \mu : \tilde{I} \to \mathbb{R}^d$ where $\tilde{I} = \{t \in \mathbb{R} : t + \tau \in I\}$
$\mu(t) = \lambda(t + \tau),\ \forall t \in \tilde{I}$ also a solution to this differential equation.

2 Existence & Uniqueness

2.1 Picard iterates

Proposition 2.1. - (Reformation as integral equation)

Consider initial value problem $\dot{x} = f(t, x), \quad x(t_0) = x_0$
for $f : D \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ continuous and $(t_0, x_0) \in D$
$\lambda : I \to \mathbb{R}^d$ a function on interval I s.t $t_0 \in I$ and $\{(t, \lambda(t)) : t \in I\} \subset D$
Following are equivalent:

(i) λ solves initial value problem
$\dot{\lambda}(t) = f(t, \lambda(t)),\ \forall t \in I$
$\lambda(t_0) = x_0$

(ii) λ continuous and

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))ds \ \forall t \in I$$

Higher dimensional derivative
for $g : \mathbb{R} \to \mathbb{R}^d$

$$\int_{t_0}^{t} g(s)ds = \begin{pmatrix} \int_{t_0}^{t} g_1(s)ds \\ \vdots \\ \int_{t_0}^{t} g_d(s)ds \end{pmatrix}$$

Definition 2.2. (Picard iterates)

Consider initial value problem; $\dot{x} = f(t, x) \quad x(t_0) = x_0$ and chosen interval J s.t $t_0 \in J$
Define **initial function**:

$$\lambda_0(t) \equiv x_0 \quad \forall t \in J$$

and inductively the **Picard iterates**:

$$\lambda_{n+1}(t) := x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))ds \quad \forall t \in J \ \forall n \in \mathbb{N}_0$$

If (λ_n) uniformly convergent sequence with limit λ_∞ obtain:

$$\lambda_\infty(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_\infty(s))ds \ \forall t \in J$$

$\implies \lambda_\infty$ a solution to integral equation $\implies$ solves initial value problem

2.2 Lipschitz Continuity

Definition

Space of continuous functions on compact interval J $:= C^0(J, \mathbb{R}^d)$
This a complete normed vector space under supremum norm. (Banach Space)

Definition 2.4. (Normed Vector Space)

Norm on a vector space V orer R a map $||\cdot|| : V \to \mathbb{R}_0$ s.t

(i) $||| = 0 \iff x = 0$

(ii) $||| =| \cdot |||, \ \forall a \in \mathbb{R}, x \in V$

(iii) $|+y|| \leqslant ||| + |||$

Normed vector space V **complete** if every cauchy sequence converges in V
Call a complete normed vector space a **Banach Space**

Definition 2.5. (Continuous + Lipschitz continuous functions)

$X \subset$ normed vector space $(V, ||\cdot||_V)$
$Y \subset$ normed vector space $(W, ||\cdot||_W)$
We say a function $f : X \to Y$

(i) **Continuous** if

$$\forall x \in X, \epsilon > 0, \exists \delta > 0, |-\bar{x}||_V < d \implies |(x) - f(\bar{x})||_W < \epsilon$$

(ii) **Lipschitz Continuous** if

$$\exists K > 0, |(x) - f(\bar{x})||_W \leqslant K|-\bar{x}||_V \ \forall x, \bar{x} \in X$$

Call K a **Lipschitz Constant**

$$\text{Lipschitz continuous} \implies \text{Continuous}$$

2.2.1 Lipschitz Continuity and MVT

Theorem 2.0. (Mean Value Theorem)

I compact interval, f continuously differentiable
$\forall x, y \in I, \ \exists \xi \in (x, y)$ s.t
$f(x) - f(y) = f'(\xi)(x - y)$
$\implies f'$ bounded $\implies f$ Lipschitz continuous

2.2.2 Lipschitz Continuity and Mean Value Inequality

Definition 2.7. (Operator norm of a matrix)

For given matrix $A \in M_n(\mathbb{R})$ Operator norm:

$$||| = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{|||}{|||} = \sup_{x \in \mathbb{R}^n \setminus \{0\}} \left|\left| A \frac{x}{|||} \right|\right| = \sup_{x \in \mathbb{R}^n, |||=1} |||$$

Theorem 2.1. (Mean Value Inequality)

Consider open set $D \subset \mathbb{R}^n$ with $f : D \to \mathbb{R}^m$ continuously differentiable
$\forall x, y \in D$ with $[x, y] \subset D$

$$\exists \xi \in [x, y] \ s.t \ |(x) - f(y)|| \leqslant |f'(\xi)|||x - y||$$

$\forall x, y \in \mathbb{R}^n$, closed line segment connecting x and y given by

$$[x, y] = \{\alpha x + (1 - \alpha)y \in \mathbb{R}^n : \alpha \in [0, 1]\}$$

Lemma 2.9. (Triangle-like inequality for integrals)

$I \subset R$ an interval
$f : I \to \mathbb{R}^m$ continuous function

$$\implies \left|\left| \int_{t_0}^{t} f(s) ds \right|\right| \leqslant \left| \int_{t_0}^{t} ||f(s)|| ds \right| \quad \forall t, t_0 \in I$$

Corollary 2.10. - (Lipschitz continuous and mean value inequality)

$U \subset \mathbb{R}^n$ open. $f : U \to \mathbb{R}^m$ continuously differentiable
Given compact and convex set $C \subset U$. Restriction is Lipschitz continuous

$$f|_C : C \to \mathbb{R}^m$$

Convex C means $\forall x, y, \in C$ closed line segment lies in C i.e. $[x, y] \subset C$

2.3 Picard-Lindelöf Theorem

Theorem 2.11. *(Picard-Lindelöf theorem - global version)*

Consider ODE $\dot{x} = f(t, x)$
$f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ continuous, satisfying global Lipschitz condition of the form

$$||f(t, x) - f(t, y)|| \leqslant K||x - y|| \quad \forall t \in \mathbb{R}, \forall x, y \in \mathbb{R}^d,\ K > 0 \text{ a const}$$

Take $h = \frac{1}{2K} \implies$ every intial value problem $x(t_0) = x_0$ admits a unique solution

$$\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$$

Definition 2.12.

(i) **Globally Lipschitz continuous**
if $\exists K > 0$ s.t $||f(t, x) - f(t, y)|| \leqslant K||x - y|| \quad \forall (t, x), (t, y) \in D$

(ii) **Locally Lipschitz continuous**
if $\forall (t_0, x_0) \in D$ and $\exists$ neighbourhood $U \subset D$ of (t_0, x_0) and $\exists L > 0$ s.t

$$||f(t, x) - f(t, y)|| \leqslant K||x - y|| \quad \forall (t, x), (t, y) \in U$$

Theorem 2.13. *(Picard-Lindelöf theorem - local version)*

$D \subset \mathbb{R} \times \mathbb{R}^d$ open
Consider function $f : D \to \mathbb{R}^d$ continuous and locally Lipschitz continuous.
For fixed $(t_0, x_0) \in D$, we have intial value problem. Following 2 hold

(i) ***Qualitative version***
Initial value problem has locally a uniquely determined solution

$$\exists h = h(t_0, x_0) \text{ s.t. there is exactly one solution on } [t_0 - h, t_0 + h]$$

(ii) ***Quantitative version***
For some τ, δ take set $W^{\tau,\delta}(t_0, x_0) := [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)}$. For $\overline{B_\delta(x_0)} := \{x \in \mathbb{R}^d : ||x - x_0|| \leqslant \delta\}$ - Closed $\delta-$neighbourhood of x_0.
Assume $W^{\tau,\delta}(t_0, x_0) \subset D$, suppose $\exists K, M > 0$ s.t

$$||f(t, x) - f(t, y)|| \leqslant K||x - y|| \quad \forall (t, x), (t, y) \in U$$

and

$$||f(t, x)|| \leqslant M \quad \forall (t, x) \in W^{\tau,\delta}(t_0, x_0)$$

$\implies$ there is exactly one solution on $[t_0 - h, t_0 + h]$ with $h(t_0, x_0) := \min\{\tau, \frac{1}{2K}, \frac{\delta}{M}\}$

Proposition 2.14. - (Continuously differentiable & Lipschitz Continuity)
$D \subset \mathbb{R} \times \mathbb{R}^d$ open. Continuously differentiable function $f : D \to \mathbb{R}^d$

$$\implies f \text{ locally Lipschitz continuous w.r.t } x$$
$$\implies \text{every intial value problem with differential equation with RHS } f \text{ solved locally uniquely.}$$

Lemma 2.15. - (Solutions cannot cross)
Let $D \subset \mathbb{R} \times \mathbb{R}^d$ open. $f : D \to \mathbb{R}^d$ continuous and locally Lipschitz continuous w.r.t x
Given 2 solutions of $\dot{x} = f(t, x)$; $\lambda : I \to \mathbb{R}^d, \mu : J \to \mathbb{R}^d$
Either $\lambda(t) = \mu(t) \quad \forall t \in I \cap J$ or $\lambda(t) \neq \mu(t) \quad \forall t \in I \cap J$

2.4 Maximal Solutions

Definition 2.16. - (Maximal existence interval)

Consider initial value problem $\dot{x} = f(t, x),\ x(t_0) = x_0$ Define

- $I_+(t_0, x_0) := \sup\{t_+ \geqslant t_0 : \text{there exists solution on } [t_0, t_+]\}$
- $I_-(t_0, x_0) := \sup\{t_- \leqslant t_0 : \text{there exists solution on } [t_-, t_0]\}$

Maximal existence interval:

$$I_{max}(t_0, x_0) := (I_-(t_0, x_0), I_+(t_0, x_0))$$

Theorem 2.17. (Existence of maximal solution + boundary behaviour)

There exists maximal solution $\lambda_{max} : I_{max}(t_0, x_0) \to \mathbb{R}^d$ to initial value problem. Having properties:

(i) $I_+(t_0, x_0)$ finite
Either - maximal solution unbounded for $t \geqslant t_0$

$$\sup_{t \in (t_0, I_+(t_0, x_0))} ||\lambda_{max}(t)|| = \infty$$

Or boundary: ∂D of D non-empty and we have

$$\lim_{t \nearrow I_+(t_0, x_0)} dist\left((t, \lambda_{max}(t)), \partial D\right) = 0$$

(ii) $I_-(t_0, x_0)$ finite
Either - maximal solution unbounded for $t \leqslant t_0$

$$\sup_{t \in (I_-(t_0, x_0), t_0)} ||\lambda_{max}(t)|| = \infty$$

Or boundary: ∂D of D non-empty and we have

$$\lim_{t \searrow I_-(t_0, x_0)} dist\left((t, \lambda_{max}(t)), \partial D\right) = 0$$

Dist function
$A \subset \mathbb{R}^n,\ dist(\cdot, A) : \mathbb{R}^n \to \mathbb{R}_0^+$

$$dist(y, A) := \inf\{|-a|| : a \in A\} \quad \forall y \in \mathbb{R}^n$$

2.5 General solutions and flows

2.5.1 General solutions

Definition 2.19. (General solution to non-autonomous differential equation)

Consider $\dot{x} = f(t, x)$. We define

$$\Omega := \{(t, t_0, x_0) \in \mathbb{R}^{1+1+d} : (t_0, x_0) \in D \text{ and } t \in I_{max}(t_0, x_0)\}$$

We say $\lambda : \Omega \to \mathbb{R}^d$ with $\lambda(t, t_0, x_0) := \lambda_{max}(t, t_0, x_0)$ a **general solution** of $\dot{x} = f(t, x)$
Solution identity:

$$\frac{\partial \lambda}{\partial t}(t, t_0, x_0) = f(t, \lambda(t, t_0, x_0)) \quad \forall (t, t_0, x_0) \in \Omega$$

Proposition 2.21. - (Properties of general solutions)

Consider $\dot{x} = f(t, x)$, $(t_0, x_0) \in D \implies \forall s \in I_{max}(t_0, x_0)$ we have

(i) $I_{max}(s, \lambda(s, t_0, x_0)) = I_{max}(t_0, x_0)$

(ii) $\lambda(t_0, t_0, x_0) = x_0$ **(Initial value property)**

(iii) $\lambda(t, s, \lambda(s, t_0, x_0)) = \lambda(t, t_0, x_0) \forall t \in I_{max}(t_0, x_0)$ **(Cocycle property)**

2.5.2 Flows

Definition 2.22. (Flow of an autonomous differential equation)

Consider $\dot{x} = f(x)$
Define for any initial value $x_0 \in D$

$$J_{max}(x_0) := I_{max}(0, x_0)$$
$$\varphi(t, x_0) = \lambda(t, 0, x_0) \quad \forall t \in J_{max}(x_0)$$

$(t, x_0) \mapsto \phi(t, 0, x_0)$ called **flow of autonomous differential equation**
Solution identity:

$$\frac{\partial \varphi}{\partial t}(t, x_0) = f(\varphi(t, x_0)) \quad \forall x_0 \in D, t \in J_{max}(0)$$

Proposition 2.24 - (Properties of the flow)

Let φ be flow of autonomous differential equation. $\implies \forall x \in D$ we have

(i) $J_{max}(\varphi(t, x)) = J_{max}(x) - t \quad \forall t \in J_{max}(x)$

(ii) $\varphi(0, x) = x$ **(Initial value property)**

(iii) $\varphi(t, \varphi(s, x)) = \varphi(t + s, x) \quad \forall t, s$ with $s, t + s \in J_{max}(x)$ **(Group property)**

(iv) $\varphi(-t, \varphi(t, x)) = x \quad \forall t \in J_{max}(x)$

Definition 2.25. (Orbits (or trajectories))

φ flow of autonomous differential equations $\forall x \in D$, we have the **Orbit** through x

$$O(x) := \{\varphi(t, x) \in D : t \in J_{max}(x)\}$$

With the positive/negative half orbits:

- $O^+(x) := \{\varphi(t, x) \in D : t \in J_{max}(x) \cap \mathbb{R}_0^+\}$
- $O^-(x) := \{\varphi(t, x) \in D : t \in J_{max}(x) \cap \mathbb{R}_0^-\}$

Types of orbits

(i) $O(x)$ singleton $\implies f(x) = 0$ and $J_{max}(x) = \mathbb{R}$
Call x the equilibrium

(ii) $O(x)$ closed curve $\exists t > 0$ s.t $\varphi(t, x) = x$ but $f(x) \neq 0 \implies J_{max}(x) = \mathbb{R}$, call x periods with $O(x)$ periodic orbit

(iii) $O(x)$ not closed curve. function $t \mapsto \varphi(t, x)$ injective on $J_{max}(x)$

Proposition 2.27. - (Orbits of one-dimensional differential equation)
Consider $\dot{x} = f(x)$ where $d = 1$
$\implies$ all solutions monotone, $\nexists$ periodic orbits
$\implies$ trajectory either an equilibrium or non-closed curve

3 Linear Systems

3.1 Matrix exponential function

Consider linear differential equation

$$\dot{x} = Ax \quad A \in \mathbb{R}^{d \times d}$$

We have $||Ax - Ay|| \leq ||A||||x - y|| \implies$ globally Lipschitz continuous with constant $||A||$

Solution to every intial value problem exists and are unique.
$\implies$ generates globally defined flow $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$

Picard iterates for local solutions
$\lambda_0(t) := x_0 \forall t \in J$
$\lambda_{n+1} = P(\lambda_n)(t) = x_0 + \int_0^t A\lambda_n(s)ds \implies \lambda_n = \sum_{k=0}^n \frac{t^k A^k}{k!} x_0$
$\implies \lambda_\infty(t) = \varphi(t, x_0)e^{At}x_0$
We have the series converge whenever $|t| \leq h$ for some $h > 0$

Definition 3.1. (Matrix exponential function)

$$t \mapsto e^{At} \qquad e^{At} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

Lemma 3.1.

$$||BC|| \leqslant ||B||||C||$$

Proposition 3.2. - (Existence of matrix exponential)
Matrix $B \in \mathbb{R}^{d\times d}$

$$e^B := \sum_{k=0}^{\infty} \frac{1}{k!} B^k \in \mathbb{R}^{d\times d}$$

exists

Theorem 3.3. (Flow of an autonomous linear differential equation)

Consider $\dot{x} = Ax, \quad A \in \mathbb{R}^{d\times d}$
Flow $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\varphi(t, x) = e^{At}x \quad \forall t \in \mathbb{R}$$

Proposition 3.4. - (Properties of matrix exponential)

(i) $C = T^{-1}BT \implies e^C = T^{-1}e^B T$

(ii) $e^{-B} = (e^B)^{-1}$

(iii) $BC = CB \implies e^{B+C} = e^B e^C$

(iv) $B = diag(B_1, \dots, B_p) \implies e^B = diag(e^{B_1}, \dots, e^{B_p})$

3.2 Planar linear systems

Consider $\dot{x} = Ax, A \in \mathbb{R}^{2\times 2}$
Transform A in Jordan normal form $\implies J = T^{-1}AT$, T invertible
$\implies e^{AT} = Te^{Jt}T^{-1}$

(C1) A has 2 distinct real eigenvalues, $a, b \in \mathbb{R} : J = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$

(C2) A has double real eigenvalues $a \in \mathbb{R}$, with 2 linearly independent eigenvectors: $J = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

(C3) A double real eigenvalues with 1 eigenvector : $J = \begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix}$

(C4) A has 2 complex eigenvalues $a \pm b$, $b \neq 0$: $J = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$

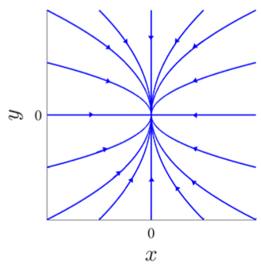
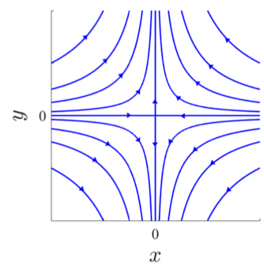
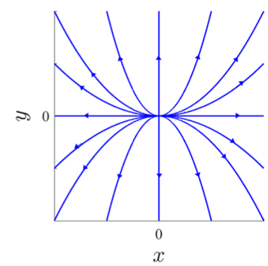
A ***not singular:***

C1

$$J = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad a, b \in \mathbb{R}\backslash\{0\},\ a \neq b$$

$$e^{Jt} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix} \quad \forall\ t \in \mathbb{R}$$

Trajectory given $O(x_0, y_0) = \{(x, y_0(\frac{x}{x_0})^{b/a} \in \mathbb{R}^2 : \frac{x}{x_0} > 0)\}$
Obtaining the following phase portraits:

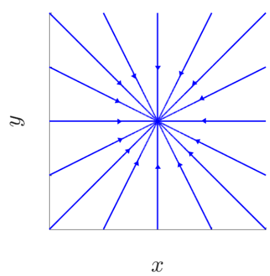
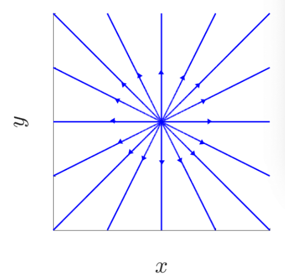
$a < b < 0$	$a < 0 < b$	$0 < a < b$
Stable knot - 2 tangents	Saddle	Unstable knot - 2 tangents
		

C2

$$J = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \quad a \in \mathbb{R}\backslash\{0\}$$

$$e^{Jt} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{at} \end{pmatrix} \quad \forall\ t \in \mathbb{R}$$

Trajectory given $O(x_0, y_0) = \{(x_0 e^{at}, y_0 e^{at}) : t \in \mathbb{R}\} = \{(x, x\frac{y_0}{x_0}) \in \mathbb{R}^2 : \frac{x}{x_0} > 0\}$
Obtaining the following phase portraits:

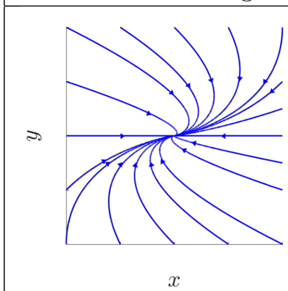
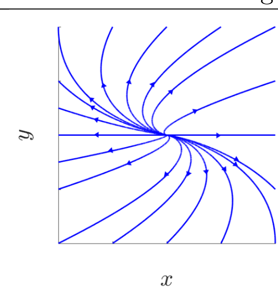
$a < 0$	$a > 0$
Stable knot - many tangents	Unstable knot - many tangents
	

C3

$$J = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix} \quad a \in \mathbb{R}\backslash\{0\}$$

$$e^{Jt} = \begin{pmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{pmatrix} \quad \forall\ t \in \mathbb{R}$$

Trajectory given $O(x_0, y_0) = \{(x_0 e^{at} + y_0 t e^{at}, y_0 e^{at}) : t \in \mathbb{R}\} = \{(\frac{x_0}{y_0} y + \frac{y}{a} \ln \frac{y}{y_0}, y) \in \mathbb{R}^2 : \frac{y}{y_0} > 0\}$
Obtaining the following phase portraits:

$a < 0$	$a > 0$
Stable knot - 1 tangents	Unstable knot - 1 tangents
	

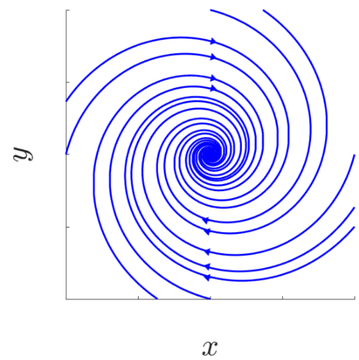
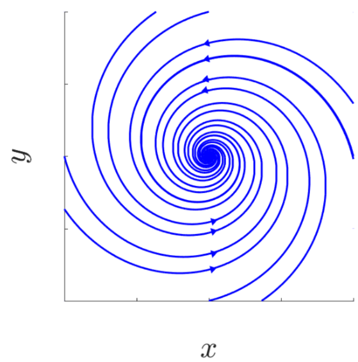
C4

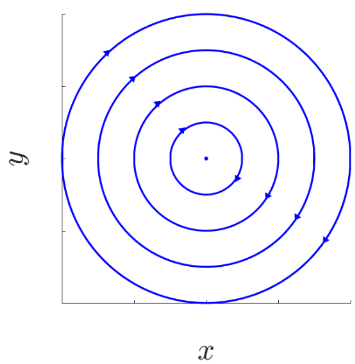
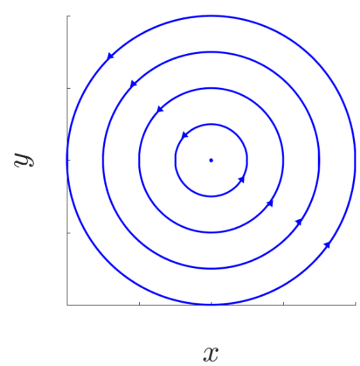
$$J = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \quad a \in \mathbb{R}\backslash\{0\}$$

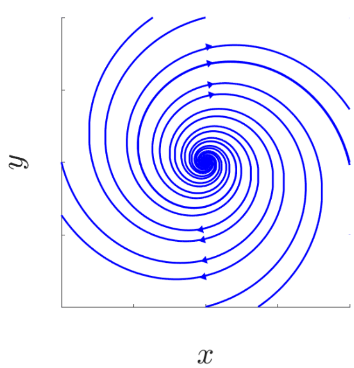
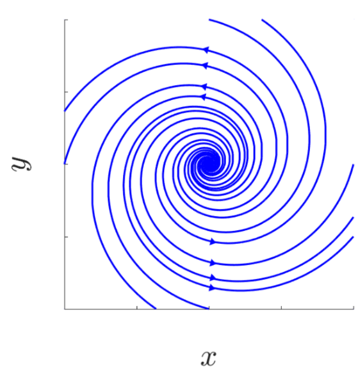
$$e^{Jt} = e^{at}\begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix} \quad \forall\ t \in \mathbb{R}$$

Trajectory given $O(x_0, y_0) = \{e^{at}\begin{pmatrix} x_0\cos(bt) + y_0\sin(bt) \\ y_0\cos(bt) - x_0\sin(bt) \end{pmatrix} : t \in \mathbb{R}\}$

Obtaining the following phase portraits:

$a < 0, b > 0$	$a < 0, b < 0$
Stable focus	Stable focus
	

$a = 0, b > 0$	$a = 0, b < 0$
Centre	Centre
	

$a > 0, b > 0$	$a > 0, b < 0$
Unstable focus	Unstable focus
	

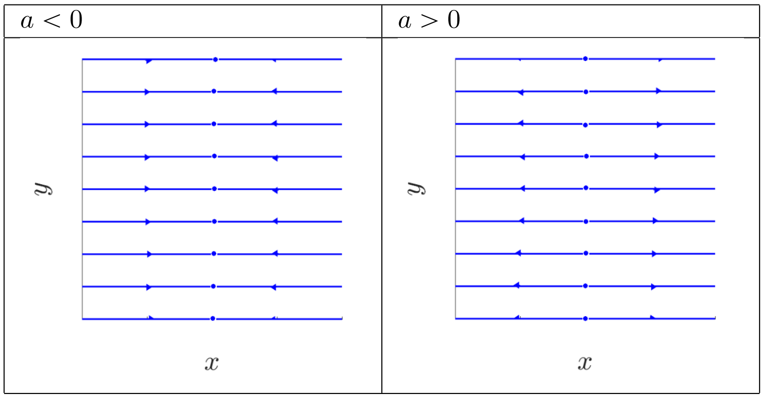
A ***singular:***

C1

$$J = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad a \in \mathbb{R}\backslash\{0\}$$

$$e^{Jt} = \begin{pmatrix} e^{at} & 0 \\ 0 & 1 \end{pmatrix} \quad \forall\ t \in \mathbb{R}$$

Trajectory given by $O(x_0, y_0) = \{(e^{at}x_0, y_0) : t \in \mathbb{R}\}$
Obtaining the following phase portraits:



C2

$$J = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

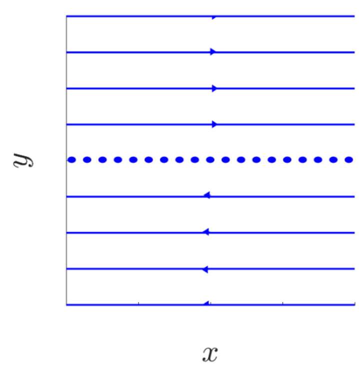
Trivially whole space is equilibria

C3

$$J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad a \in \mathbb{R}\backslash\{0\}$$

$$e^{Jt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \forall\ t \in \mathbb{R}$$

Trajectory given by $O(x_0, y_0) = e^{Jt}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \{(x_0 + ty_0, y_0) : t \in \mathbb{R}\}$
Obtaining the following phase portraits:



C4
Can't happen as a 2D matrix of real eigenvalus can't have eigenvalue of 0.

Remark 3.5 - *(Meaning of real + imaginary parts of e.vals of A)*

(i) Rate of exponential growth
Re[*e.val*] - determines rate of exponential growth behaviour of solution

$$\lambda(t) = e^{At}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Obtain exponential growth rate for $\mu(t) = e^{at}$

$$\lim_{t\to\infty} \frac{\ln e^{at}}{t} = a$$

Lyapunov exponent
For solution λ with initial condition $(x_0, y_0) \neq (0,0)$

$$\sigma_{lyap}(\lambda) = \lim_{t\to\infty} \frac{\ln \|\lambda(t)\|}{t}$$

We have a solution decay if $\sigma_{lyap} < 0$, grow if $\sigma_{lyap} > 0$

(ii) Rate of Rotation
Solution rotates is e.vals not real.
For $a + bi$ an e.val

- $|b|$ - speed of rotation
- $sign(b)$ - orientation of rotation $b > 0 \implies \circlearrowright$, $b < 0 \implies \circlearrowleft$

3.3 Jordan Normal Form

Theorem 3.6 - *Complex Jordan Normal Form*

$A \in \mathbb{R}^{d\times d}, \exists T \in^{d\times d}$ s.t we get

$$J := T^{-1}AT = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{pmatrix}$$

With Jordan blocks

$$J_j = \begin{pmatrix} \rho_j & 1 & & 0 & 0 \\ 0 & \rho_j & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \rho_j & 1 \\ 0 & 0 & & 0 & \rho_j \end{pmatrix} \quad \text{for all } j \in \{1,\dots,p\}$$

For $p_j, j \in \{1,\dots,p\}$ complex e.vals of A
Theorem 3.7 - *Real Jordan Form*

$A \in \mathbb{R}^{d\times d}$, $\exists T \in \mathbb{R}^{d\times d}$ s.t

$$J := T^{-1}AT = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{pmatrix}$$

J_j as in 3.6 if ρ_j real
if ρ_j complex $\implies$

$$J_j = \begin{pmatrix} C_j & I_2 & & 0 & 0 \\ 0 & C_2 & I_2 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & C_j & I_2 \\ 0 & 0 & & 0 & C_j \end{pmatrix} \quad \text{with } C_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix} \ \rho_j = a_j + ib_j$$

3.4 Explicit representation of matrix exponential function

$A \in \mathbb{R}^{d\times d}$

Assume invertible $T \in \mathbb{R}^{d\times d}$ transforms A into real $J := T^{-1}AT = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{pmatrix}$

$$\implies e^{At} = Te^{Jt}T^{-1} = T \begin{pmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_p t} \end{pmatrix} T^{-1}$$

Proposition 3.8

$A \in \mathbb{R}^{d\times d}$ $J_j, j \in \{1, \dots, p\}$

Jordan blocks for real Jordan normal form with eigenvalues ρ_j

(i) ρ_j real

$$\exp\left\{\begin{pmatrix} \rho_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \rho_j \end{pmatrix} t\right\} = e^{\rho_j t} \begin{pmatrix} 1 & t & t^2/2 & \dots & \frac{t^{d_j-1}}{(d_j-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & t^2/2 \\ 0 & & & 1 & t \\ 0 & 0 & & 0 & 1 \end{pmatrix}$$

(ii) $\rho_j = a_j + ib_j \in$

$$\exp\left\{\begin{pmatrix} C_j & I_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & I_2 \\ 0 & & & C_j \end{pmatrix} t\right\} = e^{a_j t} \begin{pmatrix} G(t) & tG(t) & \frac{t^2}{2}G(t) & \dots & \frac{t^{d_j-1}}{(d_j-1)!}G(t) \\ 0 & G(t) & tG(t) & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2}G(t) \\ 0 & & & G(t) & tG(t) \\ 0 & 0 & & 0 & G(t) \end{pmatrix}$$

Where $G(t) = \begin{pmatrix} \cos(b_j t) & \sin(b_j t) \\ -\sin(b_j t) & \cos(b_j t) \end{pmatrix} \quad \forall t \in \mathbb{R}$

3.5 Exponential growth behaviour

Definition 3.2.

Spectrum of A

$$A \in \mathbb{R}^{d\times d} \quad \Sigma(A) = \{Re(\rho) : \rho \text{ eval of } A\} = \{s_1, \dots, s_p\}$$

For $\dot{x} = Ax$ we have decomposition

$$\mathbb{R}^d = E_1 \oplus \dots \oplus E_q$$

E_j invariant

- $x \in E_j \implies \varphi(t, x) \in E_j \ \forall t \in \mathbb{R}$
- $x \in E_j \backslash \{0\} \implies \sigma_{lyap}(\varphi(\cdot, x)) = \lim_{t\to\infty} \frac{\|\varphi(t,x)\|}{t} = s_j$

Definition 3.3.

semi-simple eigenvalue

If all Jordan blocks associated to eval in real Jordan normal form are:
- 1 dim. for real e.val
- 2 dim. for non-real e.val

Proposition 3.9 - *Exponential estimate for matrix exponential function*

$A \in \mathbb{R}^{d\times d}$, Choose $\gamma > \max\Sigma(A)$

If all e.vals ρ with $Re(\rho) = \max\Sigma(A)$, semi-simple $\implies$ take $\gamma = \max\Sigma(A)$

$$\implies \exists K > 0 s.t \|e^{At}\| \leqslant Ke^{\gamma t} \quad \forall t \geqslant 0$$

3.6 Variation of constants formula

Proposition 3.10 - *(Variation of constants formula)*

General solution to $\dot{x} = Ax + g(t)$ given by

$$\lambda(t, t_0, x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}g(s)ds \quad \forall t, t_0 \in I, x_0 \in \mathbb{R}^d$$

4 Non-linear systems

4.1 Stability

4.1.1 Basic definitions

Definition 4.1.

x^* an equilibrium of $\dot{x} = f(x) \implies f(x^*) = 0$

(i) x^* **stable** if $\forall \epsilon > 0, \exists \delta > 0$ s.t

$$\|\varphi(t, x) - x^*\| < \epsilon \quad \forall x \in B_\delta(x^*) \text{ and } t \geqslant 0$$

(ii) x^* **unstable** if not stable

(iii) x^* **attractive** if $\exists \delta > 0$ s.t

$$\lim_{t\to\infty} \varphi(t, x) = x^* \quad \forall x \in B_\delta(x^*)$$

(iv) x^* **asymptotically stable** if x^* stable and attractive

(v) x^* **exponentially stable** if $\exists \delta > 0, K \geqslant 1$ and $\gamma < 0$ s.t

$$\|\varphi(t, x) - x^*\| \leqslant Ke^{\gamma t}\|$$

(vi) x^* **repulsive** if $\exists \delta > 0$ s.t $\lim_{t\to-\infty} \varphi(t, x) = x^*, \ \forall x \in B_\delta(x^*)$

$$INSERTFIGURESHERE$$

Definition 4.4. (Homoclinic and heteroclinic orbits)

$\dot{x} = f(x)$ $f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ locally Lipschitz continuous, with flow φ

Orbit $O(x)$ for some $x \in D$

(i) **Homoclinic** orbit if $\exists$ equilibrium $x^* \in D \backslash \{x\}$ s.t

$$\lim_{t\to\infty} \varphi(t, x) = x^* \text{ and } \lim_{t\to-\infty} \varphi(t, x) = x^*$$

(ii) **Heteroclinic** orbit if $\exists$ 2 distinct equilibria $x_1^* \neq x_2^*$ s.t

$$\lim_{t\to\infty} \varphi(t, x) = x_1^* \text{ and } \lim_{t\to-\infty} \varphi(t, x) = x_2^*$$

Theorem 4.5. (Stability of linear systems

Consider autonomous linear system, $\dot{x} = Ax, A \in \mathbb{R}^{d\times d}$
Have trivial equilibrium $x^* = 0$

(i) stable $\iff$
- $Re(\rho) \leqslant 0$ $\forall \rho$ e.vals of A
- e.val ρ semi-simple $\forall$ e.vals ρ of A with $Re(\rho) = 0$

(ii) exponentially stable $\iff$ $Re(\rho) < 0$ $\forall$ e.vals ρ of A

4.1.3 Hyperbolicity

Definition 4.7.

$A \in \mathbb{R}^{d\times d}$ **hyperbolic** if $Re(\lambda) \neq 0 \ \forall \lambda$ e.vals of A
Equilibrium x^* of differential equation $\dot{x} = f(x)$ $f : D \subset \mathbb{R}^d \to \mathbb{R}^d$ continuously differentiable, is **hyperbolic** if matrix $f'(x^*) \in \mathbb{R}^{d\times d}$ hyperbolic.
Lemma 4.9 - *Gronwall Lemma*

Consider continuous function $u : [a, b] \to \mathbb{R}$, let $c, d \geqslant 0$
Assume u satisfies implicit inequality

$$0 \leqslant u(t) \leqslant c + d\int_a^t u(s)ds \quad \forall t \in [a, b]$$

Theorem 4.10. (Linearised stability)

$\dot{x} = f(x)$ $f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ continuously differentiable.
Assume x^* equilibrium of above s.t $\forall$ e.vals $\lambda \in$ of linearisation of $f'(x^*) \in \mathbb{R}^{d\times d}$ we have $Re(\lambda) < 0 \implies x^*$ is exponentially stable.

4.1.5 Stable and unstable sets, invariant sets

Definition 4.12. (Stable + unstable set)

$\dot{x} = f(x)$ $f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ locally lipschitz continuous, with flow φ and equilibria x^*

Stable set of x^*

$$W^s(x^*) = \{x \in D : \lim_{t\to\infty} \varphi(t, x) = x^*\}$$

Unstable set of x^*

$$W^u(x^*) = \{x \in D : \lim_{t\to-\infty} \varphi(t, x) = x^*\}$$

Definition 4.15. (Invariance)

$\dot{x} = f(x)$ $f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ locally lipschitz continuous..

(i) **positively invariant** if $\forall x \in M, O^+(x) \subset M$

(ii) **negatively invariant** if $\forall x \in M, O^-(x) \subset M$

(iii) **invariant** if $\forall x \in M, O(x) \subset M$

4.2 Limit Sets

Definition 4.16. (Omega and alpha limit sets)

$\dot{x} = f(x)$ $f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ locally lipschitz continuous, with flow $\varphi, x \in D$

1. $x_w \in D$ an **omega limit point** of x
   If $\exists$ sequence $\{t_n\}_{n\in\mathbb{N}}$ s.t $\lim_{n\to\infty} t_n = \infty$ and
   $$x_\omega = \lim_{n\to\infty} \varphi(t_n, x)$$
   $\omega(x)$= {all omega limit points of x}

2. $x_\alpha \in D$ an **alpha limit point** of x
   if $\exists$ sequence $\{t_n\}_{n\in\mathbb{N}}$ s.t $\lim_{n\to\infty} t_n = -\infty$ and
   $$x_\alpha = \lim_{n\to\infty} \varphi(t_n, x)$$
   $\alpha(x)$= {all alpha limit points of x}

Proposition 4.19 - *(Alternative characterisation of limit sets)*

φ flow of differential from above $x \in D$

$$\omega(x) = \bigcap_{t \geqslant 0} \overline{O^+(\varphi(t,x))}$$
$$\alpha(x) = \bigcup_{t \leqslant 0} \overline{O^-(\varphi(t,x))}$$

Proposition 4.21 - *(properties of ω, α limit sets)*

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ locally lipschitz continuous, $x \in D$

(i) $\omega(x)$ invariant
if $O^+(x)$ bounded and $\overline{O^+(x)} \subset D \implies \omega(x) \neq \varnothing$ compact

(ii) $\alpha(x)$ invariant if $O^-(x)$ bounded and $\overline{O^-(x)} \subset D \implies \alpha(x) \neq \varnothing$ compact

4.3 Lyapunov functions

Definition 4.22. (Orbital derivatives)

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ locally lipschitz continuous
$V : D \to \mathbb{R}$ continuosly differentiable function.
Define **orbital derivative** $\dot{V}$ of V

$$\dot{V}(x) := V'(x) \cdot f(x) = \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}(x) f_i(x)$$

$V'(x) \in \mathbb{R}^{1 \times d}$ the gradient of V at $x \in D$
$\dot{V}$ describes derivative of V along solution $\mu : ID$ of $\dot{x} = f(x)$

Definition 4.24. (Lyapunov functions)

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ Locally Lipschitz continuous.
$V : D \to \mathbb{R}$ continuously differentiable function
V a Lyapunov function if $\dot{V}(X) \leqslant 0\ \forall x \in D$

Remark.
Lyapunov function decrease along solutions

$$V(\varphi(t,x)) \leqslant V(x)\ \forall t \in [0, \sup J_{max}(x))$$

Proposition 4.25. - *(Sublevel sets of Lyapunov functions are positively invariant)*
$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ Locally Lipschitz continuous, with Lyapunov function $V : D \to \mathbb{R}$
Any sublevel set of form

$$S_c := \{x \in D : V(x) \leqslant c\},\ c \in \mathbb{R}$$

is positively invariant

Theorem 4.26. (Lyapunov's direct method for stability)

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ Locally Lipschitz continuous, x^* an equilibria and $V : D \to \mathbb{R}$ lyapunov funtion s.t

$$V(x^*) = 0, V(x) > 0\ \forall x \in D \backslash \{x^*\} \implies x^* \text{ stable}$$

Theorem 4.28. (La Salle's invariance principal)

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ Locally Lipschitz continuous, with Lyapunov function $V : D \to \mathbb{R}$

$$\omega(x) \subset \{y \in D : \dot{V}(y) = 0\}\ \forall x \in D$$

Corollary 4.30 - *(Reformation of La Salle's invariance principle)*

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ Locally Lipschitz continuous, with Lyapunov function $V : D \to \mathbb{R}$

$$\forall x \in D\omega(x) \subset \underbrace{\text{largest invariant subset of } \{y \in D : \dot{V}(y) = 0\}}_{=\bigcup \text{ invariant subsets of } \{y \in D: \dot{V}(y)=0\}}$$

Theorem 4.31. (Lyapunov's direct method for asymptotic stability)

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ Locally Lipschitz continuous, $x^* \in D, V : D \to \mathbb{R}$ Lyapunov function s.t

$$V(x^*) = 0 \text{ and } V(x) > 0\ \forall x \in D \backslash \{x^*\}$$
$$\dot{V}(x^*) = 0 \text{ and } \dot{V}(x) < 0\ \forall x \in D \backslash \{x^*\}$$

$\implies x^*$ asymptotically stable

Corollary 4.33 - *(Sublevel sets of Lyapunov functions are subsets of domain of attraction)*
$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^d}_{\text{open}} \to \mathbb{R}^d$ Locally Lipschitz continuous, $x^* \in D, V : D \to \mathbb{R}$ Lyapunov function
Consider sublevel sets of Lyapunov function V

$$S_c := \{x \in D : V(x) \leqslant c\}\ \forall c > 0$$

$\implies S_c$ subset of domain of attraction $W^s(x^*)$ if $S_c \subset D$ compact

4.4 Poincaré-Bendixson Theorem

Theorem 4.34. (Poincare-Bendixson Theorem

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^2}_{\text{open}} \to \mathbb{R}^2$ with flow φ continuously differentiable.
Assume for some $x \in D, O^+(x) \subset K$ compact $\subset D$
K containing not more than finitely many equilibria.

One of the following 3 hold for $\omega(x)$

(i) $\omega(x)$ a singleton, consisting of an equilibrium

(ii) $\omega(x)$ a periodic orbit

(iii) $\omega(x)$ consists of equilibria + non-closed orbits
non-closed orbits in $\omega(x)$ converge forward and backward in time to equilibria in $\omega(x)$
$\implies$ either homoclinic or heteroclinic orbits.

Corollary 4.35 - *(Existence of a periodic orbit)*

$\dot{x} = f(x)\ f : \underbrace{D \subset \mathbb{R}^2}_{\text{open}} \to \mathbb{R}^2$ continuously differentiable with flow φ.
Assume for $x \in D$, $O^+(x) \subset K$ compact $\subset D$
D not containing an equilibrium $\implies \omega(x)$ periodic orbit.