MATH50004
Differential Equations

Lecture Notes

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Contents

Chapter 1

Introduction

An algebraic equation over the real numbers is solved by real numbers. For instance, $x^2 - 1 = 0$ is solved by $x = 1$ and $x = -1$. In contrast, ordinary differential equations have functions as their solutions. Let us look at an example.

Example 1.1 (A first example)**.** We consider the differential equation

$$\dot{x} = ax\,, \tag{1.1}$$

where $a \in \mathbb{R}$ is a constant and $\dot{x}$ means $\frac{\mathrm{d}x}{\mathrm{d}t}$. We say that a function $\lambda : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval, solves this differential equation if

$$\dot{\lambda}(t) = \frac{\mathrm{d}\lambda}{\mathrm{d}t}(t) = a\lambda(t) \quad \text{for all } \; t \in I\,.$$

The argument t of the solution λ typically stands for time, and x in (1.1) typically represents the state of a physical, ecological, or other system, so the solution λ describes the *evolution* of the state x in time. It should be noted that there are also lots of applications where t does not stand for time.

The differential equation (1.1) is a very simple but realistic model of applications in nature and society.

For instance, if $a > 0$, this models growth of capital with a interest rate linked to a (note that normally interest rates are given as yearly rates, which correspond to a discrete-time model; a here is a continuous-time interest rate; try as an exercise to convert both rates). For positive capital x, the right hand side of (1.1) is positive, so $\dot{x}$ is increasing, and importantly for the model of capital growth, the increase in capital is *proportional* to the amount of capital available.

In contrast, if $a < 0$, then, if $x > 0$, the right hand side of (1.1) is negative, and $\dot{x}$ is decreasing, proportional to x. This models, for instance, radioactive decay, which describes the decay of certain atoms such as Uranium 238.

It is easy to see that for a given $b \in \mathbb{R}$, the function $\lambda_b : \mathbb{R} \to \mathbb{R}$,

$$\lambda_b(t) = be^{at} \quad \text{for all } \ t \in \mathbb{R}\,,$$

solves (1.1), see Figure 1.1. Are there more solutions to this differential equation? Assume there is another solution $\mu : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mu(t)e^{-at}) = \dot{\mu}(t)e^{-at} - \mu(t)ae^{-at} = a\mu(t)e^{-at} - \mu(t)ae^{-at} = 0$$

for all $t \in I$. Hence, $\mu(t)e^{-at} \equiv b$ for some $b \in \mathbb{R}$, so $\mu(t) = be^{at} = \lambda_b(t)$ for all $t \in I$, which is not a new solution, so all solutions to (1.1) are known to us.

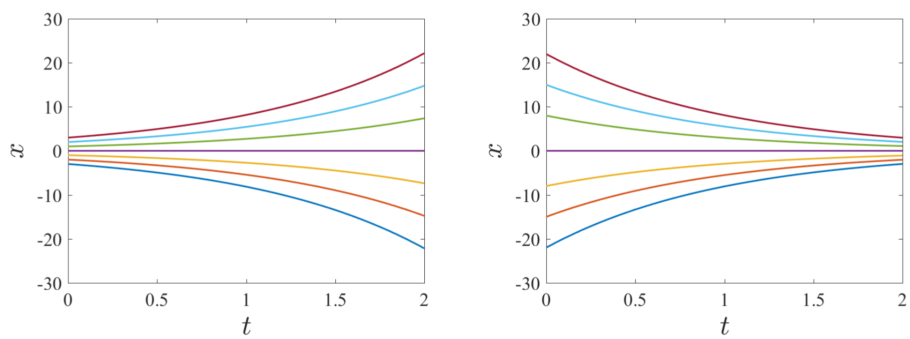


Figure 1.1. Solutions to the differential equation (1.1) for $\alpha = 1 > 0$ (left) and $\alpha = -1 < 0$ (right).

So in contrast to (simple) algebraic equations whose the solutions are in finite-dimensional vector spaces (such as $\mathbb{R}^d$), differential equations are solved by functions, and spaces of functions are typically infinite-dimensional and studied in the mathematical discipline *functional analysis*. Infinite-dimensional spaces are more difficult to grasp in general. However, for a vast majority of the material covered in this course, a finite-dimensional thinking and visualisation is enough to understand the material very well. In places, however, we will need some material from functional analysis to understand differential equations better.

You have encountered ordinary differential equations already last year. In particular, you have learned how to solve certain types of ordinary differential equations. It should be noted that for the most interesting ordinary differential equations, it is not possible to find solutions analytically. In this course, we will learn techniques how to still understand the solutions to these equations without knowing them explicitly. We also address the important question when solutions to ordinary differential equations exist and are unique.

1. Ordinary differential equations and initial value problems

In this section, we look at the definition of an ordinary differential equation and an initial value problem, and we study basic examples.

In Example 1.1, we have studied a differential equation of the form

$$\dot{x} = f(x) \tag{1.2}$$

with $f : \mathbb{R}^d \to \mathbb{R}^d$, where $d = 1$ and $f(x) = \alpha x$. Although such type of equations (i.e. *autonomous* and *first order*) will be mostly studied in this course, we would also like to deal with *nonautonomous differential equations*, i.e. where the right hand side of (1.2) depends on time t, for instance, $\dot{x} = tx^2$ (see Example 1.8 below). We note that *higher-order differential equations* also generalise the situation in (1.2), and you have studied such differential equations already in Year 1. It is demonstrated in *Repetition Material 1* that such differential equations can always be transformed to first-order differential equations, so no separate treatment (with regard to the general theory) is necessary. An example of a higher-order differential equation is given by the harmonic oscillator $\ddot{x} = -x$ (see Example 1.10 below).

The setup for *nonautonomous first-order* differential equations is explained in the next definition.

Definition 1.2 (Ordinary differential equation)**.** *Consider $d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^d$, and a function $f : D \to \mathbb{R}^d$. An equation of the form*

$$\dot{x} = f(t, x) \tag{1.3}$$

is called a d-dimensional (first-order) ordinary differential equation*. A differentiable function $\lambda : I \to \mathbb{R}^d$ on an interval $I \subset \mathbb{R}$ is called a* solution *to the differential equation* (1.3) *if $\big(t, \lambda(t)\big) \in D$ and*

$$\dot{\lambda}(t) = f\big(t, \lambda(t)\big) \quad \text{for all } t \in I\,. \tag{1.4}$$

An ordinary differential equation (1.3) is called *autonomous* if the right hand side does not depend on t, i.e. (1.3) is of the form

$$\dot{x} = f(x)\,,$$

where $f : D \to \mathbb{R}^d$ for some open set $D \subset \mathbb{R}^d$. In this case, we also use the symbol D for the domain of the right hand side f, here as a subset of $\mathbb{R}^d$ instead of $\mathbb{R} \times \mathbb{R}^d$, but this should not cause confusion, as it will be clear from the context. We note that any autonomous differential equation can be interpreted as a nonautonomous differential equation (1.3), and the domain

$D \subset \mathbb{R}^d$ then translates to the domain $\mathbb{R} \times D$, which is an open set if and only if D is open.

We will only treat *ordinary* differential equations (ODEs) in this course. Of great importance are also *partial* differential equations (PDEs), which are solved by functions depending on more than one variable, so, in contrast to ordinary differential equations, *partial* differentiation is needed to even define a partial differential equation.

The easiest types of solutions are constant solutions, which are also called equilibrium solutions. If the differential equation is autonomous, they are found algebraically, by zeros of the right hand side.

Proposition 1.3 (Constant solutions to autonomous differential equations)**.** *Consider an open set $D \subset \mathbb{R}^d$ and a function $f : D \to \mathbb{R}^d$, and consider the autonomous differential equation*

$$\dot{x} = f(x)\,.$$

Then there exists a constant solution $\lambda : \mathbb{R} \to \mathbb{R}^d$ of this differential equation with $a \in \mathbb{R}^d$, i.e. $\lambda(t) = a$ for all $t \in \mathbb{R}$, if and only if $f(a) = 0$.

Proof. ($\Rightarrow$) Suppose that $\lambda : I \to \mathbb{R}^d$ is a constant solution, i.e. $\lambda(t) = a$ for all $t \in I$. The solution identity yields

$$\dot{\lambda}(t) = f(\lambda(t)) \quad \text{for all } \; t \in I\,, \tag{1.5}$$

which implies $f(a) = 0$.

($\Leftarrow$) Suppose that $f(a) = 0$ for some $a \in \mathbb{R}^d$. Then for the constant function $\lambda : \mathbb{R} \to \mathbb{R}^d$, $\lambda(t) = a$, the solution identity (1.5) is clearly fulfilled, and thus, the constant function λ is a solution to the given differential equation. □

This proposition says that constant solutions to autonomous ordinary differential equations are easy to find. For many differential equations, constant solutions are the only solutions that can be given explicitly, which means that there are no formulas for all other solutions. In fact, most of the interesting differential equations cannot be solved analytically. There are two approaches to overcome this deficit. Firstly, there are numerous schemes to numerically approximate solutions of differential equations – this will not be covered in this course. Secondly, the so-called *qualitative theory* of ordinary differential equations provides insights into how solutions behave without knowing them explicitly – you will learn some basic elements of this theory in this course.

We are interested now to understand in solutions for a given pair of initial time and initial conditions. For the model discussed in Example 1.1, this would mean that we are interested in the time evolution of capital, given that we have x_0 capital at time t_0.

Definition 1.4 (Initial value problem)**.** *Consider $d \in \mathbb{N}$, an open set $D \subset \mathbb{R} \times \mathbb{R}^d$, and a function $f : D \to \mathbb{R}^d$. The combination of the ordinary differential equation*

$$\dot{x} = f(t, x)$$

with an initial condition of the form

$$x(t_0) = x_0 \,, \tag{1.6}$$

where $(t_0, x_0) \in D$, is called an initial value problem*, and* (1.6) *is called* initial condition*. A solution to the above initial value problem is a solution $\lambda : I \to \mathbb{R}^d$ to the differential equation such that t_0 is in the interior of I and*

$$\lambda(t_0) = x_0 \,.$$

We now solve an initial value problem for the simple differential equation $\dot{x} = ax$.

Example 1.5 (A first example revisited)**.** Consider the ordinary differential equation (1.1) from Example 1.1 with the solutions λ_b for $b \in \mathbb{R}$. For fixed $t_0, x_0 \in \mathbb{R}$, we show that there exists a unique solution $\mu : \mathbb{R} \to \mathbb{R}$ to (1.1) solving the initial condition $x(t_0) = x_0$. This follows from

$$\lambda_b(t_0) = x_0 \Leftrightarrow be^{at_0} = x_0 \Leftrightarrow b = x_0 e^{-at_0} \,.$$

Hence, the solution to this initial value problem is given by $\mu(t) = x_0 e^{a(t-t_0)}$ for all $t \in \mathbb{R}$.

We will later find conditions for ordinary differential equations that guarantee that all initial value problems have a unique solution. These conditions are rather weak and apply to large classes of applications.

2. Examples

We have seen that the differential equation $\dot{x} = x$ behaves as nicely as one can imagine:

(i) a solution *exists* for every initial value problem,

(ii) the solution to each initial value problem is *unique*,

(iii) the solution to each initial value problem *exists globally*, i.e. can be defined on $I = \mathbb{R}$.

In this section, we look at examples for which not all of this properties are satisfied. The first example demonstrates that solutions to initial value problems do not need to exist.

Example 1.6 (No solution to an initial value problem)**.** Consider the one-dimensional initial value problem

$$\dot{x} = f(x) = \begin{cases} 1 & : \quad x < 0 \\ -1 & : \quad x \geq 0 \end{cases}, \qquad x(0) = 0\,,$$

which has a discontinuous right hand side. Show as an exercise that this initial value problem does not have any solutions.

There may exist more than one solution to an initial value problem.

Example 1.7 (Many solutions to an initial value problem)**.** Consider the one-dimensional initial value problem

$$\dot{x} = f(x) := \sqrt{|x|}\,, \qquad x(0) = 0\,. \tag{1.7}$$

Since $f(0) = 0$, Proposition 1.3 implies that there exists a constant solution with value 0. In addition, for any $b \geq 0$, the function $\lambda_b : \mathbb{R} \to \mathbb{R}$,

$$\lambda_b(t) = \begin{cases} 0 & : \quad t \leq b \\ \frac{1}{4}(t-b)^2 & : \quad t > b \end{cases},$$

is a solution to this initial value problem. To check this, we have to verify the solution identity for $t < b$, $t = b$ and $t > b$. Clearly, $\dot{\lambda}(t) = 0 = f(\lambda(t))$ for all $t < b$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{4}(t-b)^2 = \frac{1}{2}(t-b)$, which is 0 at $t = b$, so the identity also holds at $t = b$. For $t > b$, we have $\dot{\lambda}(t) = \frac{1}{2}(t-b) = \sqrt{|\lambda(t)|}$, which finishes the proof.

Question: can you find even more solutions to this initial value problem?

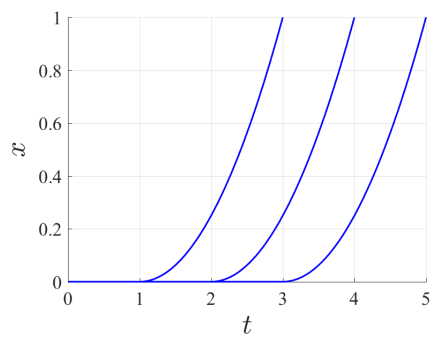


Figure 1.2. Three different solutions to the initial value problem (1.7) ($b = 1, 2, 3$).

We now study a differential equation for which there are solutions that do not exist for all times, i.e. they can only be defined on a proper subset $I \subsetneq \mathbb{R}$ of the real numbers. To solve this differential equation, we need the *separation of variables* technique that you have learned in your first year. This technique is useful to compute initial value problems of the form

$$\dot{x} = g(t)h(x)\,, \qquad x(t_0) = x_0\,, \tag{1.8}$$

where $g : I \to \mathbb{R}$ and $h : J \to \mathbb{R}$ are two continuous functions, with two open intervals $I, J \subset \mathbb{R}$, and we assume that $h(x_0) \neq 0$ (otherwise the initial value problem has the constant solution).

The formal procedure is as follows:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = g(t)h(x)\,, \quad x(t_0) = x_0 \quad \Longrightarrow \quad \frac{\mathrm{d}x}{h(x)} = g(t)\mathrm{d}t\,, \quad x(t_0) = x_0$$
$$\Longrightarrow \quad \int_{x_0}^{x} \frac{1}{h(y)}\,\mathrm{d}y = \int_{t_0}^{t} g(s)\mathrm{d}s\,,$$

and solving this equation with respect to x will give a solution to the differential equation (1.8).

We study the separation of variables procedure for the following example.

Example 1.8 (Solutions do not need to exist for all times)**.** Consider the initial value problem

$$\dot{x} = tx^2\,, \qquad x(t_0) = x_0\,,$$

where $x_0 \neq 0$. Using the above procedure, we get

$$\frac{\mathrm{d}x}{x^2} = t\mathrm{d}t\,, \quad x(t_0) = x_0 \quad \Longrightarrow \quad \frac{1}{x_0} - \frac{1}{x} = \frac{t^2}{2} - \frac{t_0^2}{2}$$
$$\Longrightarrow \quad x = \frac{2x_0}{2 + x_0(t_0^2 - t^2)}\,.$$

It is easy to see that all solutions with $x_0 > 0$ are defined only on a bounded subinterval of $\mathbb{R}$: there always exist two times $t \in \mathbb{R}$ such that the denominator $2 + x_0(t_0^2 - t^2)$ is equal to 0, and the solution converges to ∞ if these times are approached, see Figure 1.3 for an illustration of the solution with $t_0 = 0$ and $x_0 = 1$. This solution exists on the interval $\left(-\sqrt{2}, \sqrt{2}\right)$.

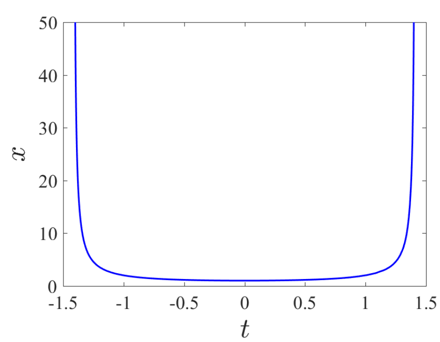


Figure 1.3. Solution to the initial value problem $\dot{x} = tx^2$, $x(0) = 1$.

Question: what happens for $x_0 < 0$ and $x_0 = 0$?

3. Visualisations

Two different ways to visualise the solutions to ordinary differential equations are discussed in this section. The first one concerns solutions curves of nonautonomous differential equations in the (t, x)-space, while the second one concerns projections of solution curves of autonomous differential equations in the x-space.

3.1. Solution portrait. We consider a function $f : D \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ and the corresponding ordinary differential equation

$$\dot{x} = f(t, x) .$$

A solution to this equation is a differentiable function $\lambda : I \to \mathbb{R}^d$ fulfilling $\dot{\lambda}(t) = f(t, \lambda(t))$ on the interval I. Then the graph of this solution, given by the so-called *solution curve*

$$G(\lambda) = \{(t, \lambda(t)) : t \in I\} \subset \mathbb{R} \times \mathbb{R}^d ,$$

is a differentiable curve. The derivative of this curve in the point $t_0 \in I$ is given by $\frac{\mathrm{d}}{\mathrm{d}t}(t, \lambda(t))\big|_{t=t_0} = (1, \dot{\lambda}(t_0)) = (1, f(t_0, \lambda(t_0)))$.

This implies in particular that the vector field

$$(t, x) \mapsto (1, f(t, x)) , \tag{1.9}$$

defined on D, is crucial for the shape of the solution curves, in the sense that the solution curves are tangential to the vector field (1.9).

A *solution portrait* is given by a visualisation of several solution curves in the (t, x)-space, the so-called *extended phase space*. The x-space is normally called phase space, and it is extended by the time axis.

See Figure 1.4 for a solution portrait of the differential equation $\dot{x} = tx^2$ (note that we have studied this differential equation in Example 1.8).

3.2. Phase portrait. A different visualisation is possible for *autonomous* differential equation, i.e. for differential equations that do not explicitly depend on time t.

We first demonstrate that in this context, solutions stay solutions when we shift them in time, which leads to some kind of redundancy when visualising them via a solution portrait. This implies that a visualisation in the phase space (which is one dimension lower) is meaningful to illustrate certain properties of solutions.

Proposition 1.9 (Translation invariance). *Let $\lambda : I \to \mathbb{R}^d$ be a solution to the autonomous differential equation*

$$\dot{x} = f(x) .$$

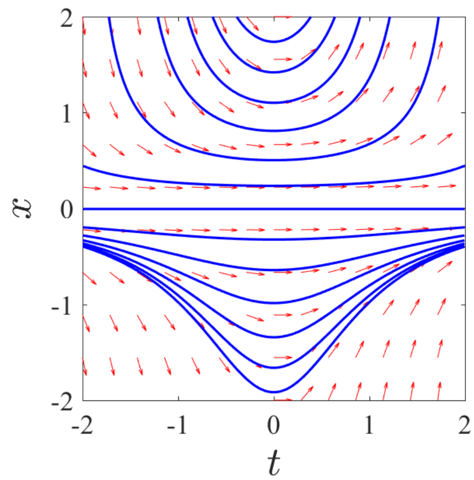


Figure 1.4. Solution portrait with several solutions to the differential equation $\dot{x} = tx^2$ (in blue), and the vector field (1.9) (in red).

Then for all $\tau \in \mathbb{R}$, the function $\mu : \tilde{I} \to \mathbb{R}^d$, where $\tilde{I} := \{t \in \mathbb{R} : t + \tau \in I\}$ and

$$\mu(t) := \lambda(t+\tau) \quad \textit{for all } t \in \tilde{I}\,,$$

is also a solution to this differential equation.

Proof. Since λ is a solution, we have $\dot{\lambda}(t) = f(\lambda(t))$ for $t \in I$. The chain rule implies that $\dot{\mu}(t) = \dot{\lambda}(t+\tau)$ for all $t \in \tilde{I}$, and we get

$$\dot{\mu}(t) = \dot{\lambda}(t+\tau) = f(\lambda(t+\tau)) = f(\mu(t)) \quad \text{for all } t \in \tilde{I}\,.$$

This finishes the proof. □

We demonstrate visualisation using phase portraits by means of the harmonic oscillator.

Example 1.10 (Harmonic oscillator)**.** Consider the differential equation of the harmonic oscillator

$$\ddot{x} = -x\,,$$

which, according to *Repetition Material 1*, can be re-written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.10}$$

Note that this is a so-called *linear* differential equation, since the right hand side of (1.10) is a matrix multiplied by the state space vector. (We have encountered another linear differential equation already in Example 1.1.) This differential equation is also autonomous (i.e. the matrix does not depend on time t), and we will see later in Chapter 3 that all such systems are solvable explicitly. The solution to the initial value problem (1.10), $(x, y)(0) = (x_0, y_0)$, is given by

$$\lambda(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} .$$

A visualisation of one solution (in the extended state space, and for the initial condition $(x, y)(0) = (1, 1)$) is given in Figure 1.5.

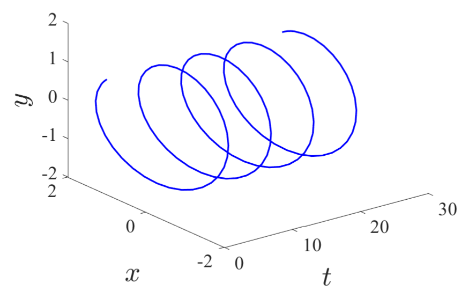


Figure 1.5. Solution to the initial value problem (1.10), $(x, y)(0) = (1, 1)$.

Note that due to the translation invariance, any translation of the above solution is also a solution to the differential equation. Therefore, a visualisation in the extended phase space contains redundant information, and it is common to visualise in the phase space itself, and not in the extended phase space. This is done by a projection of all solutions to the x-space, and we obtain the so-called *phase portrait*, see Figure 1.6 for the differential equation (1.10). Note that a projected solution is called an *orbit* or *trajectory*, see Definition 2.25 below for a precise definition.

Question: What information is lost if we do such a projection?

Let us consider now an arbitrary autonomous differential equation

$$\dot{x} = f(x) ,$$

with a right-hand side $f : D \subset \mathbb{R}^d \to \mathbb{R}^d$. Similarly to the vector field (1.9) for the solution portrait, also a vector field is tangential to the trajectories in the phase portrait. It is simply given by the projected version of vector field (1.9), given by

$$x \mapsto f(x) ,$$

defined on D, indicated by the red arrows in Figure 1.6.

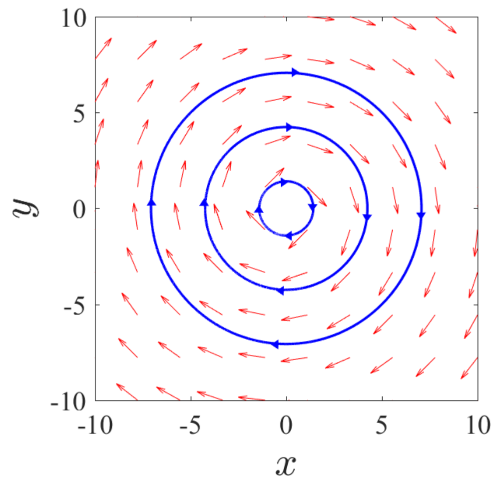


Figure 1.6. Phase portrait of the differential equation (1.10).

Chapter 2

Existence and uniqueness

We have seen in Example 1.6 that it is not guaranteed to have solutions to an initial value problem, and we have seen in Example 1.7 that solutions to an initial value problem do not need to be unique. In this chapter, we present a theory that guarantees existence and uniqueness for solutions to initial value problems. Luckily, the conditions we need to impose to obtain existence and uniqueness are rather weak and fulfilled by the vast majority of differential equation that come from applications.

1. Picard iterates

We first want to establish a procedure to show that solutions to specific initial value problems exist. The following proposition is (maybe surprisingly) extremely helpful for this purpose, although it looks a bit like a triviality: we just reformulate the differential equation equivalently as an *integral equation* by integrating it.

Proposition 2.1 (Reformulation as integral equation)**.** *Consider the initial value problem*

$$\dot{x} = f(t,x)\,, \qquad x(t_0) = x_0\,, \tag{2.1}$$

where $f : D \subset \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and $(t_0, x_0) \in D$, and let $\lambda : I \to \mathbb{R}^d$ be a function on an interval I such that $t_0 \in I$ and $\big\{(t, \lambda(t)) : t \in I\big\} \subset D$. Then the following two statements are equivalent.

(i) *λ solves the initial value problem* (2.1)*, i.e.*

$$\dot{\lambda}(t) = f(t, \lambda(t)) \quad \text{for all } t \in I\,, \text{ and } \lambda(t_0) = x_0\,. \tag{2.2}$$

(ii) *λ is continuous, and we have*

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\, \mathrm{d}s \quad \textit{for all } t \in I\,. \tag{2.3}$$

Proof. We integrate (2.2) from t_0 to t and obtain (2.3), and differentiating (2.3) with respect to t yields (2.2). □

Note that the Riemann integration in (2.3) is higher-dimensional in general, and higher-dimensional integration of a function $g : \mathbb{R} \to \mathbb{R}^d$ is defined componentwise by

$$\int_{t_0}^{t} g(s)\, \mathrm{d}s = \begin{pmatrix} \int_{t_0}^{t} g_1(s)\, \mathrm{d}s \\ \vdots \\ \int_{t_0}^{t} g_d(s)\, \mathrm{d}s \end{pmatrix},$$

provided that all integrals on the right hand side exist.

Why is (2.3) so helpful to obtain the solution λ to the initial value problem (2.1)? The reason is that λ appears both on the left and right hand side of (2.3), and we can exploit an iterative scheme to approximate it.

We first study this in a simpler setting, i.e. for an algebraic equation. Consider for $a > 0$ the algebraic equation

$$a = \frac{a}{2} + \frac{1}{a}\,. \tag{2.4}$$

This equation is obviously equivalent to $a^2 = 2$, and as above for λ, the unknown quantity a appears on the left and the right hand side of (2.4).

We define the iterative scheme

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \quad \text{for all } n \in \mathbb{N}_0\,, \tag{2.5}$$

with an arbitrary starting value $a_0 > 0$. This defines a sequence $\{a_n\}_{n\in\mathbb{N}_0}$. If we can show that this sequence converges with limit a_∞, i.e. $\lim_{n\to\infty} a_n = a_\infty$, then we have

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \left(\frac{a_n}{2} + \frac{1}{a_n}\right) \quad \Longrightarrow \quad a_\infty = \frac{a_\infty}{2} + \frac{1}{a_\infty}\,,$$

so a_∞ solves (2.4).

It is easy to see that, indeed, the sequence $\{a_n\}_{n\in\mathbb{N}_0}$ converges, since it is both monotone (for $n \geq 1$) and bounded.

Question: Can you establish this rigorously? Figure 2.1 will help you with that.

Motivated by Proposition 2.1 and the above iterative scheme for the algebraic equation (2.4), we define the Picard iterates for the initial value problem (2.1).

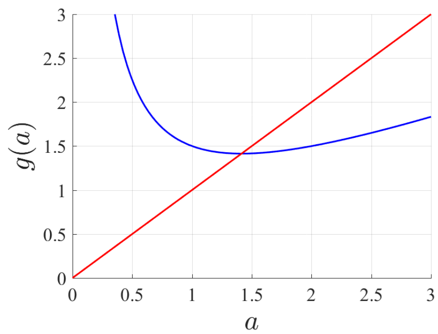


Figure 2.1. In blue: the function $g(a) = \frac{a}{2} + \frac{1}{a}$, so that (2.5) reads as $a_{n+1} = g(a_n)$. In red: the identity function. Both graphs intersect at $a = \sqrt{2}$.

Definition 2.2 (Picard iterates)**.** *Consider the initial value problem* (2.1), *and choose an interval J that contains t_0. We define a initial function*

$$\lambda_0(t) \equiv x_0 \quad \textit{for all } t \in J\,,$$

and inductively, the Picard iterates

$$\lambda_{n+1}(t) := x_0 + \int_{t_0}^{t} f(s, \lambda_n(s))\,\mathrm{d}s \quad \textit{for all } t \in J \textit{ and } n \in \mathbb{N}_0\,. \tag{2.6}$$

Note that the interval J has to be chosen appropriately, but we do not worry about this now, since we are interested in the general principle in the first instance. It will follow from a question on the second problem sheet that if this sequence is uniformly convergent with the limiting function λ_∞, we obtain

$$\lambda_\infty(t) = x_0 + \int_{t_0}^{t} f(s, \lambda_\infty(s))\,\mathrm{d}s \quad \text{for all } t \in J\,.$$

Note that uniform convergence is needed for $\lim_{n\to\infty} \int_{t_0}^{t} f(s, \lambda_n(s))\,\mathrm{d}s = \int_{t_0}^{t} \lim_{n\to\infty} f(s, \lambda_n(s))\,\mathrm{d}s$. Thus, Proposition 2.1 yields that λ_∞ is solution to the integral equation (2.3) and therefore solves the initial value problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$.

Please remind yourself what uniform convergence means! Can you give an example of a sequence of functions that does not converge uniformly?

We study the procedure of Picard iteration for a simple example.

Example 2.3 (Picard iterates for $\dot{x} = ax$)**.** We would like to compute the Picard iterates for the initial value problem

$$\dot{x} = ax\,, \qquad x(t_0) = x_0\,,$$

where $a \in \mathbb{R}$ is fixed. The first three iterates are

$$\lambda_0(t) = x_0\,,$$
$$\lambda_1(t) = x_0 + \int_{t_0}^{t} a x_0 \,\mathrm{d}s = x_0\big(1 + a(t-t_0)\big)\,,$$
$$\lambda_2(t) = x_0 + \int_{t_0}^{t} a x_0\big(1 + a(s-t_0)\big)\,\mathrm{d}s = x_0\big(1 + a(t-t_0) + \tfrac{1}{2}a^2(t-t_0)^2\big)\,.$$

It is easy to prove by induction that

$$\lambda_n(t) = x_0 \sum_{i=0}^{n} \frac{a^i (t-t_0)^i}{i!} \quad \text{for all } n \in \mathbb{N}_0 \text{ and } t \in \mathbb{R}\,,$$

and this sequence of functions converges to the limit function

$$\lambda_\infty(t) = x_0 e^{a(t-t_0)} \quad \text{for all } t \in \mathbb{R}\,.$$

This coincides with the solution to this initial value problem we have identified in Example 1.5.

Note that we obtain *global convergence* of the Picard iterates for this example (i.e. convergence for all $t \in \mathbb{R}$, although this convergence is not uniform on $\mathbb{R}$, but uniform on any compact interval). We will see later that, in general, under suitable but weak conditions, we obtain local convergence, i.e. we choose a small enough compact interval J around t_0. This will establish existence and uniqueness of solutions to initial value problems on a theoretical level. From a practical perspective, however, it is not a standard approach to look at Picard iterates to find solutions for specific systems.

2. Lipschitz continuity

We aim at uniform convergence of the Picard iterates $\{\lambda_n : J \to \mathbb{R}^d\}_{n\in\mathbb{N}_0}$ to a limit function $\lambda_\infty : J \to \mathbb{R}^d$, where J is compact interval. How can we express this type of uniform convergence? We will do so by considering the space of continuous functions on a compact interval J, denoted by $C^0(J, \mathbb{R}^d)$. As you have seen in the analysis course, this space is a complete normed vector space when equipped with the supremum norm, and convergence in this norm corresponds to uniform convergence (see *Repetition Material 2*).

We will see later in Section 3 that Lipschitz continuity plays an important role in establishing uniform convergence of the Picard iterates. Firstly, the above required uniform convergence for the Picard iterates will follow from an application of Banach's fixed point theorem (see *Repetition Material 3*), and we need a Lipschitz constant less than 1 for this. Secondly, we will see that a Lipschitz condition for Banach's fixed point theorem holds due to a certain Lipschitz condition on the right hand side f of the differential equation under consideration (and we note that this Lipschitz constant does not necessarily need to be less than 1).

Recall that a vector space V over the reals is an abelian group $(V, +)$ with an additional scalar multiplication $(V, \cdot)$. In particular, for $x, y \in V$, we have $ax + by \in V$ for all $a, b \in \mathbb{R}$.

Definition 2.4 (Normed vector space)**.** *A* norm *on a vector space V over the reals is a map $\|\cdot\| : V \to \mathbb{R}_0^+$ such that*

(i) $\|x\| = 0 \Leftrightarrow x = 0$ *(positive definiteness),*

(ii) $\|ax\| = |a|\|x\|$ *for all $a \in \mathbb{R}$ and $x \in V$ (absolute homogeneity),*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all $x, y \in V$ (triangle inequality).*

A vector space with norm is called a normed vector space*.*

Examples for normed vector spaces are the finite-dimensional Euclidean spaces $\mathbb{R}^d$, where $d \in \mathbb{N}$. In Euclidean spaces $\mathbb{R}^d$, we will normally use the Euclidean norm $\|x\| := \sqrt{x_1 + \cdots + x_d}$ for $x \in \mathbb{R}^d$.

As motivated above, we will be also interested in the infinite-dimensional normed vector space $C^0(J, \mathbb{R}^d)$, the space of continuous functions on a compact interval J.

The norm $\|\cdot\|$ of a normed vector space V naturally describes the distance between two vectors $x, y \in V$. This distance is given by $\|x-y\|$. In fact, every normed vector space is a metric space, where the metric $d : V \times V \to \mathbb{R}_0^+$ is given by

$$d(x, y) := \|x - y\| \quad \text{for all } x, y \in V\,.$$

The normed vector space $(V, \|\cdot\|)$ is called *complete* if every Cauchy sequence converges in V. A complete normed vector space is called a *Banach space.*

As mentioned above, Lipschitz continuity is crucial for rigorously establishing convergence of Picard iterates.

Definition 2.5 (Continuous and Lipschitz continuous functions)**.** *Let X be a subset of a normed vector space $(V, \|\cdot\|_V)$ and Y be a subset of a normed vector space $(W, \|\cdot\|_W)$. Then a function $f : X \to Y$ is called*

(i) continuous *if for all $x \in X$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\|x - \bar{x}\|_V < \delta \quad \Longrightarrow \quad \|f(x) - f(\bar{x})\|_W < \varepsilon\,.$$

(ii) Lipschitz continuous *if there exists a constant $K > 0$ such that*

$$\|f(x) - f(\bar{x})\|_W \leq K\|x - \bar{x}\|_V \quad \textit{for all } x, \bar{x} \in X\,.$$

The constant K is called a Lipschitz constant*.*

It is easy to see that Lipschitz continuous functions are continuous (for a given $\varepsilon > 0$, choose $\delta := \frac{\varepsilon}{K}$), but the reverse is not true, as Example 2.6 below shows.

2.1. Lipschitz continuity and the mean value theorem. In this subsection, we explore in dimension one how Lipschitz continuity is related to the mean value theorem. Consider a differentiable function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$ is an interval. Recall that the mean value theorem says that for any $x, y \in I$, there exists an ξ between x and y such that

$$f(x) - f(y) = f'(\xi)(x - y)\,.$$

This implies

$$|f(x) - f(y)| = |f'(\xi)||x - y|\,, \tag{2.7}$$

and it is clear if the derivative f' is bounded on the interval I, then f is Lipschitz continuous. In particular, this holds when I is compact and f is continuously differentiable. We now look at some examples.

Example 2.6 (Lipschitz continuity in dimension one)**.** We consider several real-valued functions defined on intervals.

(i) The function $x \mapsto \sqrt{x}$, where $x \in [0, 1]$, is continuous, but not Lipschitz continuous. Note that the function is differentiable in the open interval $(0, 1)$ with unbounded derivative, and one can prove this rigorously using this fact and the mean value theorem.

(ii) The function $x \mapsto x^2$, where $x \in \mathbb{R}$, is not Lipschitz continuous. Note that the derivative $x \mapsto 2x$ of this function is unbounded, and one can argue as outlined in (i).

(iii) The function $x \mapsto x^2$, where $x \in [0, 1]$, is Lipschitz continuous with Lipschitz constant 2, since the derivative $x \mapsto 2x$ of this function is bounded by 2 on the interval $[0, 1]$; this follows from the mean value theorem as outlined above.

2.2. Lipschitz continuity and the mean value inequality. The above example shows that a Lipschitz condition is closely connected to derivatives (although, in general, Lipschitz continuous functions do not need to be differentiable). In the one-dimensional context of this example, this followed from the mean value theorem. We explore now in what sense this result can be generalised to higher dimensions. It turns out that we only get a mean value inequality (in contrast to an equality that holds in dimension one), but that is good enough to obtain Lipschitz continuity.

Although our main interest are nonlinear mappings, we first look at linear mappings. It is clear that for any matrix $A \in \mathbb{R}^{m \times n}$, the linear mapping $x \mapsto Ax$ is continuous. To see that this mapping is Lipschitz continuous, we introduce the so-called operator norm, which is a norm on the vector space of all matrices in $\mathbb{R}^{m\times n}$.

Definition 2.7 (Operator norm of a matrix)**.** *For a given matrix $A \in \mathbb{R}^{m\times n}$, the* operator norm *of A is defined by*

$$\|A\| := \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|}\,. \tag{2.8}$$

Note that the three norms used in (2.8) are different (unless $n = m$). In addition to the operator norm (which we define), we also use the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^m$ and $\mathbb{R}^n$ here. Show as an exercise that $A \mapsto \|A\|$ is indeed a norm.

Note that due to linearity of A, we have

$$\|A\| = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \frac{\|Ax\|}{\|x\|} = \sup_{x\in\mathbb{R}^n\setminus\{0\}} \Big\|A\tfrac{x}{\|x\|}\Big\| = \sup_{x\in\mathbb{R}^n,\|x\|=1} \|Ax\|\,.$$

In particular, since the linear mapping $x \mapsto Ax$ is continuous, the above supremum is a maximum (on the compact set $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$), so $\|A\|$ is a finite real number. We have $\|Ax - Ay\| = \|A(x - y)\| \le \|A\|\|x - y\|$, so the mapping $x \mapsto Ax$ is even Lipschitz continuous with Lipschitz constant $\|A\|$.

The following result is the appropriate analogue of the mean value theorem in higher dimensions. As mentioned above, we do not have an equality in general. Can you find an example? Note that an equality can be established in the case of $m = 1$.

Theorem 2.8 (Mean value inequality)**.** *Consider an open set $D \subset \mathbb{R}^n$, and let $f : D \to \mathbb{R}^m$ be continuously differentiable. Then for all $x, y \in D$ with $[x, y] \subset D$, there exists a $\xi \in [x, y]$ such that*

$$\|f(x) - f(y)\| \le \|f'(\xi)\|\|x - y\|\,.$$

Here, for any $x, y \in \mathbb{R}^n$, the closed line segment connecting x and y *is given by $[x, y] := \big\{\alpha x + (1-\alpha)y \in \mathbb{R}^n : \alpha \in [0, 1]\big\}$.*

Proof. Consider the function $g : [0, 1] \to \mathbb{R}^m$, $g(\alpha) := f\big((\alpha x + (1-\alpha)y\big)$. By the fundamental theorem of calculus, we have

$$f(x) - f(y) = g(1) - g(0) = \int_0^1 g'(\alpha)\,\mathrm{d}\alpha = \int_0^1 f'\big(\alpha x + (1-\alpha)y\big)(x - y)\,\mathrm{d}\alpha\,.$$

This implies

$$\begin{aligned}
\|f(x)-f(y)\| &= \Big\| \int_0^1 f'\big(\alpha x+(1-\alpha)y\big)(x-y)\,\mathrm{d}\alpha\Big\| \\
&\overset{\text{Lemma 2.9}}{\leq} \int_0^1 \big\|f'\big(\alpha x+(1-\alpha)y\big)(x-y)\big\|\,\mathrm{d}\alpha \\
&\overset{(2.8)}{\leq} \int_0^1 \big\|f'\big(\alpha x+(1-\alpha)y\big)\big\|\,\mathrm{d}\alpha\|x-y\| \\
&\leq \max_{\alpha\in[0,1]} \big\|f'\big(\alpha x+(1-\alpha)y\big)\big\|\|x-y\| \\
&= \|f'(\xi)\|\|x-y\|
\end{aligned}$$

for some $\xi \in [x,y]$. Note that continuous differentiability of f was used in the last step of this proof. □

It remains to prove the triangle-like inequality we used in the above proof.

Lemma 2.9 (Triangle-like inequality for integrals)**.** *Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}^m$ be a continuous function. Then*

$$\left\|\int_{t_0}^t f(s)\,\mathrm{d}s\right\| \leq \left|\int_{t_0}^t \|f(s)\|\,\mathrm{d}s\right| \quad \textit{for all } t, t_0 \in I\,.$$

Proof. We first show the case $t_0 < t$. For $n \in \mathbb{N}$, we look at the Riemann sum $\frac{t-t_0}{n}\sum_{i=0}^{n-1} f\big(t_0+\frac{i}{n}(t-t_0)\big)$, which in the limit $n \to \infty$ converges to $\int_{t_0}^t f(s)\,\mathrm{d}s$. The triangle inequality implies

$$\left\|\frac{t-t_0}{n}\sum_{i=0}^{n-1} f\left(t_0+\frac{i}{n}(t-t_0)\right)\right\| \leq \frac{t-t_0}{n}\sum_{i=0}^{n-1}\left\|f\left(t_0+\frac{i}{n}(t-t_0)\right)\right\|\,,$$

and since the right hand side converges to $\int_{t_0}^t \|f(s)\|\,\mathrm{d}s$, the statement follows. The case $t_0 \geq t$ follows due to $\int_{t_0}^t f(s)\,\mathrm{d}s = -\int_t^{t_0} f(s)\,\mathrm{d}s$. □

The following statement is an immediate corollary from the mean value inequality. It follows from the fact that continuous functions attain their (finite) maximum on compact sets.

Corollary 2.10 (Lipschitz continuity and the mean value inequality)**.** *Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^m$ be continuously differentiable. Then given a compact and convex set $C \subset U$, the restricted function $f|_C : C \to \mathbb{R}^m$ is Lipschitz continuous.*

Note here that convexity of C means that for any two points $x, y \in C$, the closed line segment lies in C, i.e. $[x,y] \subset C$.

3. Picard–Lindelöf theorem

We aim at an easily verifiable condition that the Picard iterates corresponding to an initial value problem converge (at least locally in a neighbourhood of the initial time). It turns out that a Lipschitz condition in the state space variable x for the right hand side of a differential equation is an appropriate condition. We first study the easiest situation where a system is globally defined and has a global Lipschitz constant in x. After understanding the global case, we then show that a local Lipschitz condition is sufficient for local existence and uniqueness of solutions.

We note that the proof of the next theorem crucially makes use of Banach's fixed point theorem, which is applied to a complete normed vector space, given by the space of continuous functions on a compact interval, see *Repetition Material 2* and *3*.

Theorem 2.11 (Picard–Lindelöf theorem, global version)**.** *Consider an ordinary differential equation*

$$\dot{x} = f(t,x) \tag{2.9}$$

such that the function $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ *is continuous and satisfies a* global Lipschitz condition *of the form*

$$\|f(t,x) - f(t,y)\| \le K\|x-y\| \quad \text{for all } t \in \mathbb{R} \text{ and } x,y \in \mathbb{R}^d\,, \tag{2.10}$$

where $K > 0$ *is a constant. Define* $h := \frac{1}{2K}$*. Then every initial value problem* (2.9)*,* $x(t_0) = x_0$*, admits a unique solution* $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$*.*

Proof. The proof is divided in three steps and relies on the construction of a contraction $P : X \to X$ on the Banach space $X := C^0\big([t_0 - h, t_0 + h], \mathbb{R}^d\big)$. It turns out that a fixed point of P, which is obtained by Banach's fixed point theorem, solves the above initial value problem.

Step 1. Definition of the function $P : X \to X$*.*

Due to Proposition 2.1, it follows that solving the initial value problem with a solution $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$ is equivalent to the finding a continuous function $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$ solving the integral equation

$$\lambda(t) = x_0 + \int_{t_0}^{t} f(s, \lambda(s))\, \mathrm{d}s \quad \text{for all } t \in [t_0 - h, t_0 + h]\,.$$

This in turn follows from finding a fixed point of the function $P : X \to X$, defined by

$$P(u)(t) := x_0 + \int_{t_0}^{t} f(s, u(s))\, \mathrm{d}s \quad \text{for all } t \in [t_0 - h, t_0 + h]\,.$$

Note that P assigns to a function $u \in X$ another function which we denote by $P(u)$. To define the function $P(u)$, we need to evaluate its value at all

$t \in [t_0 - h, t_0 + h]$, and this is what $P(u)(t)$ means. Note that X is a Banach space with the supremum norm

$$\|u\|_\infty = \sup_{t \in [t_0 - h, t_0 + h]} \|u(t)\| \quad \text{for all } \ u \in X \tag{2.11}$$

(see also *Repetition Material 2*). Note that the operator P is well-defined, since the continuity of f guarantees that the integral exists. Note that P is the function for iteratively constructing the Picard iterates; in fact, the sequence $\{\lambda_n\}_{n \in \mathbb{N}_0}$ from Definition 2.2 satisfies $\lambda_{n+1} = P(\lambda_n)$ for all $n \in \mathbb{N}_0$.

Step 2. P is a contraction.

We will prove that $\|P(u_1) - P(u_2)\|_\infty \leq \frac{1}{2}\|u_1 - u_2\|_\infty$ for all $u_1, u_2 \in X$. To do so, let $u_1, u_2 \in X$. Then for all $t \in [t_0 - h, t_0 + h]$, we have

$$\begin{aligned} \|P(u_1)(t) - P(u_2)(t)\| &= \left\| \int_{t_0}^{t} \big(f(s, u_1(s)) - f(s, u_2(s)) \big)\, \mathrm{d}s \right\| \\ &\overset{\text{Lemma 2.9}}{\leq} \left| \int_{t_0}^{t} \big\| f(s, u_1(s)) - f(s, u_2(s)) \big\|\, \mathrm{d}s \right| \\ &\overset{(2.10)}{\leq} K \left| \int_{t_0}^{t} \|u_1(s) - u_2(s)\|\, \mathrm{d}s \right| \overset{(2.11)}{\leq} K \left| \int_{t_0}^{t} \|u_1 - u_2\|_\infty\, \mathrm{d}s \right| \\ &\leq Kh\|u_1 - u_2\|_\infty = \frac{1}{2}\|u_1 - u_2\|_\infty \,. \end{aligned}$$

This implies by taking the supremum over all $t \in [t_0 - h, t_0 + h]$ that

$$\|P(u_1) - P(u_2)\|_\infty = \sup_{t \in [t_0 - h, t_0 + h]} \|P(u_1)(t) - P(u_2)(t)\| \leq \frac{1}{2}\|u_1 - u_2\|_\infty \,,$$

which shows that P is a contraction on X.

Step 3. Application of the Banach fixed point theorem.

Since P is a contraction on a Banach space (which naturally is a complete metric space), the Banach fixed point theorem (see *Repetition Material 3*) implies that there exists a unique fixed point $\lambda : [t_0 - h, t_0 + h] \to \mathbb{R}^d$. As outlined in Step 1, with the help of Proposition 2.1, such a fixed point solves the initial value problem under consideration, and since it is the *unique* fixed point, the solution to this initial value problem is unique on the interval $[t_0 - h, t_0 + h]$. □

Note that the Lipschitz condition (2.10) is a strong assumption, and although some very important differential equations fulfill it (such as autonomous linear systems, studied later in Chapter 3), it is not true for most interesting differential equations. As an example, consider, for instance, the differential equation $\dot{x} = tx^2$ from Example 1.8. However, it turns out that for this differential equation, solutions to any initial value problem exist locally and are unique. The reason is that the existence of a local Lipschitz condition is enough. In the following definition, we distinguish between the global Lipschitz condition (2.10) and its local version.

Definition 2.12 (Global and local Lipschitz continuity)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$.*

(i) *f is said to be* globally Lipschitz continuous *(with respect to x) if there exists a constant $K > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le K\|x - y\| \quad \textit{for all } (t,x),(t,y) \in D\,.$$

(ii) *f is said to be* locally Lipschitz continuous *(with respect to x) if for all $(t_0, x_0) \in D$, there exists a neighbourhood $U \subset D$ of (t_0, x_0) and a constant $K > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le K\|x - y\| \quad \textit{for all } (t,x),(t,y) \in U\,.$$

The global version of the Picard–Lindelöf theorem says that, under global Lipschitz continuity, solutions to all initial value problems exist locally and are unique. Such a statement is true even under the weaker assumption of local Lipschitz continuity. However, in contrast to Theorem 2.11, the interval length $2h$ on which the solution exists will depend on the specific initial value.

Theorem 2.13 (Picard–Lindelöf theorem, local version)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$ that is continuous and locally Lipschitz continuous with respect to x. Consider for a fixed $(t_0, x_0) \in D$ the initial value problem*

$$\dot{x} = f(t,x)\,, \qquad x(t_0) = x_0\,. \tag{2.12}$$

Then the following two statements hold:

(i) Qualitative version*. The initial value problem* (2.12) *has locally a uniquely determined solution, i.e. there exists a $h = h(t_0, x_0) > 0$ such that* (2.12) *has exactly one solution on $[t_0 - h, t_0 + h]$.*

(ii) Quantitative version. *Consider for some $\tau, \delta > 0$ the set $W^{\tau,\delta}(t_0, x_0) := [t_0 - \tau, t_0 + \tau] \times \overline{B_\delta(x_0)}$, where $\overline{B_\delta(x_0)} := \big\{x \in \mathbb{R}^d : \|x - x_0\| \le \delta\big\}$ is the closed δ-neighbourhood of x_0. We assume that $W^{\tau,\delta}(t_0, x_0) \subset D$, and we suppose that there exist $K, M > 0$ such that*

$$\|f(t,x) - f(t,y)\| \le K\|x - y\| \quad \textit{for all } (t,x),(t,y) \in W^{\tau,\delta}(t_0, x_0) \tag{2.13}$$

and

$$\|f(t,x)\| \le M \quad \textit{for all } (t,x) \in W^{\tau,\delta}(t_0, x_0)\,. \tag{2.14}$$

Then (2.12) *has exactly one solution on $[t_0 - h, t_0 + h]$, where $h = h(t_0, x_0) := \min\{\tau, \frac{1}{2K}, \frac{\delta}{M}\}$.*

The proof of the local version of the Picard–Lindelöf theorem is similar to the global version, and it will be skipped here for this reason. The full proof is given in *Extra Material 1*.

The following proposition shows that if the right hand side of a differential equation is continuously differentiable, then it is locally Lipschitz continuous, and local existence and uniqueness of solution holds.

Proposition 2.14 (Continuous differentiability and Lipschitz continuity)**.** *Consider an open set $D \subset \mathbb{R} \times \mathbb{R}^d$ and a continuously differentiable function $f : D \to \mathbb{R}^d$. Then f is locally Lipschitz continuous with respect to x, and thus, every initial value problem involving a differential equation with right hand side f can be solved locally uniquely.*

Proof. Since D is open, for each fixed $(t_0, x_0) \in D$, there exists a compact and convex neighbourhood U of (t_0, x_0). Since f is continuously differentiable, and U is compact, there exists a $K > 0$ such that

$$\left\| \frac{\partial f}{\partial x}(t, \xi) \right\| \leq K \quad \text{for all } (t, \xi) \in U\,. \tag{2.15}$$

Due to the mean value inequality (Theorem 2.8), for any $(t, x), (t, y) \in U$, there exists a $\xi \in [x, y]$ with

$$\|f(t, x) - f(t, y)\| \leq \left\| \frac{\partial f}{\partial x}(t, \xi) \right\| \|x - y\| \overset{(2.15)}{\leq} K\|x - y\|\,.$$

Hence, f is locally Lipschitz continuous (with respect to x), and thus, Theorem 2.13 implies the assertion. □

The following lemma shows that two solutions cannot cross.

Lemma 2.15 (Solutions cannot cross)**.** *Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$ that is continuous and locally Lipschitz continuous with respect to x. Consider two solutions of*

$$\dot{x} = f(t, x)\,,$$

given by $\lambda : I \to \mathbb{R}^d$ and $\mu : J \to \mathbb{R}^d$, where I and J are intervals. Then either

$$\lambda(t) = \mu(t) \quad \textit{for all } t \in I \cap J$$

or

$$\lambda(t) \neq \mu(t) \quad \textit{for all } t \in I \cap J\,.$$

Proof. Assume to the contrary that $\lambda(t_0) = \mu(t_0)$ for some $t_0 \in I \cap J$, and $\lambda(\tilde{t}) \neq \mu(\tilde{t})$ for some $\tilde{t} \in I \cap J$. Without loss of generality, we assume that $\tilde{t} > t_0$. Define

$$t^* := \sup \big\{ t > t_0 : \lambda(t') = \mu(t') \text{ for all } t' \in [t_0, t] \big\}\,.$$

Due to Theorem 2.13, we get $t^* > t_0$, and due to continuity of λ and μ, we have $\lambda(t^*) = \mu(t^*)$. Now both λ and μ solve the differential equation with the initial condition $x(t^*) = \lambda(t^*)$. Thus, Theorem 2.13 implies unique solvability of this initial value problem around t^*. This contradicts the definition of t^* and finishes the proof of this lemma. □

4. Maximal solutions

Let $D \subset \mathbb{R} \times \mathbb{R}^d$ be open, and consider a function $f : D \to \mathbb{R}^d$ that is continuous and locally Lipschitz continuous with respect to x. For a given initial pair $(t_0, x_0) \in D$, consider the initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0. \tag{2.16}$$

In the last section, we have shown that we obtain a local solution to (2.16), given by an interval of length $2h$ around t_0. In this section, we prove that there exists a maximal time interval around t_0 (containing $[t_0 - h, t_0 + h]$) on which the solution to (2.16) exists. We also discuss the behaviour of this solution when time approaches the left and right points of this maximal time interval.

Definition 2.16 (Maximal existence interval). *Consider the initial value problem* (2.16). *We define*

$$I_+(t_0, x_0) := \sup\big\{t_+ \geq t_0 : \textit{there exists a solution to } (2.16) \textit{ on } [t_0, t_+]\big\},$$
$$I_-(t_0, x_0) := \inf\big\{t_- \leq t_0 : \textit{there exists a solution to } (2.16) \textit{ on } [t_-, t_0]\big\},$$

and the interval $I_{max}(t_0, x_0) := (I_-(t_0, x_0), I_+(t_0, x_0))$ *is called the* maximal existence interval *for the initial value problem* (2.16).

In the following theorem, we clarify the question of existence of a solution on the time interval $I_{max}(t_0, x_0)$ and the boundary behaviour of this solution.

Theorem 2.17 (Existence of the maximal solution and boundary behaviour). *There exists a* maximal solution $\lambda_{max} : I_{max}(t_0, x_0) \to \mathbb{R}^d$ *to the initial value problem* (2.16), *i.e. any other solution to this initial value problem is defined on an interval that is a subset of* $I_{max}(t_0, x_0)$. *The maximal solution has the following two properties:*

(i) *If* $I_+(t_0, x_0)$ *is finite, then either the maximal solution is unbounded for* $t \geq t_0$, *i.e.*

$$\sup_{t \in (t_0, I_+(t_0, x_0))} \|\lambda_{max}(t)\| = \infty, \tag{2.17}$$

or the boundary ∂D *of* D *is nonempty, and we have*

$$\lim_{t \nearrow I_+(t_0, x_0)} \operatorname{dist}\big((t, \lambda_{max}(t)), \partial D\big) = 0. \tag{2.18}$$

(ii) *If $I_-(t_0, x_0)$ is finite, then either the maximal solution is unbounded for $t \le t_0$, i.e.*

$$\sup_{t\in(I_-(t_0,x_0),t_0)} \|\lambda_{max}(t)\| = \infty\,,$$

or the boundary ∂D of D is nonempty, and we have

$$\lim_{t\searrow I_-(t_0,x_0)} \operatorname{dist}\big((t, \lambda_{max}(t)), \partial D\big) = 0\,.$$

Here, for a given set $A \subset \mathbb{R}^n$, the function $\operatorname{dist}(\cdot, A) : \mathbb{R}^n \to \mathbb{R}_0^+$ *is defined by*

$$\operatorname{dist}(y, A) := \inf\big\{\|y - a\| : a \in A\big\} \quad \textit{for all } y \in \mathbb{R}^n\,.$$

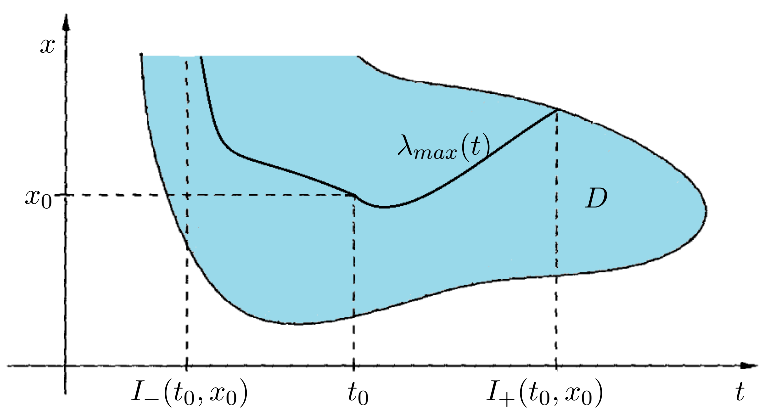


Figure 2.2. Illustration of possible boundary behaviour of the maximal solution.

Proof. *Step 1. The existence of the maximal solution.*
Choose $\bar{t} \in I_{max}(t_0, x_0)$. Due to Definition 2.16, there exists a solution $\mu : I \to \mathbb{R}^d$ of the initial value problem (2.16) such that $\bar{t} \in I$ and $t_0 \in I$. We note that due to Lemma 2.15, all solutions to this initial value problem having $\bar{t}$ in their domain must coincide at the time $\bar{t}$, and we define $\lambda_{max}(\bar{t}) = \mu(\bar{t})$, and clearly also $\dot{\lambda}_{max}(\bar{t}) = f(\bar{t}, \lambda_{max}(\bar{t}))$. Since $\bar{t}$ was chosen arbitrarily, this defines the maximal solution in the open interval $I_{max}(t_0, x_0)$. We also note that the maximal solution cannot be defined on the endpoints of the (open) interval $I_{max}(t_0, x_0)$, since via the local version of the Picard–Lindelöf theorem, we would be able to extend this solution beyond either $I_-(t_0, x_0)$ or $I_+(t_0, x_0)$, and thus contradicting Definition 2.16.
Step 2. Proof of (i).
Assume that both (2.17) and (2.18) do not hold. Hence, there exists an

$M > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = I_+(t_0, x_0)$ and

$$\|\lambda_{max}(t_n)\| \le M \quad \text{and} \quad \operatorname{dist}\big((t_n, \lambda_{max}(t_n)), \partial D\big) \ge \frac{1}{M} \quad \text{for all } n \in \mathbb{N}\,. \tag{2.19}$$

The sequence $\{(t_n, \lambda_{max}(t_n))\}_{n\in\mathbb{N}}$ is bounded, and thus, there exists a convergent subsequence $(t_{n_k}, \lambda_{max}(t_{n_k})) \to (t^*, x^*) \in D$ as $k \to \infty$ (we have $t^* = I_+(t_0, x_0)$). Note that $(t^*, x^*) \in D$, because of the second part of (2.19). Due to an exercise on the current problem sheet, in a neighbourhood W of (t^*, x^*), there exists a $h = h(W)$ such that all initial value problems with initial values $(t', x') \in W$, there exists a solution on the interval $[t'-h, t'+h]$. Due to $\big(t_{n_k}, \lambda_{max}(t_{n_k})\big) \to (t^*, x^*) \in D$, there exists an $N \in \mathbb{N}$ such that $(t_{n_k}, \lambda_{max}(t_{n_k})) \in W$ for all $k \ge N$. This implies that $t_{n_k} + h > t^*$ for large $k \in \mathbb{N}$, and thus, the solution can be extended beyond t^*, which contradicts the maximality of the solution λ_{max}.

The proof of (ii) is analogous. □

Example 2.18. For a fixed parameter $\alpha > 0$, we consider the autonomous differential equation

$$\dot{x} = x^\alpha\,, \tag{2.20}$$

the right hand side of which is defined for all $x > 0$ (and formally for all $t \in \mathbb{R}$), so the domain of (2.20) is given by $D = \mathbb{R} \times \mathbb{R}^+$. We consider the initial condition $x(0) = 1$. Using separation of variables (see description before Example 1.8), we can compute the maximal solution depending on the parameter $\alpha > 0$. It is clear that for the linear case $\alpha = 1$, we have $\lambda_{1,max}(t) = e^t$, and for $\alpha \ne 1$, we get

$$\lambda_{\alpha,max}(t) = \big(1 + (1-\alpha)t\big)^{\frac{1}{1-\alpha}}\,.$$

Note that the maximal existence intervals, on which these solutions exist, are given by

$$I_{\alpha,max}(0,1) = \begin{cases} \big(\frac{1}{\alpha-1}, \infty\big) & : \ \alpha \in (0,1)\,, \\ (-\infty, \infty) & : \ \alpha = 1\,, \\ \big(-\infty, \frac{1}{\alpha-1}\big) & : \ \alpha \in (1,\infty)\,, \end{cases}$$

and depend on α (see also Figure 2.3).

Both situations described in Theorem 2.17 (explosion or convergence against boundary of D) can occur. For $\alpha \in (0,1)$, $I_{\alpha,max}$ is bounded below, and the solution convergence to the boundary of D (which is the t-axis $\mathbb{R} \times \{0\}$). On the other hand, for $\alpha \in (1,\infty)$, $I_{\alpha,max}$ is bounded above, and we see that the solution converges to infinity when approaching the upper boundary of $I_{\alpha,max}$.

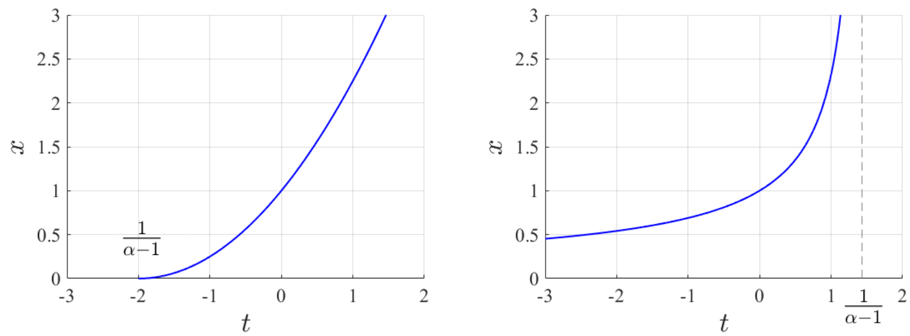


Figure 2.3. $\lambda_{\alpha,max}$ for $\alpha \in (0,1)$ (left), and $\lambda_{\alpha,max}$ for $\alpha \in (1,\infty)$ (right).

5. General solutions and flows

After studying solutions for specific initial value problems, we introduce notion of a general solution and a flow that comprise all solutions of a differential equation (provided conditions for local existence and uniqueness are satisfied).

5.1. General solutions. We first will deal with nonautonomous differential equations before providing a simpler approach for the autonomous special case.

Consider an open subset $D \subset \mathbb{R} \times \mathbb{R}^d$ and a continuous and locally Lipschitz continuous right hand side $f : D \to \mathbb{R}^d$ of the differential equation

$$\dot{x} = f(t, x)\,. \tag{2.21}$$

The notion of a general solution is explained in the following definition.

Definition 2.19 (General solution to a nonautonomous differential equation)**.** *Consider the nonautonomous differential equation* (2.21)*, and define*

$$\Omega := \left\{(t, t_0, x_0) \in \mathbb{R}^{1+1+d} : (t_0, x_0) \in D \text{ and } t \in I_{max}(t_0, x_0)\right\}.$$

Then the function $\lambda : \Omega \to \mathbb{R}^d$, defined by

$$\lambda(t, t_0, x_0) := \lambda_{max}(t, t_0, x_0)\,,$$

where λ_{max} is defined as in Theorem 2.17, is called the general solution *of* (2.21)*.*

Note that in the setting of Theorem 2.17, the initial pair (t_0, x_0) was fixed, but here we vary it to combine the maximal solutions to all initial value problems in one notion, so we indicate the dependence of $\lambda_{max}(t, t_0, x_0)$ on this initial pair here.

This means that a general solution is a function that brings together all maximal solutions of initial value problems. Using the notion of a general solution, the solution identity then reads as

$$\frac{\partial \lambda}{\partial t}(t, t_0, x_0) = f\big(t, \lambda(t, t_0, x_0)\big) \quad \text{for all } (t, t_0, x_0) \in \Omega\,, \tag{2.22}$$

where a partial derivative has to be used, since the time t is not the only argument in the general solution.

We first study a simple example.

Example 2.20. We consider the ordinary differential equation (1.1) from Example 1.1, given by

$$\dot{x} = ax\,,$$

where $a \in \mathbb{R}$. We have seen already in Example 1.1 that each initial condition $x(t_0) = x_0$ leads to a unique solution

$$\lambda_{max}(t) = x_0 e^{a(t-t_0)} \quad \text{for all } t \in \mathbb{R}\,,$$

and we have $I_{max}(t_0, x_0) = \mathbb{R}$. Hence the general solution is given by

$$\lambda(t, t_0, x_0) = x_0 e^{a(t-t_0)} \quad \text{for all } (t, t_0, x_0) \in \Omega\,,$$

where the domain Ω is given by $\Omega = \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$.

The general solution has the following fundamental and important properties.

Proposition 2.21 (Properties of the general solution)**.** *Consider the nonautonomous differential equation* (2.21)*, and let* $(t_0, x_0) \in D$*. Then for all* $s \in I_{max}(t_0, x_0)$*, we have*

$$I_{max}(s, \lambda(s, t_0, x_0)) = I_{max}(t_0, x_0)\,, \tag{2.23}$$

$$\lambda(t_0, t_0, x_0) = x_0\,, \tag{2.24}$$

$$\lambda\big(t, s, \lambda(s, t_0, x_0)\big) = \lambda(t, t_0, x_0) \quad \textit{for all } t \in I_{max}(t_0, x_0)\,. \tag{2.25}$$

The identity (2.24) *is called* initial value property*, while* (2.25) *is called* cocycle property*.*

Question: Can you explain the identities (2.24) and (2.25) in words?

Proof. The identity (2.24) is clear. Due to $s \in I_{max}(t_0, x_0)$, we have $(s, \lambda(s, t_0, x_0)) \in D$, and we consider this as initial pair. Then the functions $\mu_1(t) := \lambda(t, t_0, x_0)$ and $\mu_2(t) := \lambda(t, s, \lambda(s, t_0, x_0))$ are maximal solutions to the initial value pairs (t_0, x_0) and $(s, \lambda(s, t_0, x_0))$, and obviously, $\mu_1(s) = \mu_2(s) = \lambda(s, t_0, x_0)$. Due to uniqueness of solutions and the fact that both μ_1 and μ_2 are maximal solutions, we get $\mu_1 = \mu_2$. This implies (2.23) and (2.25). □

5.2. Flows. A simpler situation is given when the differential equation under consideration is autonomous. We consider an open subset $D \subset \mathbb{R}^d$ and a locally Lipschitz continuous right hand side $f : D \to \mathbb{R}^d$ of the differential equation

$$\dot{x} = f(x)\,. \tag{2.26}$$

The simplification is due to the fact that, because of the translation invariance of autonomous differential equations, as proved in Proposition 1.9, the general solution to an autonomous differential equation does not depend on the actual time and initial time separately, but only on the *elapsed time* (which is the difference between these two times). More precisely, let λ denote the general solution to (2.26). Then we get

$$\lambda(t, t_0, x_0) = \lambda(t - t_0, 0, x_0) \quad \text{for all } t \in I_{max}(t_0, x_0)\,,$$

as well as

$$I_{max}(t_0, x_0) = I_{max}(0, x_0) + t_0 := \big\{t_0 + t : t \in I_{max}(0, x_0)\big\}\,, \tag{2.27}$$

and this motivates the definition of a flow of the autonomous differential equation (2.26).

Definition 2.22 (Flow of an autonomous differential equation)**.** *Consider the autonomous differential equation* (2.26)*, and define for any initial value* $x_0 \in D$*,*

$$J_{max}(x_0) := I_{max}(0, x_0) \tag{2.28}$$

and

$$\varphi(t, x_0) = \lambda(t, 0, x_0) \quad \text{for all } t \in J_{max}(x_0)\,.$$

The function $(t, x_0) \mapsto \varphi(t, x_0)$ *is called the* flow *of the autonomous differential equation* (2.26)*.*

Analogously to (2.22) in the nonautonomous case and for the general solution, using the flow of an autonomous differential equation, the solution identity reads as

$$\frac{\partial \varphi}{\partial t}(t, x_0) = f\big(\varphi(t, x_0)\big) \quad \text{for all } x_0 \in D \text{ and } t \in J_{max}(x_0)\,.$$

We first consider a simple example.

Example 2.23. The differential equation

$$\dot{x} = ax$$

considered in the recent Example 2.20 to illustrate the notion of a general solution is an autonomous differential equation, so we can consider this in the setting of flows also. As explained above, the general solution of this differential equation is given by

$$\lambda(t, t_0, x_0) = x_0 e^{a(t-t_0)} \quad \text{for all } (t, t_0, x_0) \in \mathbb{R}^{1+1+d}\,.$$

Hence, the flow of this differential equation is given by the function

$$\varphi(t, x_0) = \lambda(t, 0, x_0) = x_0 e^{at} .$$

We note that normally flows are denoted as $\varphi(t, x)$ rather than $\varphi(t, x_0)$, and we will use the more usual notation in the following.

Flows have the following properties, which correspond naturally to the properties of general solutions studied in Proposition 2.21.

Proposition 2.24 (Properties of the flow)**.** *Let φ be the flow of the autonomous differential equation* (2.26)*. Then for any $x \in D$, the following statements hold.*

$$J_{max}(\varphi(t, x)) = J_{max}(x) - t \quad \textit{for all } t \in J_{max}(x) , \tag{2.29}$$

$$\varphi(0, x) = x , \tag{2.30}$$

$$\varphi(t, \varphi(s, x)) = \varphi(t + s, x) \quad \textit{for all } t, s \textit{ with } s, t + s \in J_{max}(x) , \tag{2.31}$$

$$\varphi(-t, \varphi(t, x)) = x \quad \textit{for all } t \in J_{max}(x) . \tag{2.32}$$

The identity (2.30) *is called* initial value condition*, while* (2.31) *is called* group property*.*

Proof. Let λ denote the general solution to (2.26), and let I_{max} denote the maximal existence intervals corresponding to λ.

(2.29): We have

$$\begin{aligned} J_{max}(\varphi(t, x)) &= J_{max}(\lambda(t, 0, x)) \overset{(2.28)}{=} I_{max}(0, \lambda(t, 0, x)) \\ &= I_{max}(0, \lambda(0, -t, x)) \overset{(2.23)}{=} I_{max}(-t, x) \\ &\overset{(2.27)}{=} I_{max}(0, x) - t = J_{max}(x) - t , \end{aligned}$$

where we used translation invariance in the third equality.

(2.30): From (2.24), it follows that $\varphi(0, x) = \lambda(0, 0, x) = x$.

(2.31): We have

$$\begin{aligned} \varphi(t, \varphi(s, x)) &= \lambda(t, 0, \lambda(s, 0, x)) = \lambda(t + s, s, \lambda(s, 0, x)) \\ &\overset{(2.25)}{=} \lambda(t + s, 0, x) = \varphi(t + s, x) . \end{aligned}$$

(2.32): This follows from $\varphi(-t, \varphi(t, x)) \overset{(2.31)}{=} \varphi(-t + t, x) \overset{(2.30)}{=} x$. □

As explained in Subsection 3.2 of Chapter 1, for visualising autonomous systems, we project solution curves from the extended phase space to objects (called *orbits* or *trajectories*) in the phase space. By doing so, we obtain the phase portrait of the differential equation.

Definition 2.25 (Orbits or trajectories)**.** *Let φ be the flow of the autonomous differential equation* (2.26)*. For all $x \in D$, we call the set*

$$O(x) := \big\{\varphi(t,x) \in D : t \in J_{max}(x)\big\}$$

the orbit (or trajectory) through x. In addition, we call $O^+(x) := \big\{\varphi(t,x) \in D : t \in J_{max}(x) \cap \mathbb{R}_0^+\big\}$ the positive half-orbit *through x, and $O^-(x) := \big\{\varphi(t,x) \in D : t \in J_{max}(x) \cap \mathbb{R}_0^-\big\}$ the* negative half-orbit *through x.*

The geometric picture is that the domain D of the right hand side f is partitioned into orbits of φ. There are essentially three different types of orbits $O(x)$ for $x \in D$.

(i) $O(x)$ is a singleton. This implies $f(x) = 0$ (see also Proposition 1.3), and $J_{max}(x) = \mathbb{R}$. The point x is called *equilibrium*.

(ii) $O(x)$ is a closed curve, i.e. there exists $t > 0$ such that $\varphi(t,x) = x$, but $f(x) \neq 0$. This implies $J_{max}(x) = \mathbb{R}$. The point x is called *periodic*, and $O(x)$ is called *periodic orbit*.

(iii) $O(x)$ is not a closed curve, i.e. the function $t \mapsto \varphi(t,x)$ is injective on $J_{max}(x)$.

We identify all three types of points in the following example.

Example 2.26. Consider the autonomous two-dimensional differential equation

$$\begin{aligned}\dot{x} &= y + x(1 - x^2 - y^2)\,,\\ \dot{y} &= -x + y(1 - x^2 - y^2)\,,\end{aligned}$$

for which it is possible (via polar coordinates leading to the differential equation $\dot{r} = r(1-r^2), \dot{\phi} = -1$) to obtain the expression

$$\varphi(t,x,y) = \frac{1}{\sqrt{x^2 + y^2 + \big(1 - x^2 - y^2\big)e^{-2t}}}\begin{pmatrix} x\cos(t) + y\sin(t)\\ y\cos(t) - x\sin(t)\end{pmatrix}$$

for the flow of the system, see Figure 2.4 for the phase portrait.

It is easy to see that all three trajectory types are present in this example. The point $(0,0)$ is the only equilibrium, and there exists a periodic orbit $O(0,1)$. All other orbits are not closed curves, and they converge in forward time to the periodic orbit. It is clear that it makes sense to call the periodic orbit *stable*, while the equilibrium $(0,0)$ is called *unstable*. We will formally introduce these so-called notions of stability later in this course.

The situation is a bit simpler in the one-dimensional case, where periodic orbits cannot occur. The proof of the following proposition is left as an exercise.

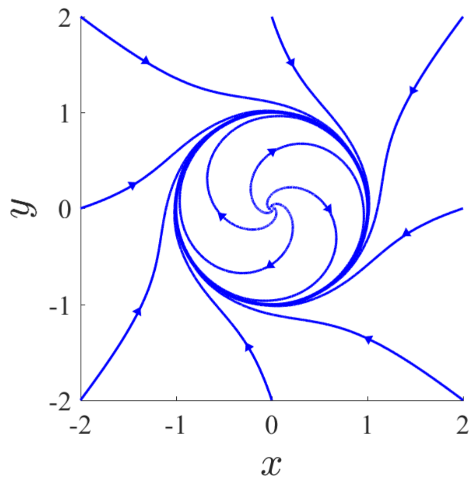


Figure 2.4. Phase portrait of the differential equation from Example 2.26.

Proposition 2.27 (Orbits of one-dimensional differential equations)**.** *Consider the autonomous differential equation* (2.26)*, where* $d = 1$*. Then all solutions are monotone, and there do not exist periodic orbits. This means that a trajectory is either a equilibrium or a non-closed curve.*

The following remark addresses smoothness properties of general solutions and flows.

Remark 2.28 (Continuity and differentiability of general solutions and flows)**.** Note that the general solution (as a function of three variables) and the flow (as a function of two variables) is continuous. The proof of this fact is lengthy and not very insightful, so we do not cover this in this course. In addition, if we have more regularity of the right hand side, such as continuous differentiability, one can even prove that the general solution and the flow are also continuously differentiable. These are important results, since in many problems, one is interested in variation of the initial conditions (and also of parameters of the system, for which similar results hold).

Chapter 3

Linear systems

Although the most interesting differential equations in applications are nonlinear, the class of linear systems is very important, because they allow to describe a first-order approximation of the behaviour of solutions close to a given reference solution. Consider such a solution $\mu : I \to \mathbb{R}^d$ of a differential equation

$$\dot{x} = f(t, x)\,. \tag{3.1}$$

Then the linearisation along the reference solution μ is given by

$$\dot{x} = \underbrace{\frac{\partial f}{\partial x}(t, \mu(t))}_{=:A(t)\in\mathbb{R}^{d\times d}} x$$

where the right hand side is a (time-dependent) linear function, the matrix of which is given by the derivative of the (differentiable) function f evaluated along the solution. In general, i.e. if the differential equation (3.1) is nonautonomous, or the solution μ is not constant, then the matrix A depends on time t, which makes the situation quite complicated. This is due to the fact that nonautonomous linear systems $\dot{x} = A(t)x$ are not solvable in general, while there exists an explicit representation for the flow of an autonomous linear system $\dot{x} = Ax$. For this reason, we focus on the autonomous case, which is obtained when we linearise an autonomous differential equation $\dot{x} = f(x)$ in an equilibrium x^*, leading to the linear system $\dot{x} = f'(x^*)x$.

1. Matrix exponential function

We consider the linear differential equation

$$\dot{x} = Ax\,, \tag{3.2}$$

where $A \in \mathbb{R}^{d\times d}$. Since $\|Ax - Ay\| = \|A(x-y)\| \le \|A\|\|x-y\|$, this system is globally Lipschitz continuous with Lipschitz constant $\|A\|$, and due to the global version of the Picard–Lindelöf theorem (and an exercise from a problem sheet), solutions to every initial value problem exist on $\mathbb{R}$ and are unique, and this generates a globally defined flow $\varphi : \mathbb{R}\times\mathbb{R}^d \to \mathbb{R}^d$.

Due to the Picard–Lindelöf theorem (Theorem 2.11), solutions can be obtained locally by convergence of Picard iterates $(\lambda_n)_{n\in\mathbb{N}_0}$ on some interval $J \subset \mathbb{R}$ as defined in (2.6). We fix an initial value $x_0 \in \mathbb{R}^d$ and are interested in how the solution corresponding to the initial condition $x(0) = x_0$ looks in a neighbourhood J of 0. Note that the following analysis generalises Example 2.3.

We define the initial function $\lambda_0(t) := x_0$ for all $t \in J$, and the iterates as defined in (2.6) read as

$$\lambda_{n+1}(t) = P(\lambda_n)(t) = x_0 + \int_0^t A\lambda_n(s)\,\mathrm{d}s\,.$$

Then for all $t \in J$, we have

$$\begin{aligned}\lambda_1(t) &= x_0 + \int_0^t A\lambda_0(s)\,\mathrm{d}s = x_0 + tAx_0\,,\\ \lambda_2(t) &= x_0 + \int_0^t A\lambda_1(s)\,\mathrm{d}s = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0\,.\end{aligned}$$

By induction, we obtain

$$\lambda_n(t) = x_0 + tAx_0 + \frac{t^2}{2}A^2x_0 + \dots + \frac{t^n}{n!}A^nx_0 = \sum_{k=0}^{n}\frac{t^kA^k}{k!}x_0\,.$$

So the solution to the initial value problem (3.2), $x(0) = x_0$, is given locally around $t = 0$ by

$$\lambda_\infty(t) = \varphi(t,x_0) = e^{At}x_0\,,\ \text{with } e^{At} := \sum_{k=0}^{\infty}\frac{t^kA^k}{k!} \text{ with } A^0 = \mathrm{Id}_d\,. \tag{3.3}$$

The proof of the Picard–Lindelöf Theorem 2.11 shows that this infinite sum exists (i.e. the series converges) whenever $|t| \le h$ for some $h > 0$. We will demonstrate later that it exists for all $t \in \mathbb{R}$.

For a given matrix $A \in \mathbb{R}^{d\times d}$, the function $t \mapsto e^{At}$ is called the *matrix exponential function*.

To prove that the matrix exponential function exists for all $t \in \mathbb{R}$, we need the following lemma (recall the definition of the operator norm of a matrix from Definition 2.7).

Lemma 3.1 (Sub-multiplicativity of the matrix norm)**.** *For two matrices $B, C \in \mathbb{R}^{d\times d}$, we have*

$$\|BC\| \le \|B\|\|C\|\,. \tag{3.4}$$

Proof. For any $0 \neq x \in \mathbb{R}^d$, we have

$$\|BCx\| \le \|B\|\|Cx\| \le \|B\|\|C\|\|x\|\,,$$

and thus, $\|B\|\|C\|$ is an upper bound for $\frac{\|BCx\|}{\|x\|}$, which finishes the proof. □

Proposition 3.2 (Existence of the matrix exponential)**.** *Consider a matrix $B \in \mathbb{R}^{d\times d}$. Then its matrix exponential*

$$e^B := \sum_{k=0}^{\infty} \frac{1}{k!} B^k$$

exists and is a matrix in $\mathbb{R}^{d\times d}$.

Proof. Due to Lemma 3.1, we have $\|B^k\| \le \|B\|^k$ for all $k \in \mathbb{N}$. This implies

$$e^{\|B\|} = \sum_{k=0}^{\infty} \frac{\|B\|^k}{k!} \ge \sum_{k=0}^{\infty} \frac{\|B^k\|}{k!}$$

Hence, using the comparison test, we see that the series $\sum_{k=0}^{\infty} \frac{\|B^k\|}{k!}$ is convergent. Define the sequences

$$a_n := \sum_{k=0}^{n} \frac{\|B^k\|}{k!} \quad \text{and} \quad b_n := \sum_{k=0}^{n} \frac{B^k}{k!} \quad \text{for all } n \in \mathbb{N}\,,$$

and note that due to the above observation, the sequence $\{a_n\}_{n\in\mathbb{N}}$ converges. For $n > m$, the triangle inequality implies that

$$\|b_n - b_m\| = \left\| \sum_{k=m+1}^{n} \frac{1}{k!} B^k \right\| \le \sum_{k=n+1}^{m} \left\| \frac{1}{k!} B^k \right\| = |a_n - a_m|\,.$$

Since $\{a_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence, this inequality shows that $\{b_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence as well, and it is convergent in $\mathbb{R}^{d\times d}$. □

We have seen that in (3.3) that the flow of (3.2) ($\dot{x} = Ax$) is given locally around $t = 0$ by $\varphi(t, x) = e^{At}x$. We show now that this holds for all $t \in \mathbb{R}$.

Theorem 3.3 (The flow of an autonomous linear differential equation)**.** *Consider the autonomous linear differential equation* (3.2) *with coefficient matrix $A \in \mathbb{R}^{d\times d}$. Then the flow $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ generated by this differential equation is given by*

$$\varphi(t, x) = e^{At}x \quad \textit{for all } t \in \mathbb{R}\,.$$

Proof. *Step 1. We show that $e^{A(t+s)} = e^{At}e^{As}$ for any $t, s \in \mathbb{R}$.*
We have

$$e^{At}e^{As} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} \sum_{\ell=0}^{\infty} \frac{s^\ell A^\ell}{\ell!} = \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{t^k s^\ell A^{k+\ell}}{k!\ell!},$$

and setting $n = k + \ell$, so that $k = n - \ell$, we get

$$\begin{aligned} e^{At}e^{As} &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty} \frac{t^k s^{n-k} A^n}{k!(n-k)!} = \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} t^k s^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{A^n (t+s)^n}{n!} = e^{A(t+s)}. \end{aligned}$$

Step 2. We show that $\varphi(t, x_0) = e^{At}x_0$ for all $t \in \mathbb{R}$.
We know already from (3.3) that there exists $h > 0$ such that

$$\varphi(t, x_0) = e^{At}x_0 \quad \text{for all } t \in [-h, h].$$

Let $t \in \mathbb{R}$ and choose $N \in \mathbb{N}$ with $\frac{t}{N} \in [-h, h]$. Then the group property (2.31) of φ implies

$$\begin{aligned} \varphi(t, x_0) &= \underbrace{\varphi(\tfrac{t}{N}, \varphi(\tfrac{t}{N}, \dots, \varphi(\tfrac{t}{N}, x_0)\dots))}_{N\text{times}} \\ &= \prod_{i=1}^{N} e^{\frac{t}{N}A} x_0 \overset{\text{Step 1}}{=} e^{AN\frac{t}{N}} x_0 = e^{At}x_0. \end{aligned}$$

This finishes the proof. □

The matrix exponential has the following important properties. The proof is left as an exercise.

Proposition 3.4 (Properties of the matrix exponential)**.** *Consider matrices $B, C, T \in \mathbb{R}^{d\times d}$ such that T is invertible. Then the following statements hold.*

(i) *If $C = T^{-1}BT$, then $e^C = T^{-1}e^B T$.*
(ii) $e^{-B} = \left(e^B\right)^{-1}$.
(iii) *If $BC = CB$, then $e^{B+C} = e^B e^C$.*
(iv) *If B is a block diagonal matrix $B = \operatorname{diag}(B_1, \dots, B_p)$ with matrices $B_1, \dots, B_p$, then $e^B = \operatorname{diag}(e^{B_1}, \dots, e^{B_p})$.*

Question: Why does (iii) of Proposition 3.4 not hold in general? Can you find matrices $B, C \in \mathbb{R}^{d\times d}$ such hat $e^{B+C} \neq e^B e^C$?

2. Planar linear systems

We look first at the two-dimensional case, and we would like to have an explicit representation of the flow e^{At} for an autonomous linear differential equation

$$\dot{x} = Ax\,,$$

where $A \in \mathbb{R}^{2\times 2}$. In addition, we would like to understand what different types of phase portraits we can get.

We transform the matrix A in Jordan normal form $J = T^{-1}AT$, where the transformation matrix $T \in \mathbb{R}^{2\times 2}$ is invertible. Using Proposition 3.4 (i), we get $e^{At} = Te^{Jt}T^{-1}$, and it remains to understand e^{Jt} for two-dimensional Jordan normal forms J.

There are four different cases:

(C1) A has two different real eigenvalues $a, b \in \mathbb{R}$: $J = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$.

(C2) A has a double real eigenvalue $a \in \mathbb{R}$ with two linearly independent eigenvectors: $J = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$.

(C3) A has a double real eigenvalue $a \in \mathbb{R}$ with only one eigenvector: $J = \left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$.

(C4) A has complex pair $a \pm ib$ of eigenvalues with $b \neq 0$: $J = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$.

We first study the situation for all cases when the matrix is not singular, i.e. 0 is not an eigenvalue of A.

I. The matrix A is not singular.

I.(C1) $J = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, where $a, b \in \mathbb{R} \setminus \{0\}$ and $a \neq b$.

We get

$$e^{Jt} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{pmatrix} \quad \text{for all } \; t \in \mathbb{R}\,.$$

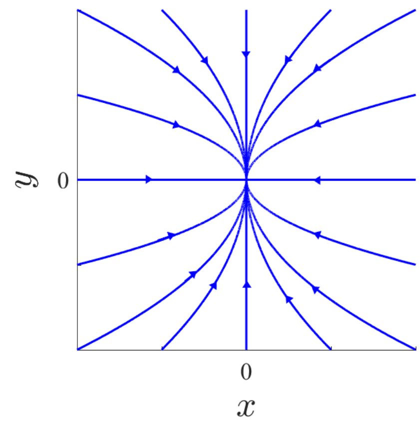
This means that for any $(x_0, y_0) \in \mathbb{R}^2$, the trajectory is given by $O(x_0, y_0) = \{e^{Jt}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) : t \in \mathbb{R}\} = \big\{\big(x_0 e^{at}, y_0 e^{bt}\big) : t \in \mathbb{R}\big\}$. We see that apart from the equilibrium $(0,0)$, the four half axes are trajectories. Outside of these trajectories, we obtain the representation

$$O(x_0, y_0) = \left\{\left(x, y_0\left(\tfrac{x}{x_0}\right)^{\frac{b}{a}}\right) \in \mathbb{R}^2 : \frac{x}{x_0} > 0\right\}.$$

We obtain the following phase portraits, depending on the order and sign of a and b.

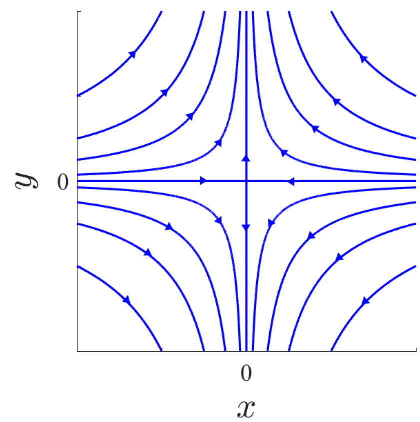
$a < b < 0$

stable knot
with two tangents



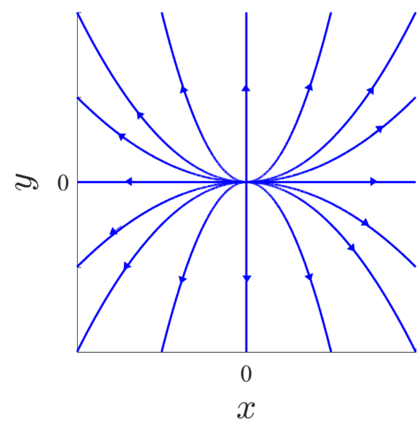
$a < 0 < b$

saddle



$0 < a < b$

unstable knot
with two tangents



I.(C2) $J = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$, where $a \in \mathbb{R} \setminus \{0\}$.
We get

$$e^{Jt} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{at} \end{pmatrix} \quad \text{for all } \ t \in \mathbb{R}\,.$$

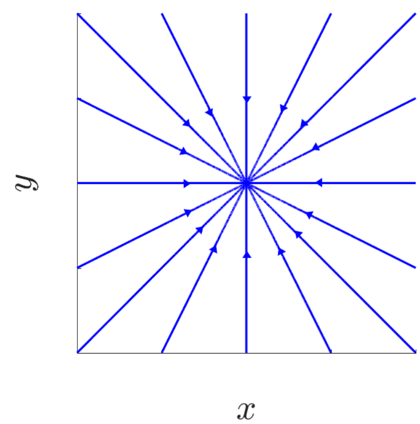
This means that for any $(x_0, y_0) \in \mathbb{R}^2$, the trajectory is given by $O(x_0, y_0) = \{e^{Jt}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) : t \in \mathbb{R}\} = \big\{\big(x_0 e^{at}, y_0 e^{at}\big) : t \in \mathbb{R}\big\}$. We see that apart from the equilibrium $(0,0)$, the four half axes are trajectories. Outside of these trajectories, we obtain the representation

$$O(x_0, y_0) = \Big\{\big(x, x\tfrac{y_0}{x_0}\big) \in \mathbb{R}^2 : \frac{x}{x_0} > 0\Big\}\,.$$

We obtain the following phase portraits, depending on the sign of a.

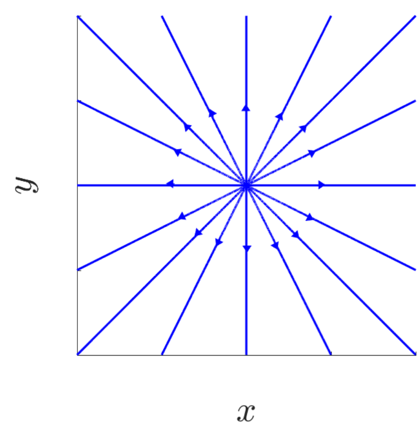
$a < 0$

stable knot
with many tangents



$a > 0$

unstable knot
with many tangents



I.(C3) $J = \left(\begin{smallmatrix} a & 1 \\ 0 & a \end{smallmatrix}\right)$, where $a \in \mathbb{R} \setminus \{0\}$.
It follows from an exercise that

$$e^{Jt} = \begin{pmatrix} e^{at} & te^{at} \\ 0 & e^{at} \end{pmatrix} \quad \text{for all } \ t \in \mathbb{R}\,.$$

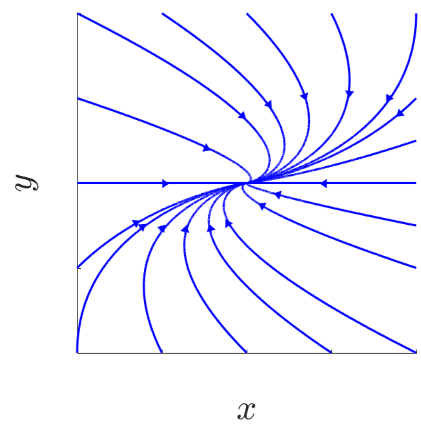
This means that for any $(x_0, y_0) \in \mathbb{R}^2$, the trajectory is given by $O(x_0, y_0) = \{e^{Jt}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) : t \in \mathbb{R}\} = \big\{\big(x_0 e^{at} + y_0 t e^{at}, y_0 e^{at}\big) : t \in \mathbb{R}\big\}$. We see that apart from the equilibrium $(0,0)$, the two half x-axes are trajectories. Outside of these trajectories, we obtain the representation

$$O(x_0, y_0) = \left\{ \left(\tfrac{x_0}{y_0} y + \tfrac{y}{a} \ln \tfrac{y}{y_0}, y \right) \in \mathbb{R}^2 : \frac{y}{y_0} > 0 \right\}.$$

We obtain the following phase portraits, depending on the sign of a.

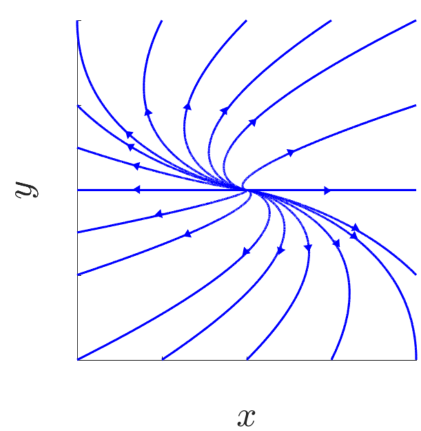
$a < 0$

stable knot
with one tangent



$a > 0$

unstable knot
with one tangent



I.(C4) $J = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where $b \in \mathbb{R} \setminus \{0\}$.
It follows from an exercise that

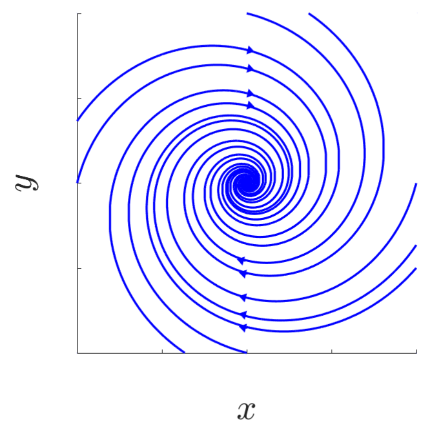
$$e^{Jt} = e^{at} \begin{pmatrix} \cos(bt) & \sin(bt) \\ -\sin(bt) & \cos(bt) \end{pmatrix} \quad \text{for all } t \in \mathbb{R}\,. \tag{3.5}$$

This means that for any $(x_0, y_0) \in \mathbb{R}^2$, the trajectory is given by

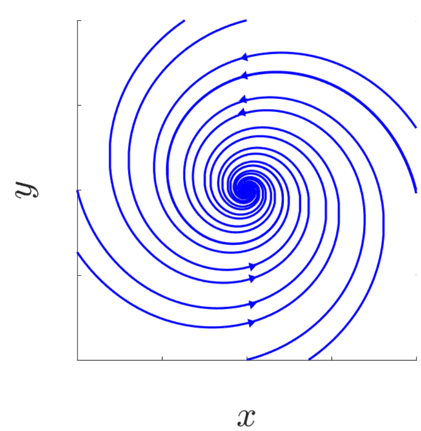
$$\begin{aligned} O(x_0, y_0) &= \left\{ e^{Jt} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} : t \in \mathbb{R} \right\} \\ &= \left\{ e^{at} \begin{pmatrix} x_0 \cos(bt) + y_0 \sin(bt) \\ y_0 \cos(bt) - x_0 \sin(bt) \end{pmatrix} : t \in \mathbb{R} \right\}. \end{aligned}$$

We obtain the following phase portraits, depending on the signs of a and b

$a < 0, b > 0$
stable focus

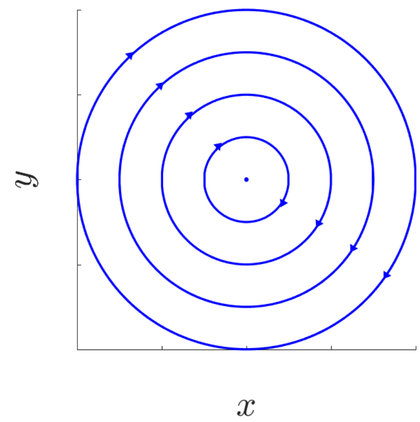


$a < 0, b < 0$
stable focus



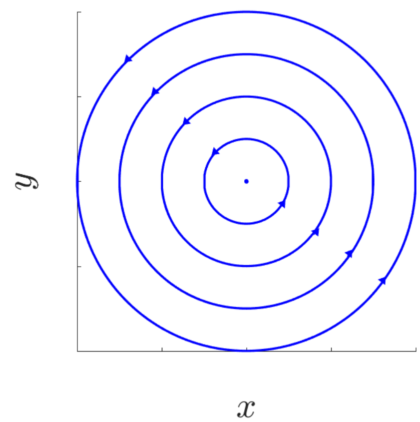
$a = 0, b > 0$

centre



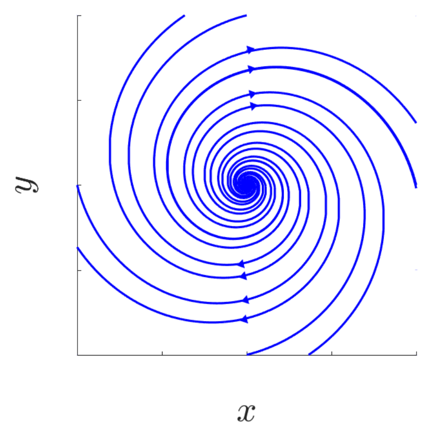
$a = 0, b < 0$

centre



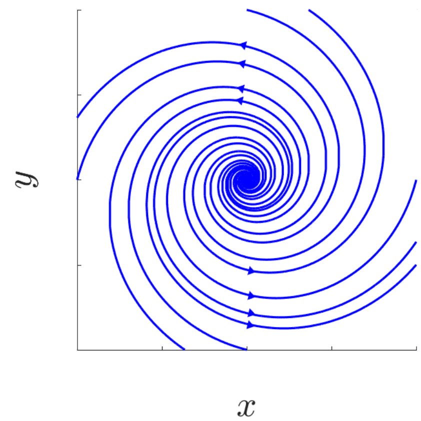
$a > 0, b > 0$

unstable focus



$a > 0, b < 0$

unstable focus



II. The matrix A is singular.

II.(C1) $J = \left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $a \in \mathbb{R} \setminus \{0\}$.

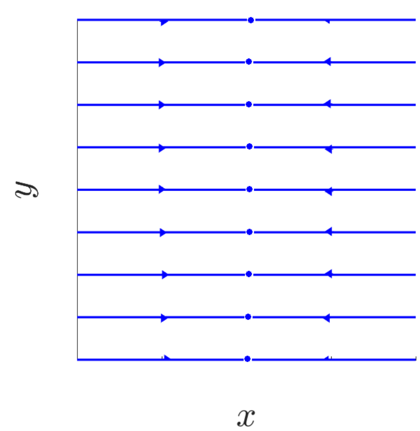
We get

$$e^{Jt} = \begin{pmatrix} e^{at} & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all } \; t \in \mathbb{R}\,.$$

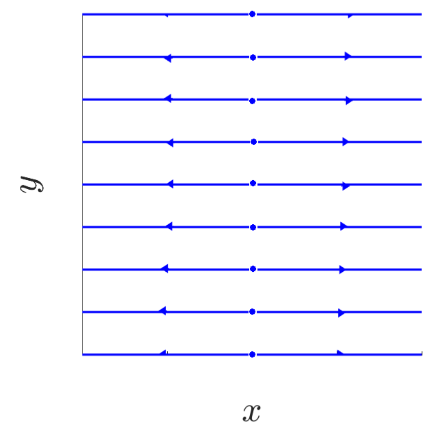
This means that for any $(x_0, y_0) \in \mathbb{R}^2$, the trajectory is given by $O(x_0, y_0) = \{e^{Jt}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) : t \in \mathbb{R}\} = \big\{(e^{at}x_0, y_0) : t \in \mathbb{R}\big\}$.

We obtain the following phase portraits, depending on the sign of a.

$a < 0$



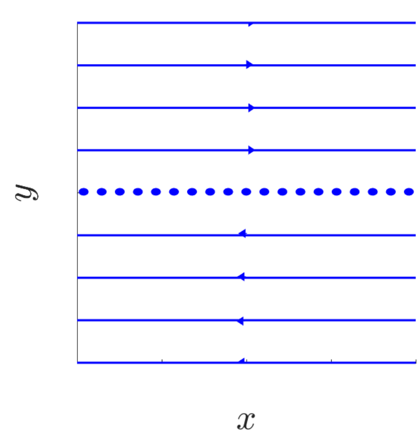
$a > 0$



II.(C2) $J = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$.
In this trivial case, the whole phase space consists of equilibria.
II.(C3) $J = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$.
We get

$$e^{Jt} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \text{for all} \ \ t \in \mathbb{R} \,.$$

This means that for any $(x_0, y_0) \in \mathbb{R}^2$, the trajectory is given by $O(x_0, y_0) = e^{Jt}\left(\begin{smallmatrix} x_0 \\ y_0 \end{smallmatrix}\right) = \big\{(x_0 + y_0 t, y_0) : t \in \mathbb{R}\big\}$. Note that the x-axis consists of equilibria. The phase portrait is given as follows.



II.(C4) This case does not exist, since a two-dimensional matrix without real eigenvalues cannot have eigenvalue zero.

We note that while we know now everything about two-dimensional phase portraits for differential equations in Jordan normal form, the question remains how phase portraits look like when the system is not in Jordan normal form. Firstly, note that, given a general two-dimensional linear system $\dot{x} = Ax$, we transform the matrix $A \in \mathbb{R}^{2\times 2}$ in Jordan normal form via $J = T^{-1}AT$, where the transformation matrix $T \in \mathbb{R}^{2\times 2}$ is invertible. Using Proposition 3.4 (i), we have $e^{At} = Te^{Jt}T^{-1}$, but what does this mean for the phase portrait? In an exercise, you show that the phase portrait of the original system $\dot{x} = Ax$ is the result of applying the linear transformation T to the phase portrait of system in Jordan normal form.

We close this section on two-dimensional linear systems with a remark concerning the importance of the eigenvalues of A with regard to both exponential growth and rotation.

Remark 3.5 (Meaning of real and imaginary part of the eigenvalues of A)**.** It turns out that in all two-dimensional examples, the eigenvalues are characteristic for both the strength of exponential growth and rotation of the solutions of the corresponding linear differential equation $\dot{x} = Ax$.

(i) *Rate of exponential growth.* The real part of the eigenvalues of A determine *rates* the exponential growth behaviour of solutions $\lambda(t) = e^{At}(x_0, y_0)^\top$, where $t \in \mathbb{R}$. Note that if a function $\mu : \mathbb{R} \to \mathbb{R} \setminus \{0\}$ is growing exponentially, for instance, if $\mu(t) = e^{at}$ for some $a \in \mathbb{R}$, then the exponential growth rate can be obtained as

$$\lim_{t\to\infty} \frac{\ln \mu(t)}{t} = \frac{\ln e^{at}}{t} = a\,.$$

This is still true if the exponential growth is not purely exponential, for instance, an easy calculation shows that the function $\mu(t) = t^n e^{at}$, where $n \in \mathbb{N}$, will give the same exponential growth rate a (note that we see such a function in Case I.(C3)).
This motivates the definition of a Lyapunov exponent corresponding to the above solution λ, where we assume that the initial condition $(x_0, y_0) \neq (0, 0)$.

$$\sigma_{Lyap}(\lambda) := \lim_{t\to\infty} \frac{\ln \|\lambda(t)\|}{t}\,,$$

provided the limit exists. As can be checked easily, it turns out in our all of our two-dimensional examples that the limit exists and is equal the real part of one of the eigenvalues of the matrix A. Note also that a solution exponentially decays if $\sigma_{Lyap}(\lambda) < 0$ and exponentially increases if $\sigma_{Lyap}(\lambda) > 0$.

(ii) *Rate of rotation.* As can be seen in the examples, solutions only rotate in Case I.(C4). This is the only case where the eigenvalues are not real, and it can be seen that the absolute value of b determines the speed of rotation, while the sign of b determines the orientation of rotation: it goes in clockwise if $b > 0$, and anti-clockwise if $b < 0$.

It will turn out in a moment that these two-dimensional observations are true in higher-dimensions as well, see the formulas for the matrix exponential in Proposition 3.8 below.

3. Jordan normal form

We now aim at studying the higher-dimensional case, and we would like to have an explicit representation of the flow e^{At} for an autonomous linear differential equation

$$\dot{x} = Ax\,,$$

where $A \in \mathbb{R}^{d\times d}$. As before, the Jordan normal form plays a crucial role, since, as we will see later, it is easy to compute the matrix exponential function corresponding to the Jordan normal form of A.

We first look at the complex Jordan normal form of A. The transformation matrix (called T below) leading to this Jordan normal form may be complex in this case when there are complex eigenvalues. When looking at the real Jordan normal form in the next step, we will be able to remove the complex entries in the transformation matrix, in order to arrive at the real Jordan normal form.

Theorem 3.6 (Complex Jordan normal form)**.** *Consider a matrix $A \in \mathbb{R}^{d\times d}$. Then there exists a matrix $T \in \mathbb{C}^{d\times d}$ so that under a basis transformation with the matrix T, we obtain the* complex Jordan normal form

$$J := T^{-1}AT = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{pmatrix}\,,$$

with the so-called Jordan blocks

$$J_j = \begin{pmatrix} \rho_j & 1 & & 0 & 0 \\ 0 & \rho_j & 1 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & \rho_j & 1 \\ 0 & 0 & & 0 & \rho_j \end{pmatrix} \quad \textit{for all } j \in \{1,\dots,p\}\,, \tag{3.6}$$

where the ρ_j, $j \in \{1,\dots,p\}$, are complex eigenvalues of the matrix A (some of which may be the same).

Note that if J_j is a 1×1 matrix, then $J_j = (\rho_j)$, and if J_j is a 2×2 matrix, then

$$J_j = \begin{pmatrix} \rho_j & 1 \\ 0 & \rho_j \end{pmatrix} .$$

You have encountered the complex Jordan normal form already, and we will not prove this theorem here, and we refer to *Repetition Material 4* for a more details how to compute the complex Jordan normal form.

We cover in more detail now the real Jordan normal form.

Theorem 3.7 (Real Jordan normal form)**.** *Consider a matrix $A \in \mathbb{R}^{d \times d}$. Then there exists a matrix $T \in \mathbb{R}^{d \times d}$ so under basis transformation with the matrix T, we obtain the* real Jordan normal form

$$J := T^{-1}AT = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_p \end{pmatrix} ,$$

where the Jordan blocks J_j are either as in Theorem 3.6, i.e. given by (3.6), *in case the eigenvalue ρ_j is real, or, in case the eigenvalue ρ_j is complex,*

$$J_j = \begin{pmatrix} C_j & \mathrm{Id}_2 & & 0 & 0 \\ 0 & C_j & \mathrm{Id}_2 & & 0 \\ & & \ddots & \ddots & \\ 0 & & & C_j & \mathrm{Id}_2 \\ 0 & 0 & & 0 & C_j \end{pmatrix} , \tag{3.7}$$

where $C_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix}$ with $\rho_j = a_j + ib_j$, and $\mathrm{Id}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note that if J_j is a 2×2 matrix, then

$$J_j = \begin{pmatrix} a_j & b_j \\ -b_j & a_j \end{pmatrix} .$$

We also do not prove this theorem, but would like to understand how to construct the matrix T to obtain the real Jordan form, given by the matrix J.

Question 1. Why does the real Jordan normal form consist of blocks of the form C_j? We demonstrate this here for the two-dimensional case. Assume that we are in the situation of (C4) in Section 2, i.e. the matrix $A \in \mathbb{R}^{2 \times 2}$ has the complex eigenvalues $\rho = a \pm ib$ with $b \neq 0$. Then the complex Jordan normal form reads as

$$J_{\text{complex}} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix} .$$

Assume that $u+iv$ with $u,v\in\mathbb{R}^2$ is complex eigenvector for the eigenvalue $a+ib$, i.e.

$$A(u+iv)=(a+ib)(u+iv)\,. \tag{3.8}$$

We note that the vectors u and v are linearly independent in $\mathbb{R}^2$. To see this, assume they are linearly dependent, i.e. $v=\gamma u$ for some real value $\gamma\neq 0$. Hence $u+iv=(1+i\gamma)u$. Then (3.8) implies that $(1+i\gamma)Au=(a+ib)(1+i\gamma)u$, which leads to

$$Au=(a+ib)u\,,$$

and we get a contradiction, since the left hand side is vector in $\mathbb{R}^2$, but not the right hand side. We now look at action of the linear mapping on the basis vectors u and v of $\mathbb{R}^2$. Comparing real part and imaginary part of (3.8) implies that

$$\begin{aligned}Au&=au-bv\\Av&=bu+av\,,\end{aligned}$$

which gives the desired Jordan normal form $\left(\begin{smallmatrix}a&b\\-b&a\end{smallmatrix}\right)$ corresponding to the basis $\{u,v\}$. Note that the sign of b is not unique in the Jordan form. In fact, taking $-v$ instead of v as basis vector changes the sign of b. Note also that taking complex conjugation on both sides of (3.8) implies that $u-iv$ is eigenvector for the eigenvalue $a-ib$.

Question 2. How do we compute the matrix $T\in\mathbb{R}^{d\times d}$? We assume that we know the complex Jordan normal form already, with the transformation matrix. Note that the non-real eigenvalues appear in complex conjugate pairs a_j+ib_j and a_j-ib_j with $b_j>0$. Now ignore all blocks with $b_j<0$ and replace the bases of complex generalised eigenvectors and eigenvectors corresponding to a_j+ib_j with $b_j>0$, given by $w_1^j,\dots,w_{d_j}^j$, by the column vectors

$$\operatorname{Re}w_1^j,\operatorname{Im}w_1^j,\dots,\operatorname{Re}w_{d_j}^j,\operatorname{Im}w_{d_j}^j\,.$$

Then we will get a real matrix $T\in\mathbb{R}^{d\times d}$ that guarantees transformation in real Jordan normal form according to Theorem 3.7.

4. Explicit representation of the matrix exponential function

Consider a matrix $A\in\mathbb{R}^{d\times d}$. We explain in this section how to obtain an explicit representation of e^{At} for all $t\in\mathbb{R}$.

We assume that the invertible matrix $T\in\mathbb{R}^{d\times d}$ transforms A into the real Jordan normal form

$$J=T^{-1}AT=\begin{pmatrix}J_1&&0\\&\ddots&\\0&&J_p\end{pmatrix},$$

as in Theorem 3.7, where the Jordan blocks J_j are given by either (3.6) (corresponding to real eigenvalues ρ_i) or (3.7) (corresponding to complex eigenvalues $\rho_j = a_j + ib_j$). Due to Proposition 3.4 (i) and (iv), we get

$$e^{At} = Te^{Jt}T^{-1} = T \begin{pmatrix} e^{J_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{J_p t} \end{pmatrix} T^{-1},$$

so it remains to find an explicit representation for the matrix exponentials $t \mapsto e^{J_j t}$ of each Jordan block J_j, where $j \in \{1, \dots, p\}$.

Proposition 3.8. *Consider the matrix $A \in \mathbb{R}^{d \times d}$, and let J_j for $j \in \{1, \dots, p\}$ be the Jordan blocks for the real Jordan normal form with eigenvalues ρ_j. The matrix exponentials $e^{J_j t}$ for $t \in \mathbb{R}$ are then given as follows.*

(i) *If ρ_j is real, i.e. $J_j \in \mathbb{R}^{d_j \times d_j}$ is of the form (3.6), we obtain*

$$e^{\begin{pmatrix} \rho_j & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \rho_j \end{pmatrix} t} = e^{\rho_j t} \begin{pmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{d_j-1}}{(d_j-1)!} \\ 0 & 1 & t & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2} \\ 0 & & & 1 & t \\ 0 & 0 & & 0 & 1 \end{pmatrix}.$$

(ii) *If $\rho_j = a_j + ib_j \in \mathbb{C}$ is not real, i.e. $J_j \in \mathbb{R}^{2d_j \times 2d_j}$ is of the form (3.7), we obtain*

$$e^{\begin{pmatrix} C_j & \mathrm{Id}_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \mathrm{Id}_2 \\ 0 & & & C_j \end{pmatrix} t} = e^{a_j t} \begin{pmatrix} G(t) & tG(t) & \frac{t^2}{2}G(t) & \cdots & \frac{t^{d_j-1}}{(d_j-1)!}G(t) \\ 0 & G(t) & tG(t) & \ddots & \vdots \\ & & \ddots & \ddots & \frac{t^2}{2}G(t) \\ 0 & & & G(t) & tG(t) \\ 0 & 0 & & 0 & G(t) \end{pmatrix},$$

where $G(t) = \begin{pmatrix} \cos(b_j t) & \sin(b_j t) \\ -\sin(b_j t) & \cos(b_j t) \end{pmatrix}$ *for all* $t \in \mathbb{R}$.

Proof. (i) If ρ_j is real, then $J_j \in \mathbb{R}^{d_j \times d_j}$ has the form $J = P + D$, where

$$P = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \rho_j & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & \rho_j \end{pmatrix}.$$

Obviously, we have $PD = \lambda P = DP$, i.e. Proposition 3.4 (ii) is applicable, and we get

$$e^{J_j t} = e^{(P+D)t} = e^{Pt} e^{Dt} \quad \text{for all } t \in \mathbb{R}\,.$$

The matrix P is nilpotent, since we have $P^\ell = 0$ for all $\ell \geq d_j$, and thus we get

$$e^{Pt} = \sum_{\ell=0}^{d_j-1} \frac{t^\ell}{\ell!} P^\ell \quad \text{for all } t \in \mathbb{R}\,,$$

and this implies the assertion, since $e^{Dt} = e^{\rho_j t} \operatorname{Id}_{d_j}$ for all $t \in \mathbb{R}$.

(ii) If $\rho_j = a_j + i b_j \in \mathbb{C}$ is not real, then $J_j \in \mathbb{R}^{2d_j \times 2d_j}$ has the form $J = Q + B$ with

$$Q = \begin{pmatrix} 0_{2\times 2} & \operatorname{Id}_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \operatorname{Id}_2 \\ 0 & & & 0_{2\times 2} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} C_j & & 0 \\ & \ddots & \\ & & \ddots & \\ 0 & & & C_j \end{pmatrix}.$$

Due to $QB = BQ$, we can apply Proposition 3.4 (ii), and we get

$$e^{J_j t} = e^{(Q+B)t} = e^{Qt} e^{Bt} \quad \text{for all } t \in \mathbb{R}\,.$$

The matrix Q is nilpotent, since we have $Q^\ell = 0$ for all $\ell \geq d_j$, and thus we get

$$e^{Qt} = \sum_{\ell=0}^{d_j-1} \frac{t^\ell}{\ell!} Q^\ell \quad \text{for all } t \in \mathbb{R}\,,$$

and this implies the assertion, since $e^{C_j t} = e^{a_j t} G(t)$ for all $t \in \mathbb{R}$ due to (3.5). □

5. Exponential growth behaviour

We are interested in the exponential growth of the flow induced by a linear system $\dot{x} = Ax$, where $A \in \mathbb{R}^{d \times d}$, and motivated by Remark 3.5, we define the spectrum of A as

$$\Sigma(A) := \big\{ \operatorname{Re} \rho : \rho \text{ is an eigenvalue of } A \big\} = \{s_1, \dots, s_q\}\,.$$

One can show that for the linear system

$$\dot{x} = Ax\,,$$

one gets a decomposition

$$\mathbb{R}^d = E_1 \oplus \cdots \oplus E_q$$

into linear spaces E_j, $j \in \{1, \dots, q\}$, which are *invariant* in the sense that

$$x \in E_j \text{ for some } j \in \{1, \dots, q\} \quad \Longrightarrow \quad \varphi(t, x) \in E_j \text{ for all } t \in \mathbb{R}\,.$$

Moreover, the Lyapunov exponent associated to non-trivial solutions starting in E_j is given by s_j, where $j \in \{1, \dots, q\}$:

$$x \in E_j \setminus \{0\} \quad \Longrightarrow \quad \sigma_{Lyap}(\varphi(\cdot, x)) = \lim_{t\to\infty} \frac{\ln \|\varphi(t,x)\|}{t} = s_j \,.$$

We do not prove this here, but note that this does not say anything about Lyapunov exponents for solutions starting in points $x \in \mathbb{R}^d$ that not in the spaces E_j, $j \in \{1, \dots, q\}$, which are the vast majority of points (if we do not consider the trivial case $q = 1$).

Question: What is the exponential growth behaviour for solutions starting in a general element $x \in \mathbb{R}^d$?

Instead of analysing the above decomposition and the associated Lyapunov exponents in detail, due to time constraints, we only aim at estimating the exponential growth of the norm of the matrix exponential function $t \mapsto e^{At}$, which will be important later when analysing nonlinear systems. To do this in a sharp way, we need the notion of a semi-simple eigenvalue. Recall that an eigenvalue is *semi-simple* if its algebraic multiplicity equals its geometric multiplicity. Equivalently, an eigenvalue is semi-simple if all the Jordan blocks associated to this eigenvalue in the real Jordan normal form from Theorem 3.7 are one-dimensional for real eigenvalues and two-dimensional for non-real eigenvalues.

Proposition 3.9 (Exponential estimate for the matrix exponential function)**.** *Consider a matrix $A \in \mathbb{R}^{d\times d}$, and choose $\gamma \in \mathbb{R}$ such that*

$$\gamma > \max\big\{\operatorname{Re}\rho : \rho \text{ is an eigenvalue of } A\big\}\,.$$

If all eigenvalues ρ with $\operatorname{Re}\rho = \max\big\{\operatorname{Re}\rho : \rho \text{ is an eigenvalue of } A\big\}$ are semi-simple, we can use a smaller γ, given by

$$\gamma := \max\big\{\operatorname{Re}\rho : \rho \text{ is an eigenvalue of } A\big\}\,.$$

Then there exists a $K > 0$ such that

$$\big\|e^{At}\big\| \le K e^{\gamma t} \quad \text{for all } t \ge 0\,.$$

Proof. Let J be the real Jordan normal form of the matrix A, i.e. there exists a $T \in \mathbb{R}^{d\times d}$ such that $J = T^{-1}AT$. We first consider exponential bounds for e^{Jt}. In an exercise, you have proved that there exists a constant $C \ge 1$ with

$$\frac{1}{C}\|B\|_\infty \le \|B\| \le C\|B\|_\infty \quad \text{for all } B \in \mathbb{R}^{d\times d}\,.$$

Here, the Euclidean operator norm $\|\cdot\|$, as introduced in (2.8), is compared to the infinity norm $\|\cdot\|_\infty$, which is defined as

$$\|B\|_\infty := \max_{i,j\in\{1,\dots,d\}} |b_{ij}| \quad \text{for all } B = (b_{ij})_{i,j\in\{1,\dots,d\}} \in \mathbb{R}^{d\times d}\,.$$

Note that a general result says that all norms on finite-dimensional spaces are equivalent. It follows now from Proposition 3.8 that all entries of e^{Jt} are of the form $g(t)t^n e^{\rho t}$, where ρ is an eigenvalue real part, $g(t)$ is a bounded function and $n \in \{0, \dots, d-1\}$. Note that in case of a semi-simple eigenvalue, n is always equal to 0. All entries of e^{Jt} can thus be estimated by $\tilde{K}e^{\gamma t}$ for $t \geq 0$, where $\tilde{K} > 0$ is chosen appropriately. Note here that we need the estimate $t^n e^{\rho t} \leq K' e^{\gamma t}$ for $\rho < \gamma$ and $t \geq 0$ here, where $K' > 0$ has to be chosen appropriately. Finally, we get the inequality

$$\begin{aligned} \|e^{At}\| &= \|Te^{Jt}T^{-1}\| \overset{(3.4)}{\leq} \|T\|\|T^{-1}\|\|e^{Jt}\| \leq C\|T\|\|T^{-1}\|\|e^{Jt}\|_\infty \\ &\leq \underbrace{C\|T\|\|T^{-1}\|\tilde{K}}_{=:K} e^{\gamma t} \quad \text{for all } t \geq 0\,, \end{aligned}$$

which finishes the proof of this theorem. □

6. Variation of constants formula

We know now that the flow φ of a general autonomous (homogeneous) linear system $\dot{x} = Ax$, where $A \in \mathbb{R}^{d \times d}$, is given by the matrix exponential function $\varphi(t, x) = e^{At}x$. We are now interested in the general solution to the corresponding inhomogeneous equation

$$\dot{x} = Ax + g(t)\,, \tag{3.9}$$

where $g : I \to \mathbb{R}^d$ is a continuous function on an interval $I \subset \mathbb{R}$.

Proposition 3.10 (Variation of constants formula)**.** *The general solution to* (3.9) *is given by*

$$\lambda(t, t_0, x_0) = e^{A(t-t_0)}x_0 + \int_{t_0}^t e^{A(t-s)}g(s)\,\mathrm{d}s \quad \textit{for all } t, t_0 \in I \textit{ and } x_0 \in \mathbb{R}^d\,.$$

Proof. We first show that for fixed $t_0 \in I$, the function $\mu_g : I \to \mathbb{R}^d$, given by

$$\mu_g(t) = \int_{t_0}^t e^{A(t-s)}g(s)\,\mathrm{d}s = e^{At}\int_{t_0}^t e^{-As}g(s)\,\mathrm{d}s$$

is a solution to (3.9). This follows from

$$\dot{\mu}_g(t) = Ae^{At}\int_{t_0}^t e^{-As}g(s)\,\mathrm{d}s + e^{At}e^{-At}g(t) = A\mu_g(t) + g(t)\,,$$

where we have used the product rule and the fundamental theorem of calculus. Note that the general solution to the homogeneous system $\dot{x} = Ax$ is given by

$$\lambda_h(t, t_0, x_0) = e^{A(t-t_0)}x_0\,,$$

which follows from the definition of a flow and Theorem 3.3. We show now that for a fixed (t_0, x_0), the function

$$\nu_{t_0,x_0}(t) := \lambda_h(t, t_0, x_0) + \mu_g(t) \quad \text{for all } t \in I$$

is a solution to (3.9). This follows from

$$\begin{aligned}\dot\nu_{t_0,x_0}(t) &= \dot\lambda_h(t, t_0, x_0) + \dot\mu_g(t) = A(t)\lambda_h(t, t_0, x_0) + A(t)\mu_g(t) + g(t) \\ &= A(t)(\lambda_h(t, t_0, x_0) + \mu_g(t)) + g(t) = A(t)\nu_{t_0,x_0}(t) + g(t)\,.\end{aligned}$$

Obviously, the function ν_{t_0,x_0} satisfies the initial condition $x(t_0) = x_0$, which proves that λ as given in the statement of the proposition is the general solution to (3.9). □

Note that a nonautonomous (homogeneous) linear system of the form

$$\dot x = A(t)x\,,$$

cannot be solved analytically in general, that means in the case when the matrices $A(t)$, $t \in I$, do not commute. Of course, one-dimensional matrices always commute, and thus, we can compute the general solution to the inhomogeneous linear differential equation

$$\dot x = a(t)x + g(t)\,,$$

where $a : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous functions. Similarly to above, one can show that its general solution is given by

$$\lambda(t, t_0, x_0) = e^{\int_{t_0}^t a(s)\,\mathrm{d}s} x_0 + \int_{t_0}^t e^{\int_s^t a(\tau)\,\mathrm{d}\tau} g(s)\,\mathrm{d}s \quad \text{for all } t, t_0 \in I \text{ and } x_0 \in \mathbb{R}\,.$$

Chapter 4

Nonlinear systems

This chapter deals with nonlinear autonomous differential equations. Such systems are not solvable in general, in contrast to the linear autonomous systems in the previous chapter, and we aim at understanding their behaviour in the spirit of the Russian mathematician and physicist Aleksandr M. Lyapunov (1857–1918) and the French mathematician Henri Poincaré (1854–1912), who are the so-called fathers of the *Qualitative Theory of Dynamical Systems*. This theory aims at understanding dynamical systems (for instance, given in the form of ordinary differential equations) from a qualitative point of view, i.e. without being able to solve them.

This chapter gives an introduction into the basic elements of this theory. In particular, the basic elements of stability theory will be explained, and the direct method of Lyapunov will be introduced. Finally, the asymptotic behaviour of two-dimensional differential equations will be analysed using the Poincaré–Bendixson theory.

1. Stability

We are interested in the dynamical behaviour of the flow of an autonomous differential equation in the vicinity of equilibria. We will distinguish between different types of stability close to equilibria, and we will learn about criteria that indicate such behaviour.

1.1. Basic definitions. We introduce different notions of stability for an autonomous differential equation

$$\dot{x} = f(x)\,, \tag{4.1}$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous and $D \subset \mathbb{R}^d$ is an open set. We denote the flow of this differential equation by φ.

Definition 4.1 (Notions of stability)**.** *Let x^* be an equilibrium of* (4.1), *i.e.* $f(x^*) = 0$.

(i) x^* *is called* stable *if for all $\varepsilon > 0$, there exists a $\delta > 0$ such that*

$$\|\varphi(t,x) - x^*\| < \varepsilon \quad \textit{for all } x \in B_\delta(x^*) \textit{ and } t \geq 0\,.$$

(ii) x^* *is called* unstable *if x^* is not stable.*

(iii) x^* *is called* attractive *if there exists a $\delta > 0$ such that*

$$\lim_{t\to\infty} \varphi(t,x) = x^* \quad \textit{for all } x \in B_\delta(x^*)\,.$$

(iv) x^* *is called* asymptotically stable *if x^* is both stable and attractive.*

(v) x^* *is called* exponentially stable *if there exist $\delta > 0$, $K \geq 1$ and $\gamma < 0$ such that*

$$\|\varphi(t,x) - x^*\| \leq K e^{\gamma t}\|x - x^*\| \quad \textit{for all } x \in B_\delta(x^*) \textit{ and } t \geq 0\,.$$

(vi) x^* *is called* repulsive *if there exists a $\delta > 0$ such that*

$$\lim_{t\to-\infty} \varphi(t,x) = x^* \quad \textit{for all } x \in B_\delta(x^*)\,.$$

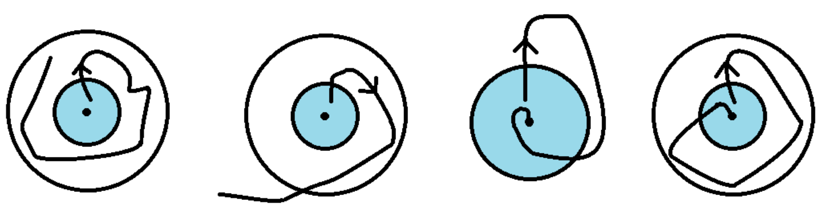


Figure 4.1. Notions of stability, from left to right: stable, unstable, attractive, asymptotically stable.

Example 4.2 (Stability of one-dimensional linear differential equations)**.** We study the trivial equilibrium $x^* = 0$ of the linear differential equation

$$\dot{x} = \alpha x\,,$$

where $\alpha \in \mathbb{R}$. This differential equation has the flow $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\varphi(t,x) = xe^{\alpha t}$. Depending on the parameter α, the equilibrium x^* has differential stability properties.

(i) x^* is stable for $\alpha \leq 0$. Choose $\delta := \varepsilon$ for a given $\varepsilon > 0$. We have $|\varphi(t,x) - x^*| = |x|e^{\alpha t} \leq |x| < \varepsilon = \delta$ for all $t \geq 0$ and $x \in (-\delta, \delta)$.

(ii) x^* is unstable for $\alpha > 0$. We fix $\varepsilon = 1$ and choose $\delta > 0$ arbitrarily. Then $|\varphi(t, \frac{\delta}{2})| = \frac{\delta}{2}e^{\alpha t} > \varepsilon$ for some $t > 0$, since $e^{\alpha t} \to \infty$ as $t \to \infty$.

(iii) x^* is exponentially stable for $\alpha < 0$, since $|\varphi(t,x)| = |x|e^{\alpha t} = Ke^{\gamma t}|x|$ for $t \geq 0$ and $x \in B_\delta(0)$ with $\gamma := \alpha < 0$ and $K := \delta = 1$.

(iv) x^* is repulsive for $\alpha > 0$. Choose $\delta := 1$. Then for all $x \in (-\delta, \delta)$, we have $|\varphi(t,x)| = |xe^{\alpha t}| = |x||e^{\alpha t}| < \delta|e^{\alpha t}| \to 0$ as $t \to -\infty$.

We see in this example that the eigenvalues for this one-dimensional linear system are crucial for the stability of the system. This applies to higher-dimensional linear systems as well, and we will make this more precise later.

It is clear from the definitions that exponential stability implies asymptotic stability. However, perhaps surprisingly, there is no relation between stability and attractivity.

Example 4.3 (On the relation between stability and attractivity)**.** We show in this example that, in general, the notions of stability and attractivity are not related.

(i) For the differential equation $\dot{x} = 0$, every point is an equilibrium, and all equilibria are stable (choose $\delta := \varepsilon$ for a given $\varepsilon > 0$). It is clear that no equilibrium is attractive in this example. Note that stability does not imply attractivity can also been seen when looking at the harmonic oscillator, which we studied in Example 1.10.

(ii) The two-dimensional differential equation

$$\dot{x} = x + xy - (x+y)\sqrt{x^2+y^2}\,,$$
$$\dot{y} = y - x^2 + (x-y)\sqrt{x^2+y^2}$$

can be understood well when looking at the corresponding system in polar coordinates

$$\dot{r} = r(1-r)\,, \quad \dot{\phi} = r(1-\cos\phi)\,.$$

The phase portrait is plotted in Figure 4.2. We note that this system has exactly two equilibria: $(0,0)$, which is an unstable knot with many tangents, and $(1,0)$, which admits a so-called homoclinic orbit, given by the unit circle. Note that an orbit is called *homoclinic* if converges forward and backward in time to the same equilibrium. It is clear, from looking at the polar coordinate system, that the equilibrium $(1,0)$ is attractive. It can be even proved that all orbits starting outside of the equilibrium $(0,0)$ converge to the other equilibrium $(1,0)$ in forward time: $\lim_{t\to\infty}\varphi(t,(x,y)) = (1,0)$ for all $(x,y) \neq (0,0)$. This is clearly seen in Figure 4.2, and can be shown easily using the Poincaré–Bendixson theory, which will be developed in Section 4 below. Although the equilibrium $(1,0)$ is attractive, it is also clear that it is not stable, since orbits starting in (x,y) on the unit circle very close to $(1,0)$, but with positive y

take very long to complete the journey round the circle to come close to $(1, 0)$ from below, and for $y \to 0$, this time converges to ∞. Hence $(-1, 0) \in \varphi(t, B_\delta(1, 0))$ for all $\delta > 0$ and t sufficiently large. This means that $(1, 0)$ is unstable.

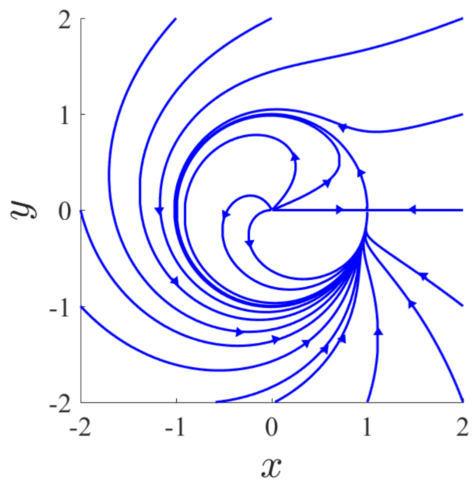


Figure 4.2. The attractive equilibrium $(0, 1)$ is not stable.

In addition to the homoclinic orbit (connecting to $(1, 0)$ in both time directions), there are also so-called heteroclinic orbits in this example (connecting to $(1, 0)$ in forward time and $(0, 0)$ in backward time). In fact, all orbits starting inside the unit circle have this property.

Definition 4.4 (Homoclinic and heteroclinic orbits)**.** *Consider the differential equation* (4.1) *with associated flow* φ*.*

(i) *An orbit* $O(x)$ *for some* $x \in D$ *is called a* homoclinic orbit *if there exists an equilibrium* $x^* \in D \setminus \{x\}$ *such that*

$$\lim_{t\to\infty} \varphi(t, x) = x^* \quad \textit{and} \quad \lim_{t\to-\infty} \varphi(t, x) = x^* .$$

(ii) *An orbit* $O(x)$ *for some* $x \in D$ *is called a* heteroclinic orbit *if there exists two different equilibria* $x_1^* \neq x_2^* \in D$ *such that*

$$\lim_{t\to\infty} \varphi(t, x) = x_1^* \quad \textit{and} \quad \lim_{t\to-\infty} \varphi(t, x) = x_2^* .$$

1.2. Stability of linear systems. Before looking more into nonlinear systems, we study stability of the trivial equilibrium of autonomous linear systems, and we note that in the Example 4.2, we have already explored this in the one-dimensional case. Recall also that in Remark 3.5, we identified that the real part of eigenvalues play a role in the exponential growth behaviour in two-dimensional linear systems, and this was confirmed for higher dimensions in Proposition 3.9.

Theorem 4.5 (Stability of linear systems)**.** *Consider the autonomous linear system*

$$\dot{x} = Ax \,, \tag{4.2}$$

where $A \in \mathbb{R}^{d\times d}$*. Then the trivial equilibrium* $x^* = 0$ *of this system is*

(i) stable *if and only if the following two statements hold:*
  (a) *the real part of all eigenvalues of* A *is non-positive, i.e. we have* $\operatorname{Re}\rho \le 0$ *for all eigenvalues* ρ *of* A*, and*
  (b) *the eigenvalue* ρ *is semi-simple for all eigenvalues* ρ *of* A *with* $\operatorname{Re}\rho = 0$*.*

(ii) exponentially stable *if and only if* $\operatorname{Re}\rho < 0$ *for all eigenvalues* ρ *of* A*.*

Proof. Let J be the real Jordan normal form of the matrix A, i.e. there exists a $T \in \mathbb{R}^{d\times d}$ such that $J = T^{-1}AT$. Using Proposition 3.4 (i), we get

$$\varphi(t,x) = e^{At}x = Te^{Jt}T^{-1}x \tag{4.3}$$

for the flow of (4.2).

(i) ($\Rightarrow$) We show that if either (a) or (b) does not hold, then $x^* = 0$ is not stable. If either (a) or (b) is false, this means that either there exists an eigenvalue ρ with positive real part, or there exists an eigenvalue ρ with real part 0 that is not semi-simple. In both cases, it follows from Proposition 3.8 that $t \mapsto e^{J_j t}$, for a Jordan block corresponding to the eigenvalue ρ is unbounded for $t \in [0,\infty)$. Hence it follows from (4.3) that one element of the matrix e^{At} is unbounded for $t \in [0,\infty)$, say the element in the k-th row and ℓ-th column. Set $\varepsilon := 1$ and choose $\delta > 0$. This implies that

$$\varphi\big(t, \underbrace{\tfrac{\delta}{2}e_\ell}_{\in B_\delta(0)}\big) = \tfrac{\delta}{2}\varphi\big(t,e_\ell\big) = \tfrac{\delta}{2}e^{At}e_\ell \notin B_\varepsilon(0) \quad \text{for some } t \ge 0\,.$$

Hence x^* is not stable.

($\Leftarrow$) It follows from Proposition 3.9 that (a) and (b) imply that there exists a $K > 0$ such that

$$\big\|e^{At}\big\| \le K \quad \text{for all } \ t \ge 0\,.$$

To prove that $x^* = 0$ is stable, choose $\varepsilon > 0$ arbitrarily, and define $\delta := \frac{\varepsilon}{K}$. Then for all $x \in B_\delta(0)$, we get

$$\|\varphi(t,x)\| = \big\|e^{At}x\big\| \le \big\|e^{At}\big\|\|x\| < K\delta = \varepsilon \quad \text{for all } \ t \ge 0\,.$$

Hence, $x^* = 0$ is stable.

(ii) The strategy of the proof is similar to (i) and is left as an exercise. □

It should be noted, although only *stability* and *exponential stability* are described in the above theorem, the results immediately give a characterisation for instability due the equivalence formulation in (i). Moreover, it can be shown easily that attractivity of autonomous linear systems is equivalent to exponential stability, and thus, all stability notions from Definition 4.1 can clearly be understood in the context of linear systems.

1.3. Hyperbolicity. Since we understand the stability of linear systems very well now, we focus our attention on nonlinear systems. As motivated in the beginning of Chapter 3 on linear systems, we aim at using linear systems to understand nonlinear systems locally in the neighbourhood of a reference solution. In the autonomous context, this works well for reference solutions that are constant, i.e. given by equilibria. To motivate the results that will follow, we consider two different two-dimensional linear systems from Section 2 in Chapter 3, to which we add some nonlinear perturbation.

Example 4.6 (Nonlinear perturbations of linear systems)**.** We first consider a perturbation of a saddle equilibrium. Let $A_1 := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_1 \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.4}$$

as well as the nonlinearly perturbed system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_1 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{5}y^2 \\ \frac{3}{10}x^2 + \frac{1}{5}y^2 \end{pmatrix}, \tag{4.5}$$

The phase portraits of both systems a neighbourhood $[-2, 2] \times [-2, 2]$ of the trivial equilibrium $(0, 0)$ are given in Figure 4.3, and we can see that the nonlinear perturbation induces just a slight perturbation of the phase portrait.

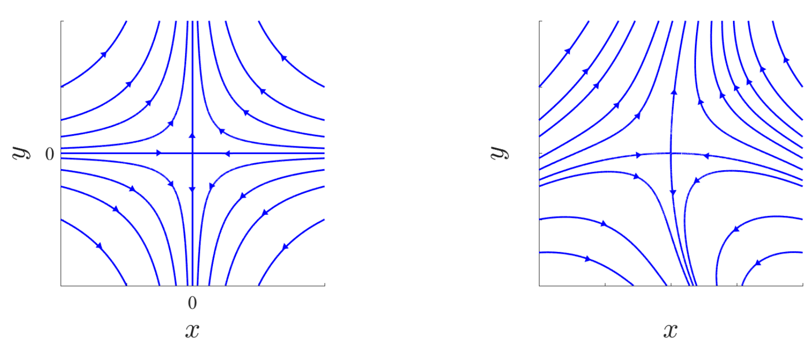


Figure 4.3. Phase portraits of (4.4) (on the left) and (4.5) (on the right).

The situation is much different when we consider a centre equilibrium. Let $A_2 := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_2 \begin{pmatrix} x \\ y \end{pmatrix}, \tag{4.6}$$

as well as the nonlinearly perturbed system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_2 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -5x(x^2+y^2) \\ -5y(x^2+y^2) \end{pmatrix}, \tag{4.7}$$

The phase portraits of both systems in a neighbourhood $[-2,2] \times [-2,2]$ of the trivial equilibrium $(0,0)$ are given in Figure 4.4, and in contrast to the example above, the nonlinearly perturbed system behaves much differently than the linear system, in the sense that all orbits converge forward to in time to the trivial equilibrium $(0,0)$.

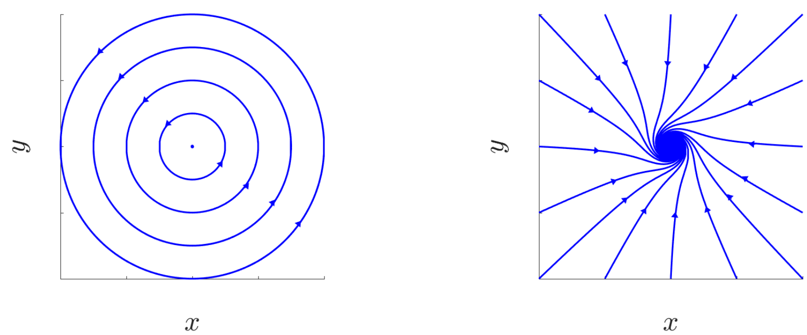


Figure 4.4. Phase portraits of (4.6) (on the left) and (4.7) (on the right).

The reason for this difference that if a linear system has only eigenvalues with nonzero real parts, this implies exponential attractivity or repulsivity in invariant linear subspaces, and this exponential behaviour cannot be destroyed locally by a nonlinear perturbation. This is formulated precisely in the so-called *Hartman–Grobman theorem*, which we will skip due to time constraints. However, the boundary between attractivity and repulsivity is given by an eigenvalue with real part 0, and different nonlinear perturbations can make systems with zero real part eigenvalue attractive or unstable (but not exponentially stable or exponentially unstable).

This observation leads to concept of hyperbolicity.

Definition 4.7 (Hyperbolicity)**.** *A matrix $A \in \mathbb{R}^{d\times d}$ is called* hyperbolic *if all eigenvalues λ of A have non-zero real part, i.e.* $\operatorname{Re}\lambda \neq 0$*. An equilibrium x^* of a differential equation*

$$\dot{x} = f(x),$$

where $f : D \subset \mathbb{R}^d \to \mathbb{R}^d$ is continuously differentiable, is called hyperbolic *if the matrix $f'(x^*) \in \mathbb{R}^{d\times d}$ is hyperbolic.*

We close this section by looking at the non-hyperbolic one-dimensional case.

Example 4.8 (One-dimensional non-hyperbolicity)**.** We consider the linear one-dimensional differential equation

$$\dot{x} = 0\,.$$

Then the trivial equilibrium $x^* = 0$ is stable (see also Example 4.3 (i)), and the derivative $f'(x^*)$ of the right hand side ($f(x) = 0$ for all $x \in \mathbb{R}$) is 0, and thus this equilibrium (as well as all the other equilibria – every point is an equilibrium) is non-hyperbolic. We consider the following nonlinear perturbations and discuss its effect on the stability of $x^* = 0$:

(i) $\dot{x} = x^2$: the equilibrium $x^* = 0$ becomes unstable, although it attracts all points starting in the negative half-line.

(ii) $\dot{x} = -x^2$: the equilibrium $x^* = 0$ becomes unstable, although it attracts all points starting in the positive half-line.

(iii) $\dot{x} = x^3$: the equilibrium $x^* = 0$ becomes unstable, and all points (except x^*) move away from x^* forward in time.

(iv) $\dot{x} = -x^3$: the equilibrium $x^* = 0$ becomes asymptotically stable, but it is not exponentially stable.

1.4. Linearised stability. In this subsection, we consider a hyperbolic equilibrium, for which all eigenvalues of the linearisation have negative real parts. This situation is significantly easier to analyse than the case when there are both positive and negative eigenvalue real parts (and due to time constraints, we will not cover this case in detail, but we give some in insights in the Subsection 1.5 below).

Under the assumption that the real parts of all eigenvalues of the hyperbolic equilibrium are negative, it follows from Theorem 4.5 that the trivial equilibrium of the linearised system is exponentially stable. We show now that this exponential stability is transferred to the nonlinear system. Importantly in the proof, we make use of the following lemma, the Gronwall lemma. Via the Gronwall lemma, we obtain an explicit exponential estimate from an implicit inequality (and such implicit inequalities appear quite frequently in the context of differential equations).

Lemma 4.9 (Gronwall lemma)**.** *We consider a continuous function $u : [a,b] \to \mathbb{R}$ defined on an interval $[a,b]$, and let $c, d \geq 0$. We assume that the function u satisfies the implicit inequality*

$$0 \leq u(t) \leq c + d \int_a^t u(s)\,\mathrm{d}s \quad \textit{for all } t \in [a,b]\,. \tag{4.8}$$

Then we have the explicit estimate

$$u(t) \leq c\,e^{d(t-a)} \quad \textit{for all } t \in [a,b]\,. \tag{4.9}$$

Proof. Because the function u is continuous on the compact set $[a, b]$, it is bounded, i.e. there exists an $M > 0$ such that

$$u(t) \leq M \quad \text{for all } \ t \in [a, b] \,.$$

Using this in (4.8), we get

$$u(t) \leq c + Md(t-a) \quad \text{for all } \ t \in [a, b] \,.$$

Using this improved estimate in (4.8), we arrive at

$$u(t) \leq c + cd(t-a) + \frac{1}{2} M d^2 (t-a)^2 \quad \text{for all } \ t \in [a, b] \,.$$

Inductively, we get after n steps that

$$u(t) \leq \underbrace{c \sum_{k=0}^{n-1} \frac{d^k (t-a)^k}{k!}}_{\to c e^{d(t-a)}} + \underbrace{\frac{M d^n (t-a)^n}{n!}}_{\to 0} \,,$$

which implies the claim in the limit $n \to \infty$. □

We obtain exponential stability for an equilibrium of a nonlinear system, for which the real parts of all eigenvalues of the linearisation are negative. This is referred to as *linearised stablity.*

Theorem 4.10 (Linearised stability)**.** *Let $D \subset \mathbb{R}^d$ be open and $f : D \to \mathbb{R}^d$ be continuously differentiable, and consider the autonomous differential equation*

$$\dot{x} = f(x) \,. \tag{4.10}$$

Assume that x^* is an equilibrium of (4.10) (i.e. $f(x^*) = 0$) such that for all eigenvalues $\lambda \in \mathbb{C}$ of the linearisation $f'(x^*) \in \mathbb{R}^{d \times d}$, we have $\operatorname{Re} \lambda < 0$. Then the equilibrium x^* of (4.10) is exponentially stable.

Proof. To simplify notation in the proof, we assume that the equilibrium is given by $x^* = 0$. To see this, one can make a change of variables using a transformation $y = x - x^*$, which yields a differential equation for y having the zero equilibrium. It should be noted that this transformation (since it is just a translation) does not change any stability properties.

Step 1. Some useful estimates.
We first write

$$\dot{x} = f(x) = \underbrace{f'(0)}_{=:A} x + r(x) \,, \tag{4.11}$$

where the term

$$r(x) := f(x) - f'(0)x = o(\|x\|) \quad \text{for all } \ x \in D$$

is a higher order term, defined on a neighbourhood of 0, and clearly satisfies $r(0) = 0$ and $r'(0) = 0$. We denote the flow of (4.11) by φ.

Since the real parts of the eigenvalues of the matrix A are negative, due to Proposition 3.9, there exist constants $K > 0$ and $\gamma < 0$ such that

$$\left\|e^{At}\right\| \leq K e^{\gamma t} \quad \text{for all } t \geq 0 . \tag{4.12}$$

We choose a positive number $M < -\frac{\gamma}{K}$. Since r is continuously differentiable, there exists $\rho > 0$ such that $\|r'(x)\| \leq M$ for all $x \in \overline{B_\rho(0)}$. Due to the mean value inequality (Theorem 2.8), we get then get

$$\|r(x)\| \leq M\|x\| \quad \text{for all } x \in \overline{B_\rho(0)} . \tag{4.13}$$

Finally, we define for each initial value $x \in B_\rho(0)$ the *escape time*

$$T_e(x) := \sup\left\{T > 0 : \|\varphi(t,x)\| \leq \rho \text{ for all } t \in [0,T)\right\} .$$

Note that $T_e(x)$ can be ∞, which is desirable case, and, as we will see later, this is true for small enough x.

Step 2. We show that for all $x_0 \in B_\rho(0)$, we have

$$\|\varphi(t,x_0)\| \leq K e^{(KM+\gamma)t}\|x_0\| \quad \textit{for all } t \in [0, T_e(x_0)) . \tag{4.14}$$

We note that the solution $t \mapsto \varphi(t,x_0)$ of the differential equation (4.10) is also a solution to the nonautonomous linear differential equation

$$\dot{x} = Ax + r(\varphi(t,x_0)) ,$$

for which the variation of constants formula can be applied (see Proposition 3.10), and we obtain

$$\varphi(t,x_0) = e^{At}x_0 + \int_0^t e^{A(t-s)} r(\varphi(s,x_0))\,\mathrm{d}s .$$

Hence, for all $t \in [0, T_e(x_0))$, we have

$$\begin{aligned}
\|\varphi(t,x_0)\| &\leq \left\|e^{At}\right\| \cdot \|x_0\| + \int_0^t \left\|e^{A(t-s)}\right\| \cdot \|r(\varphi(s,x_0))\|\,\mathrm{d}s \\
&\overset{(4.12)}{\leq} K e^{\gamma t}\|x_0\| + \int_0^t K e^{\gamma(t-s)} \|r(\varphi(s,x_0))\|\,\mathrm{d}s \\
&\overset{(4.13)}{\leq} K e^{\gamma t}\|x_0\| + \int_0^t K e^{\gamma(t-s)} M \|\varphi(s,x_0)\|\,\mathrm{d}s .
\end{aligned}$$

We multiply this inequality with $e^{-\gamma t}$ and obtain the implicit estimate

$$u(t) := e^{-\gamma t}\|\varphi(t,x_0)\| \leq K\|x_0\| + KM \int_0^t \underbrace{e^{-\gamma s}\|\varphi(s,x_0)\|}_{u(s)}\,\mathrm{d}s .$$

We use the Gronwall lemma (Lemma 4.9) for the above defined function u, and get the explicit estimate

$$e^{-\gamma t}\|\varphi(t,x_0)\| \leq K\|x_0\| e^{KMt} \quad \text{for all } t \in [0, T_e(x_0)) ,$$

which finishes the proof of this step.

Step 3. Finalisation of the proof.
From the fact that $t \mapsto e^{(KM+\gamma)t}$ is monotonically decreasing, we get

$$\begin{aligned}\|\varphi(t,x_0)\| &\overset{(4.14)}{\leq} Ke^{(KM+\gamma)t}\|x_0\| \\ &\overset{t=0}{\leq} K\|x_0\| \leq \rho \quad \text{for all } t \geq 0 \text{ and } x_0 \in B_{\rho/K}(0)\,,\end{aligned}$$

and it follows that $T_e(x_0) = \infty$. Hence, (4.14) implies that (4.11) is exponentially stable, since $KM + \gamma < 0$. □

Question: Does this result imply anything for hyperbolic equilibria, for which the eigenvalues of the linearisation have only positive real parts? Yes, a time-reversed version of Theorem 4.10 implies that such equilibria are exponentially attractive backward in time, which means that they are repulsive.

A more complicated situation is given when there eigenvalues with both positive and negative real parts, and we will discuss this in the next subsection, but firstly, we now show that this result can be applied to the pendulum system, which turns out to have an exponentially stable equilibrium whenever friction is taken into account.

Example 4.11 (Pendulum, exponentially stable equilibrium)**.** Consider a pendulum moving along a circle of radius $r > 0$, with a mass $m > 0$ and friction coefficient $k > 0$. Let x denote the angle from the vertical. The force tangential to the circle depends on both the position x and the speed $\dot{x}$ of the pendulum, and is given by

$$F_{\tan}(x,\dot{x}) = -mg\sin(x) - kr\dot{x}\,,$$

see Figure 4.5.

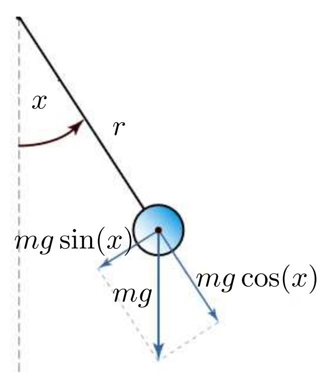


Figure 4.5. The pendulum.

Newton's law reads as $mr\ddot{x} = F_{\tan}(x, \dot{x})$, and we thus get the second-order one-dimensional differential equation

$$\ddot{x} = -\frac{g}{r}\sin(x) - \frac{k}{m}\dot{x}\,,$$

which we can transform into the first-order two-dimensional system

$$\begin{aligned}\dot{x} &= y\,,\\ \dot{y} &= -\frac{g}{r}\sin(x) - \frac{k}{m}y\,.\end{aligned}$$

(see *Repetition Material 1*). The equilibria of this system are given by $(n\pi, 0)$, where $n \in \mathbb{Z}$, which corresponds to the pendulum being in vertical position (pointing down for even n and pointing up for odd n. We linearise this system in $(0, 0)$ (or equivalently in $(n\pi, 0)$ for even n), and obtain the linearisation

$$\begin{pmatrix} 0 & 1 \\ -\frac{g}{r} & -\frac{k}{m} \end{pmatrix},$$

which gives two eigenvalues $\lambda_\pm := \frac{1}{2}\Big(-\frac{k}{m} \pm \sqrt{\big(\frac{k}{m}\big)^2 - 4\frac{g}{r}}\Big)$. It follows that the real parts of both eigenvalues $\lambda_\pm$ are negative, and hence, Theorem 4.10 implies that the equilibria $(n\pi, 0)$ for n even are exponentially stable. Note that if $(\frac{k}{m})^2 - 4\frac{g}{r} < 0$ then both eigenvalues are complex, and if $(\frac{k}{m})^2 - 4\frac{g}{r} \geq 0$, then both eigenvalues are real and negative. The stability of the equilibria $(n\pi, 0)$ for n odd will be discussed below in Example 4.16.

1.5. Stable and unstable sets, invariant sets. In the previous subsection, we have looked at the case where all eigenvalues of the linearisation in an equilibrium are negative, which implies exponential stability of the equilibrium. We now look at situations where some eigenvalues have positive and some eigenvalues have negative real parts.

Definition 4.12 (Stable and unstable set)**.** *Consider the differential equation* (4.1) *with associated flow* φ*, and let* x^* *be an equilibrium. We define the* stable set *of* x^* *as*

$$W^s(x^*) := \Big\{x \in D : \lim_{t\to\infty} \varphi(t, x) = x^*\Big\}\,,$$

and the unstable set *of* x^* *is defined as*

$$W^u(x^*) := \Big\{x \in D : \lim_{t\to-\infty} \varphi(t, x) = x^*\Big\}\,,$$

Note that if x^* is an attractive equilibrium, then $W^s(x^*)$ is called *domain of attraction*, and it follows from the definition of attractivity that this is a neighbourhood of x^*. Moreover, $W^s(x^*)$ is an open set in this case (see exercises).

Example 4.13 (Stable and unstable set of a linear system)**.** Given a matrix $A \in \mathbb{R}^{d\times d}$, we consider the linear system

$$\dot{x} = Ax\,,$$

which we assume to be hyperbolic, i.e there are no eigenvalues with zero real part (which is equivalent to $\{s_1, \dots, s_q\} \cap \{0\} = \emptyset$, where $\Sigma(A) = \{s_1, \dots, s_q\}$). It was mentioned in Section 5 of Chapter 3 that one can show that

$$\mathbb{R}^d = E_1 \oplus \dots \oplus E_q\,,$$

with Lyapunov exponents $s_1 < \dots < s_q$ associated to these spaces. Due to hyperbolicity, there exists an $k \in \{1, \dots, q+1\}$ such that $s_\ell < 0$ for all $\ell < k$ and $s_\ell > 0$ for all $\ell \geq k$. One can show that

$$W^s(0) = \bigoplus_{i=1}^{k-1} E_i \quad \text{and} \quad W^u(0) = \bigoplus_{i=k}^{q} E_i\,,$$

so that the above decomposition can be rewritten as

$$\mathbb{R}^d = W^s(0) \oplus W^u(0)\,.$$

We now look at nonlinear perturbations of these objects in a two-dimensional example.

Example 4.14 (Stable and unstable set of a hyperbolic equilibrium)**.** Recall the first two systems discussed in Example 4.6. With $A_1 := \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, we consider the linear system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_1 \begin{pmatrix} x \\ y \end{pmatrix}\,, \tag{4.15}$$

as well as the nonlinearly perturbed system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_1 \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{5}y^2 \\ \frac{3}{10}x^2 + \frac{1}{5}y^2 \end{pmatrix}\,, \tag{4.16}$$

From Example 4.13, it follows for the linear example (4.15) that $\mathbb{R}^2 = E_1 \oplus E_2$ with $W^s(0) = E_1$ being the x-axis and $W^u(0) = E_2$ being the y-axis, which are both one-dimensional objects. If we look at the phase portrait of the nonlinearly perturbed system (4.16), given in Figure 4.3, we notice that this one-dimensional object survives – both the stable and unstable set seem to be given by the union of two trajectories and the equilibrium point, and the trajectories are tangential to the linear spaces E_1 and E_2 in the equilibrium 0 respectively.

We note that stable and unstable sets in these two examples are also called stable and unstable manifolds of the equilibrium 0. We do not give a proper definition of manifolds here, and only note that manifolds are objects that locally look like Euclidean spaces in each point.

Stable and unstable manifolds are special cases of so-called *invariant* manifolds, and the notion of an invariant set is very important in the theory of differential equations. We require from invariant sets that orbits starting in it do not leave the set. Recall the definition of orbits $O(x)$ and half-orbits $O^+(x)$, $O^-(x)$ from Definition 2.25.

Definition 4.15 (Invariance)**.** *Consider the differential equation* (4.1)*. Then a set $M \subset D$ is called*

(i) positively invariant *if for all $x \in M$, we have $O^+(x) \subset M$,*

(ii) negatively invariant *if for all $x \in M$, we have $O^-(x) \subset M$,*

(iii) invariant *if for all $x \in M$, we have $O(x) \subset M$.*

Note that sets that consist of equilibria or periodic orbits are invariant. Stable and unstable sets are also invariant, and any union of orbits is invariant. Accordingly, unions of half-orbits of the form $O^+(x)$ or $O^-(x)$ are positively invariant or negatively invariant, respectively.

We identify stable and unstable sets for the pendulum.

Example 4.16 (Pendulum, saddle equilibrium)**.** We consider again the pendulum from Example 4.11, given by the one-dimensional system of order two

$$\ddot{x} = -\frac{g}{r}\sin(x) - \frac{k}{m}\dot{x}\,,$$

which we can transform due to Extra Material 1 into the first-order two-dimensional system

$$\dot{x} = y\,, \qquad \dot{y} = -\frac{g}{r}\sin(x) - \frac{k}{m}y$$

We have seen that the equilibria of this system are given by $(n\pi, 0)$, where $n \in \mathbb{Z}$, and the analysis in Example 4.11 showed that the equilibria $(n\pi, 0)$ for even $n \in \mathbb{Z}$ are exponentially stable. We analyse now the equilibria $(n\pi, 0)$ for odd $n \in \mathbb{Z}$ and obtain the the linearisation

$$\begin{pmatrix} 0 & 1 \\ \frac{g}{r} & -\frac{k}{m} \end{pmatrix},$$

which gives two eigenvalues $\lambda_\pm := \frac{1}{2}\Big(-\frac{k}{m} \pm \sqrt{\big(\frac{k}{m}\big)^2 + 4\frac{g}{r}}\Big)$. It follows that we have $\lambda_- < 0 < \lambda_+$, and one can see in the phase portrait in Figure 4.6 that the equilibria $(n\pi, 0)$ for n odd have stable and unstable sets/manifolds that are given by one-dimensional curves.

Question: Can you give a physical interpretation of these two invariant sets/manifolds?

We note that under the assumption that the right hand side of a differential equation is continuously differentiable, the so-called stable and unstable

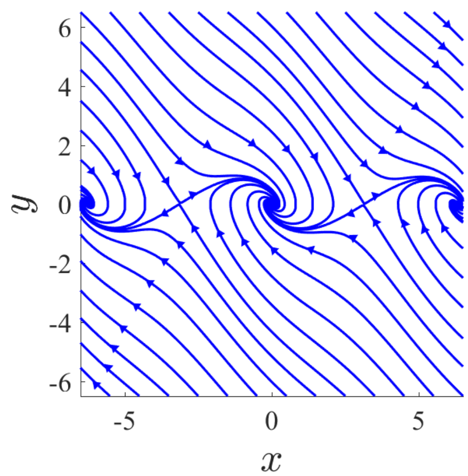


Figure 4.6. Phase portrait of the pendulum.

manifold theorem says that for a hyperbolic equilibrium in dimension two with one negative and one positive eigenvalue, the stable and unstable sets are given by one-dimensional curves that intersect in the equilibrium and are tangential to the eigenspaces of the linearisation. A similar statement holds also in higher dimensions, where the curves need to be replaced by higher-dimensional manifolds. We do not attempt to formulate the stable and unstable manifold theorem here, but note that an application of this theorem yields that instability of the linearisation (in form of an eigenvalue with positive real part) carries over to the equilibrium of the nonlinear system.

2. Limit sets

The asymptotic behaviour of differential equations (that is the limiting behaviour for $t \to \infty$ and $t \to -\infty$) is determined by certain types of invariant set, so-called limit sets. We study this for an autonomous differential equation of the form

$$\dot{x} = f(x)\,, \tag{4.17}$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous and $D \subset \mathbb{R}^d$ is an open set. We denote the flow of this differential equation by φ.

We now introduce two important classes of invariant sets, so-called omega and alpha limits sets. Their importance is due to the fact that they describe the asymptotic behaviour. Note that invariance is not part of the following definition, but it will follow later from it.

Definition 4.17 (Omega and alpha limit sets)**.** *Consider the flow φ of the differential equation* (4.17)*, and let $x \in D$.*

(i) *A point $x_\omega \in D$ is called* omega limit point *of x, if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that* $\lim_{n\to\infty} t_n = \infty$ *and*

$$x_\omega = \lim_{n\to\infty} \varphi(t_n, x)\,.$$

We denote by $\omega(x)$ the set of all omega limit points of x. $\omega(x)$ is called omega limit set *of x.*

(ii) *A point $x_\alpha \in D$ is called* alpha limit point *of x, if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ such that* $\lim_{n\to\infty} t_n = -\infty$ *and*

$$x_\alpha = \lim_{n\to\infty} \varphi(t_n, x)\,.$$

We denote by $\alpha(x)$ the set of all alpha limit points of x. $\alpha(x)$ is called alpha limit set *of x.*

Note that the omega limit set of a point x is empty if $\sup J_{max}(x) < \infty$, and the alpha limit set to be nonempty requires $\inf J_{max}(x) = -\infty$.

We look at omega limit sets for the differential equation from Example 2.26.

Example 4.18. Consider the autonomous two-dimensional differential equation

$$\begin{aligned} \dot{x} &= y + x(1 - x^2 - y^2)\,, \\ \dot{y} &= -x + y(1 - x^2 - y^2)\,. \end{aligned} \tag{4.18}$$

We have seen already in Example 2.26 that this differential equation has the trivial equilibrium $(0, 0)$, which is unstable, and there exists a periodic orbit, given by the unit circle $\mathbb{S}^1$. All orbits of this system that do not start in the unstable trivial equilibrium approach this periodic orbit in forward time. It is possible to show that

$$\omega\big((x, y)\big) = \begin{cases} \{(0,0)\} & : \quad (x, y) = (0, 0)\,, \\ \mathbb{S}^1 & : \quad (x, y) \neq (0, 0)\,, \end{cases}$$

and

$$\alpha\big((x, y)\big) = \begin{cases} \{(0,0)\} & : \quad \|(x, y)\| < 1\,, \\ \mathbb{S}^1 & : \quad \|(x, y)\| = 1\,, \\ \emptyset & : \quad \|(x, y)\| > 1\,. \end{cases}$$

Question: Can you establish this rigorously? You can use the explicit representation of the flow from Example 2.26. Proving this will become easier when we have established the Poincaré–Bendixson theorem, see Section 4 below.

We now derive an alternative characterisation of omega and alpha limit sets.

Proposition 4.19 (Alternative characterisation of limit sets)**.** *Consider the flow φ of the differential equation* (4.17)*, and let $x \in D$. Then we have*

$$\omega(x) = \bigcap_{t\geq 0} \overline{O^+\big(\varphi(t,x)\big)} \quad \textit{and} \quad \alpha(x) = \bigcap_{t\leq 0} \overline{O^-\big(\varphi(t,x)\big)}\,. \tag{4.19}$$

Before proving this proposition, we would like to understand its content better using the following example.

Example 4.20. We again consider the differential equation (4.18). As described above, for any point $(x,y) \neq 0$, the half-orbit $O^+\big((x,y)\big)$ approaches the periodic orbit $\mathbb{S}^1$, see Figure 4.7, so we have that

$$\overline{O^+\big((x,y)\big)} = O^+\big((x,y)\big) \cup \mathbb{S}^1\,.$$

Hence,

$$\begin{aligned}\bigcap_{t\geq 0} \overline{O^+\big(\varphi(t,(x,y))\big)} &= \bigcap_{t\geq 0} \Big(O^+\big(\varphi(t,(x,y))\big) \cup \mathbb{S}^1\Big)\\ &= \mathbb{S}^1 \cup \bigcap_{t\geq 0} O^+\big(\varphi(t,(x,y))\big) = \mathbb{S}^1\,.\end{aligned}$$

This follows, since $\bigcap_{t\geq 0} O^+\big(\varphi(t,(x,y))\big)$ is either $\emptyset$ (if $\|(x,y)\| \neq 1$) or $\mathbb{S}^1$ (if $\|(x,y)\| = 1$).

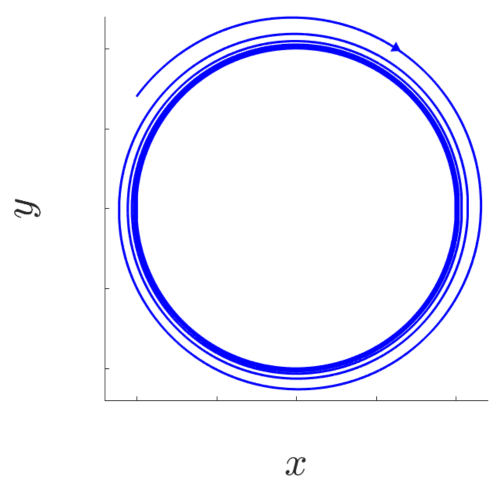


Figure 4.7. Explanation of Proposition 4.19 using a half-orbit approaching a periodic orbit.

Proof of Proposition 4.19. We will only prove $\omega(x) = \bigcap_{t\geq 0} \overline{O^+\big(\varphi(t,x)\big)}$, since the statement concerning alpha limit sets can be shown similarly.

($\subset$). Choose $y \in \omega(x)$. Then there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n \to \infty$ and $y = \lim_{n\to\infty} \varphi(t_n, x)$. This implies that for any $t \geq 0$, we have

$$y \in \overline{O^+\big(\varphi(t,x)\big)}\,,$$

since there exists an $n_0 = n_0(t)$ such that $\varphi(t_n, x) \in O^+\big(\varphi(t,x)\big)$ for all $n \geq n_0$. Hence $y \in \bigcap_{t\geq 0} \overline{O^+\big(\varphi(t,x)\big)}$.

($\supset$). Choose $y \in \bigcap_{t\geq 0} \overline{O^+\big(\varphi(t,x)\big)}$. This implies that $y \in \overline{O^+\big(\varphi(t,x)\big)}$ for all $t \geq 0$, and in particular this means that

$$B_{1/n}(y) \cap O^+\big(\varphi(n,x)\big) \neq \emptyset \quad \text{for all } n \in \mathbb{N}\,.$$

Hence, for all $n \in \mathbb{N}$, there exists $t_n \geq n$ such that $\varphi(t_n, x) \in B_{1/n}(y)$. This implies that $\lim_{n\to\infty} t_n \to \infty$ and $y = \lim_{n\to\infty} \varphi(t_n, x)$, and the proof is finished, since $y \in \omega(x)$. □

Omega and alpha limit sets have the following important properties.

Proposition 4.21 (Properties of omega and alpha limit sets)**.** *Consider the differential equation* (4.17)*, and let $x \in D$. Then the following statements hold.*

(i) *The omega limit set $\omega(x)$ is invariant. In addition, if $O^+(x)$ is bounded and $\overline{O^+(x)} \subset D$, then $\omega(x)$ is non-empty and compact.*

(ii) *The alpha limit set $\alpha(x)$ is invariant. In addition, if $O^-(x)$ is bounded and $\overline{O^-(x)} \subset D$, then $\alpha(x)$ is non-empty and compact.*

Proof. We only prove (i), since (ii) can be shown similarly, and we denote the flow of (4.17) by φ.

Step 1. $\omega(x)$ is nonempty.
The sequence $\big\{\varphi(n,x)\big\}_{n\in\mathbb{N}}$ is bounded and thus has a convergent subsequence $\big\{\varphi(n_k,x)\big\}_{k\in\mathbb{N}}$ with limit in D. The limit of this subsequence is an omega limit point of x, and thus, $\omega(x)$ is nonempty.

Step 2. $\omega(x)$ is compact.
This follows directly from (4.19), since each of the sets $\overline{O^+\big(\varphi(t,x)\big)} \subset O^+(x)$ are compact, since they are bounded and closed, and the intersection $\omega(x)$ over these sets is also bounded and closed, and thus compact.

Step 3. $\omega(x)$ is invariant.
To show that $\omega(x)$ is invariant, we need to show that for all $x_0 \in \omega(x)$, we have $O(x_0) \subset \omega(x)$. Choose $\tau \in J_{max}(x_0)$. Since $x_0 \in \omega(x)$, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ and

$$x_0 = \lim_{n\to\infty} \varphi(t_n, x)\,. \tag{4.20}$$

Now consider the sequence $s_n := t_n + \tau$, where $n \in \mathbb{N}$, which also converges to ∞ as $n \to \infty$. Note that the flow φ is a continuous function (see Remark 2.28). This implies that

$$\begin{aligned}\lim_{n\to\infty} \varphi(s_n, x) &= \lim_{n\to\infty} \varphi(t_n + \tau, x) = \lim_{n\to\infty} \varphi(\tau, \varphi(t_n, x)) \\ &= \varphi\Big(\tau, \lim_{n\to\infty} \varphi(t_n, x)\Big) \overset{(4.20)}{=} \varphi(\tau, x_0)\,,\end{aligned}$$

and this means that $\varphi(\tau, x_0) \in \omega(x)$, which finishes the proof of this proposition. □

3. Lyapunov functions

We have seen in Subsection 1.4 and Subsection 1.5 that stability can be deduced from the linearisation around hyperbolic equilibria. In particular, if all eigenvalues of the linearisation $f'(x^*)$ around an equilibrium x^* of a differential equation have negative real parts, then x^* is asymptotically stable (or even exponentially stable), and it follows that the domain of attraction $W^s(x^*)$ contains a neighbourhood of the equilibrium x^*.

In that sense, the method of linearisation provides *local* information, but often it is useful to know more about *global* properties of the domain of attraction $W^s(x^*)$. In addition, sometimes the method of linearisation can not be used to determine stability of nonlinear systems, and this is the case when the equilibrium is non-hyperbolic. It turns out that so-called *Lyapunov functions* can be of help with regard to these restrictions to the methods discussed so far. They are useful tools to prove stability (or instability) of (not necessarily hyperbolic) equilibria and to determine their basin of attraction.

In this section, we consider autonomous differential equations of the form

$$\dot{x} = f(x)\,, \tag{4.21}$$

where $f : D \to \mathbb{R}^d$ is locally Lipschitz continuous and $D \subset \mathbb{R}^d$ is an open set. We denote the flow of this differential equation by φ.

Lyapunov functions are real-valued functions $V : D \to \mathbb{R}$ and can be thought of as energy functions. Their main property is that they decrease along solutions of (4.21), for instance in systems with friction where energy is lost. To model this, we will define the notion of an orbital derivative, i.e. the derivative of V along solutions.

Definition 4.22 (Orbital derivative)**.** *Consider the differential equation* (4.21)*, and let $V : D \to \mathbb{R}$ be a continuously differentiable function. Then* *the orbital derivative* $\dot{V}$ *of the function* V *is defined by*

$$\dot{V}(x) := V'(x) \cdot f(x) = \sum_{i=1}^{d} \frac{\partial V}{\partial x_i}(x) f_i(x)\,.$$

Here, the row vector $V'(x) \in \mathbb{R}^{1\times d}$ *is the gradient of* V *at* $x \in D$*.*

Note that while the gradient V' does not depend on (4.21), the orbital derivative $\dot{V}$ does. In fact, $\dot{V}$ describes the derivative of V along solutions $\mu : I \to D$ of (4.21). This follows from

$$\frac{\mathrm{d}}{\mathrm{d}t} V(\mu(t)) = V'(\mu(t)) \cdot \dot{\mu}(t) = \dot{V}(\mu(t)) \quad \text{for all } \ t \in I\,, \tag{4.22}$$

where we have used the chain rule.

We consider the energy of the pendulum and study its orbital derivative.

Example 4.23 (Pendulum, orbital derivative of energy function)**.** We consider the pendulum

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\frac{g}{r}\sin(x) - \frac{k}{m} y\,. \end{aligned}$$

which we have studied first in Example 4.11 and then in Example 4.16. Its kinetic energy is given by $\frac{1}{2}m(ry)^2$, and its potential energy is given by $mg(r - r\cos(x))$. The sum of these two parts constitutes the function

$$V(x, y) := \frac{1}{2}m(ry)^2 + mgr\big(1 - \cos(x)\big)\,.$$

V is equal to 0 in the asymptotically stable equilibria $(n\pi, 0)$ for even n, and V is positive outside of these equilibria. We compute the orbital derivative of V, given by

$$\dot{V}(x, y) = (mgr\sin(x), mr^2 y) \begin{pmatrix} y \\ -\frac{g}{r}\sin(x) - \frac{k}{m}y \end{pmatrix} = -kr^2y^2\,.$$

We considered so far only the case with positive friction $k > 0$, and in this case $V(x, y) < 0$ whenever $y \neq 0$. Interesting is also the case without friction ($k = 0$), since in this case, $\dot{V}(x, y) = 0$. This immediately implies that solutions stay on the level sets of V, see Figure 4.8 for the phase portrait.

We concentrate only on functions V that do not increase along solutions, which are so-called Lyapunov functions.

Definition 4.24 (Lyapunov function)**.** *Consider the differential equation* (4.21)*, and let* $V : D \to \mathbb{R}$ *be a continuously differentiable function. Then* V *is called a* Lyapunov function *if*

$$\dot{V}(x) \leq 0 \quad \textit{for all } \ x \in D\,.$$

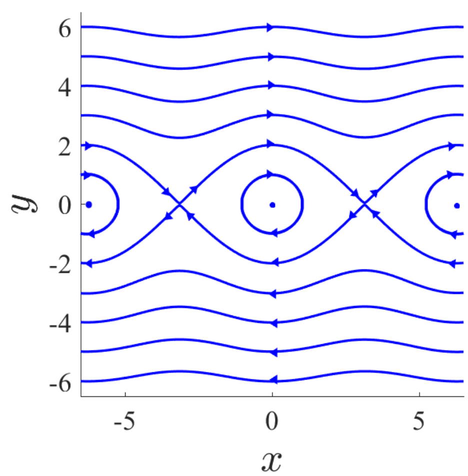


Figure 4.8. Phase portrait of the pendulum without friction.

Note that any Lyapunov function decreases along solutions, i.e.

$$V(\varphi(t,x)) \leq V(x) \quad \text{for all } \ t \in [0, \sup J_{max}(x))\,, \tag{4.23}$$

which can be seen as follows. Integrating (4.22) implies that for all $t \in [0, \sup J_{max}(x))$, we have

$$V(\varphi(t,x)) - V(\underbrace{\varphi(0,x)}_{=x}) = \int_0^t \underbrace{\dot{V}(\varphi(s,x))}_{\leq 0}\, \mathrm{d}s \leq 0\,,$$

which shows (4.23). This property implies immediately that sublevel sets of Lyapunov functions are positively invariant.

Proposition 4.25 (Sublevel sets of Lyapunov functions are positively invariant)**.** *Consider the differential equation* (4.21) *with a Lyapunov function* $V : D \to \mathbb{R}$*. Then any sublevel set of the form*

$$S_c := \big\{x \in D : V(x) \leq c\big\}\,,$$

where $c \in \mathbb{R}$*, is positively invariant.*

Proof. Assume that S_c is not positively invariant. Then there exists an $x \in S_c$ and $t > 0$ such that $\varphi(t,x) \notin S_c$. Since $x \in S_c$ implies $V(x) \leq c$, and $\varphi(t,x) \notin S_c$ implies that $V(\varphi(t,x)) > c$, this contradicts (4.23) and finishes the proof. $\square$

The following theorem says that if a Lyapunov function has a strict local minimum in an equilibrium, then the equilibrium is stable. The result is called a *direct method*, since it can get information on the stabilty behaviour of solutions directly from the right hand side of the differential equation, and there is no need to solve the differential equation.

Theorem 4.26 (Lyapunov's direct method for stability)**.** *Consider the differential equation* (4.21) *with an equilibrium* $x^* \in D$*, and let* $V : D \to \mathbb{R}$ *be a Lyapunov function such that*

$$V(x^*) = 0 \quad \text{and} \quad V(x) > 0 \quad \text{for all } x \in D \setminus \{x^*\}\,.$$

Then the equilibrium x^* *is stable.*

Proof. To prove stability of x^*, we choose $\varepsilon > 0$. Since D is open, there exists an $\tilde\varepsilon \in (0, \varepsilon]$ such that $\overline{B_{\tilde\varepsilon}(x^*)} \subset D$, and we define

$$m := \min\big\{V(x) : \|x - x^*\| = \tilde\varepsilon\big\} > 0\,,$$

which is positive due to $V(x) > 0$ outside of $x = x^*$. Since V is a continuous function, there exists a $\delta = \delta(\tilde\varepsilon) \in (0, \tilde\varepsilon)$ such that

$$0 \le V(x) \le \frac{m}{2} \quad \text{for all } x \in B_\delta(x^*)\,.$$

This implies with (4.23) that

$$V(\varphi(t,x)) \le V(x) \le \frac{m}{2} \quad \text{for all } x \in B_\delta(x^*) \text{ and } t \ge 0\,.$$

The orbit starting in $x \in B_\delta(x^*)$ can thus not leave the $\tilde\varepsilon$-neighbourhood of x^*, since the V is at least m on the boundary of this neighbourhood. Since $\varepsilon \ge \tilde\varepsilon$, this implies that x^* is stable. □

Proving stability of an equilibrium is often checked by looking at the linearisation, and exponential stability follows if the real parts of the eigenvalues of the linearisation are negative (see Theorem 4.10). The above theorem using Lyapunov functions can be helpful if the linearisation is non-hyperbolic and thus not amenable to an analysis using Theorem 4.10.

Example 4.27 (Application of Lyapunov's direct method for stability)**.** We consider the two-dimensional differential equation

$$\begin{aligned} \dot x &= -y - xy^2\,, \\ \dot y &= x - yx^2\,. \end{aligned}$$

The only equilibrium of this system is the trivial equilibrium $(0, 0)$. Indeed, if $\dot x = 0$, then either $y = 0$ or $1 + xy = 0$, but if $y = 0$, then $x(1 - xy) = 0$ implies $x = 0$, and if $1 + xy = 0$, then $0 = x(1 - xy) = 2x$ implies $x = 0$ which contradicts $1 + xy = 0$. The linearisation in $(0, 0)$ is given by $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and thus the system is non-hyperbolic and stability cannot be deduced from Theorem 4.10.

We first show that the trivial equilibrium is stable by considering the quadratic function $V(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$. Then

$$\dot{V}(x,y) = (2x, 2y)\begin{pmatrix} -y - xy^2 \\ x - yx^2 \end{pmatrix} = -4x^2y^2 \leq 0$$

so $(0,0)$ is stable. We will see later in Example 4.32 that $(0,0)$ is even asymptotically stable.

We now show that the existence of a Lyapunov function gives information about the location of omega limit sets.

Theorem 4.28 (La Salle's invariance principle)**.** *Consider the differential equation* (4.21) *with a Lyapunov function* $V : D \to \mathbb{R}$*. Then*

$$\omega(x) \subset \big\{y \in D : \dot{V}(y) = 0\big\} \quad \textit{for all } x \in D\,.$$

Proof. Assume to the contrary that

$$z \in \omega(x) \quad \text{and} \quad \dot{V}(z) < 0\,.$$

Then for some $\tau > 0$, we have $V(\varphi(\tau, z)) < V(z)$. Since $z \in \omega(x)$, there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} t_n = \infty$ such that

$$\lim_{n\to\infty} \varphi(t_n, x) = z\,.$$

It is clear that we can choose the sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $t_{n+1} - t_n > \tau$ for all $n \in \mathbb{N}$. Due to $\dot{V}(y) \leq 0$ for all $y \in D$, this implies that

$$V(\varphi(t_{n+1}, x)) \leq V(\varphi(\tau + t_n, x)) = V(\varphi(\tau, \varphi(t_n, x))) \quad \text{for all } n \in \mathbb{N}\,.$$

Taking the limit $n \to \infty$ on both sides of this inequality gives

$$V(z) \leq V(\varphi(\tau, z))\,,$$

which contradicts $V(\varphi(\tau, z)) < V(z)$. Note that here we have used that the flow φ is continuous in x (see also Remark 2.28) □

We use La Salle's principle to understand the asymptotic behaviour of the pendulum better.

Example 4.29 (Pendulum, all orbits converge to equilibria)**.** For a positive friction coefficient $k > 0$, we consider the pendulum

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\frac{g}{r}\sin(x) - \frac{k}{m}y\,. \end{aligned}$$

which we have studied several times (in Examples 4.11, 4.16 and 4.23). In particular, we have analysed the stability of all equilibria $(n\pi, 0)$, where $n \in \mathbb{Z}$, and we proved that the equilibria with n even are exponentially stable and the equilibria with n odd are unstable. We also found the Lyapunov function

$$V(x,y) := \frac{1}{2}m(ry)^2 + mgr\big(1-\cos(x)\big)\,,$$

and we showed that

$$\dot{V}(x,y) = -kr^2y^2\,.$$

Hence, $\dot{V}(x,y) = 0$ if and only if $y = 0$. Then La Salle's principle implies that

$$\omega((x,y)) \subset \mathbb{R}\times\{0\} \quad \text{for all } \ (x,y)\in\mathbb{R}^2\,. \tag{4.24}$$

It is possible to show, but this is left as a challenging exercise, that $O^+((x,y))$ is bounded for all $(x,y)\in\mathbb{R}^2$. Given this boundedness, Proposition 4.21 implies that $\omega(x)$ is nonempty. Proposition 4.21 also implies that $\omega((x,y))$ is invariant. Suppose now that $\omega((x,y))$ contains a point that is not an equilibrium. Then (4.24) implies that there exists a $(\bar{x},0)\in\omega((x,y))$ with $\bar{x}\neq n\pi$ for $n\in\mathbb{Z}$. At this point $(\bar{x},0)$, we thus have $\dot{y} = -\frac{g}{r}\sin(\bar{x})\neq 0$. This implies that the the flow starting in $(\bar{x},0)$ will leave the x-axis $\mathbb{R}\times\{0\}$ immediately (both forward and backward in time). Since the omega limit set $\omega((x,y))$ is invariant and we have (4.24), this cannot happen, and this implies that such a point $(\bar{x},0)$ is not part of the the omega limit set $\omega((x,y))$, so

$$\omega((x,y)) \subset \big\{(n\pi,0) : n\in\mathbb{Z}\big\} \quad \text{for all } \ (x,y)\in\mathbb{R}^2\,.$$

Since $\omega((x,y))$ is connected (see exercises), this implies that $\omega((x,y))$ is a singleton. Since it is possible to show that when $\omega((x,y))$ is a singleton, then the flow starting in any $(x,y)\in\mathbb{R}^2$ converges in forward time to an equilibrium (do this as an exercise, or look into the proof of Theorem 4.31 below).

These observations lead to a better formulation of La Salle's principle.

Corollary 4.30 (Reformulation of La Salle's invariance principle)**.** *Consider the differential equation* (4.21) *with a Lyapunov function* $V : D\to\mathbb{R}$*. Then for any* $x\in D$*, the omega limit set* $\omega(x)$ *is contained in the largest invariant subset of* $\big\{y\in D : \dot{V}(y) = 0\big\}$*. Here the largest invariant subset is given by the union of invariant subsets of* $\big\{y\in D : \dot{V}(y) = 0\big\}$

Proof. Let M be the largest invariant subset of $\big\{y\in D : \dot{V}(y) = 0\big\}$. Assume that for some $x\in D$, we have $\omega(x)\setminus M\neq\emptyset$. Since $\omega(x)$ is invariant due to Proposition 4.21, the set $\omega(x)\cup M$ is invariant. Because of $\omega(x)\cup M\supsetneq M$, this contradicts the maximality of M. □

We apply La Salle's principle to obtain asymptotic stability via Lyapunov functions.

Theorem 4.31 (Lyapunov's direct method for asymptotic stability)**.** *Consider the differential equation* (4.21) *with an equilibrium $x^* \in D$, and let $V : D \to \mathbb{R}$ be a Lyapunov function such that*

$$\begin{aligned} V(x^*) = 0 \quad &\textit{and} \quad V(x) > 0 \quad \textit{for all } x \in D \setminus \{x^*\}\,, \\ \dot V(x^*) = 0 \quad &\textit{and} \quad \dot V(x) < 0 \quad \textit{for all } x \in D \setminus \{x^*\}\,. \end{aligned}$$

Then the equilibrium x^* is asymptotically stable.

Proof. The proof is divided in two steps.
Step 1. We show that there exists a $\delta > 0$ such that $\omega(x) = \{x^*\}$ for all $x \in B_\delta(x^*)$.
Note first that x^* is stable due to Theorem 4.26. Since D is open, there exists an $\varepsilon > 0$ such that $B_{2\varepsilon}(x^*) \subset D$, and since x^* is stable, there exists a $\delta > 0$ such that for all $\delta > 0$ such that for all $x \in B_\delta(x^*)$, we have

$$\varphi(t, x) \in B_\varepsilon(x^*) \quad \text{for all } t \geq 0\,.$$

We fix an $x \in B_\delta(x^*)$. Since $\varphi(t,x) \in \overline{B_\varepsilon(x^*)} \subset D$ for all $t \geq 0$, we get $\overline{O^+(x)} \subset D$, and this means that Proposition 4.21 implies that $\omega(x)$ is nonempty. La Salle's principle (Theorem 4.28) implies that $\omega(x) \subset \big\{\bar x \in D : \dot V(\bar x) = 0\big\} = \{x^*\}$, and $\omega(x)$ being nonempty means that $\omega(x) = \{x^*\}$.
Step 2. We show that for all $x \in B_\delta(x^*)$, we have $\lim_{t\to\infty} \varphi(t,x) = x^*$.
Assume to the contrary that we do not have $\lim_{t\to\infty} \varphi(t,x) = x^*$. This means that there exists an $\eta > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ converging to ∞ such that

$$\|\varphi(t_n, x) - x^*\| \geq \eta \quad \text{for all } n \in \mathbb{N}\,,$$

and since the sequence $\{\varphi(t_n, x)\}_{n\in\mathbb{N}}$ is bounded and bounded away from the boundary of D, it has an accumulation point in D. This accumulation point is an omega limit point of x, which contradicts the above observation that the omega limit set $\omega(x)$ is the singleton $\{x^*\}$. This finishes the proof of this theorem. □

We use a slightly modified version of Theorem 4.31 (in the spirit of the reformulation of La Salle's principle in Corollary 4.30) to prove asymptotic stability for the differential equation considered in Example 4.27.

Example 4.32 (Application of Lyapunov's direct method for asymptotic stability)**.** We reconsider the differential equation

$$\begin{aligned} \dot x &= -y - xy^2\,, \\ \dot y &= x - yx^2\,, \end{aligned}$$

from Example 4.27. We found a Lyapunov function for this differential equation, given by $V(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R}^2$, and we showed that

$$V(0,0) = 0 \quad \text{and} \quad V(x,y) > 0 \quad \text{for all } (x,y) \in \mathbb{R} \setminus \{(0,0)\}\,.$$

In addition,

$$\dot{V}(x,y) = -4x^2y^2 \quad \text{for all } (x,y) \in \mathbb{R}^2$$

shows that in the equilibrium $(0,0)$, we have $\dot{V}(0,0) = 0$, but

$$\big\{(x,y) \in \mathbb{R}^2 : \dot{V}(x,y) = 0\big\} = \big(\{0\} \times \mathbb{R}\big) \cup \big(\mathbb{R} \times \{0\}\big)\,.$$

This means that Theorem 4.31 is not applicable directly. However, in Step 1 of the proof of Theorem 4.31, La Salle's principle was applied to show that the omega limit set is a singleton, and we will demonstrate now that instead, we can apply the reformulation of this principle given by Corollary 4.30. This is possible, since the largest invariant subset of

$$\big\{(x,y) \in \mathbb{R}^2 : \dot{V}(x,y) = 0\big\} = \big(\{0\} \times \mathbb{R}\big) \cup \big(\mathbb{R} \times \{0\}\big)$$

is given by the equilibrium $\{(0,0)\}$. This follows from the fact that if we start on the y-axis outside of trivial equilibrium (i.e. $x = 0$ and $y \neq 0$), then $\dot{x} = -y \neq 0$, so we leave the y-axis immediately. The same holds for the x-axis outside of trivial equilibrium (i.e. $y = 0$ and $x \neq 0$). In this case, we get $\dot{y} = x \neq 0$, so we leave the x-axis immediately. It follows that a modified version of Step 1 of Theorem 4.31 then implies that $\omega((x,y)) = \{(0,0)\}$, for (x,y) from a neighbourhood around $(0,0)$. This implies that $(0,0)$ is asymptotically stable.

The following corollary to Theorem 4.31 shows that sublevel sets of Lyapunov functions are part of the domain of attraction.

Corollary 4.33 (Sublevel sets of Lyapunov functions are subsets of the domain of attraction)**.** *Under the assumptions of Theorem 4.31, we consider the sublevel sets of the Lyapunov function V, which are of the form*

$$S_c := \big\{x \in D : V(x) \le c\big\}\,,$$

where $c > 0$. Then S_c is a subset of the domain of attraction $W^s(x^*)$ if $S_c \subset D$ is compact.

Proof. Let $x \in S_c$, and we need to show that $\lim_{t\to\infty} \varphi(t,x) = x^*$. Proposition 4.25 implies that $\varphi(t,x) \in S_c$ for all $t \ge 0$, and since S_c is compact, we get $\overline{O^+(x)} \subset D$. This means due to Proposition 4.21, $\omega(x)$ is nonempty. La Salle's principle implies that $\omega(x) \subset \big\{x \in D : \dot{V}(x) = 0\big\} = \{x^*\}$, and $\omega(x)$ being nonempty means that $\omega(x) = \{x^*\}$. Then Step 2 of Theorem 4.31 finishes the proof. □

4. Poincaré–Bendixson theorem

In this final section, we complete the set of tools that helps us to analyse two-dimensional autonomous differential equations. This analysis decomposes into a *local* and a *global* analysis.

In the local analysis, we

(i) locate the fixed points, and

(ii) determine their stability by linearising (which is possible if they are hyperbolic).

In the global analysis, we

(i) look at nullclines in order to understand the global behaviour better (see exercises),

(ii) try to find Lyapunov functions to understand stability of equilibria (if non-hyperbolic) and domains of attractions,

(iii) locate periodic orbits (done in this section).

We consider the two-dimensional differential equation

$$\dot{x} = f(x)\,, \tag{4.25}$$

where $f : D \to \mathbb{R}^2$ is continuously differentiable function on an open set $D \subset \mathbb{R}^2$. We denote the flow of (4.25) by φ. Note that one can show (and this is important for us) that the flow φ is continuously differentiable (see Remark 2.28).

Theorem 4.34 (Poincaré–Bendixson theorem)**.** *Consider the differential equation* (4.25)*, and assume that for some $x \in D$, the positive half-orbit $O^+(x)$ lies in a compact subset K of D, which contains not more than finitely many equilibria. Then one of the following three statements holds for the omega-limit set $\omega(x)$.*

(i) *$\omega(x)$ is a singleton consisting of an equilibrium.*

(ii) *$\omega(x)$ is a periodic orbit.*

(iii) *$\omega(x)$ consists of equilibria and non-closed orbits. The non-closed orbits in $\omega(x)$ converge forward and backward in time to equilibria in $\omega(x)$, so they are either homoclinic or heteroclinic orbits.*

We note that an analogous statement holds for alpha limit sets, when we look at the negative half-orbit. The theorem of Poincaré–Bendixson shows in particular that in two-dimensional differential equation, there is only very regular behaviour and no chaos. Chaotic differential equations occur in dimension three, for instance in the famous Lorenz system.

The proof of the Poincaré–Bendixson theorem is quite involved, and we skip it due to time constraints. The full proof is written down in *Extra Material 2*.

The Poincaré–Bendixson theorem is often applied in the following form to prove the existence of a periodic orbit.

Corollary 4.35 (Existence of a periodic orbit)**.** *Consider the differential equation* (4.25)*, and assume that for some $x \in D$, the positive half-orbit $O^+(x)$ lies in a compact subset K of D that does not contain an equilibrium. Then $\omega(x)$ is a periodic orbit.*

We can apply the Poincaré–Bendixson theorem to prove the existence of a periodic orbit.

Example 4.36 (Existence of a periodic orbit)**.** We consider the two-dimensional differential equation

$$\begin{aligned} \dot{x} &= y \,, \\ \dot{y} &= -x + y(1 - x^2 - 2y^2) \,. \end{aligned}$$

We first show that $M := \big\{(x, y) \in \mathbb{R}^2 \; : \frac{1}{3} \le x^2 + y^2 \le 2\big\}$ is positively invariant. We show that the vector field of the right hand side points inwards at the boundary of M. More precisely, we consider the scalar-valued function $V(x, y) = x^2 + y^2$ and show that the orbital derivative $\dot{V}$ satisfies $\dot{V}(x, y) < 0$ for $x^2 + y^2 = 2$ and $\dot{V}(x, y) > 0$ for $x^2 + y^2 = \frac{1}{3}$. Firstly, the orbital derivative reads as

$$\dot{V}(x, y) = 2xy + 2y\big(-x + y(1 - x^2 - 2y^2)\big) = 2y^2(1 - x^2 - 2y^2) \,.$$

For $x^2 + y^2 = 2$, we have $\dot{V}(x, y) = 2y^2(-1 - y^2) < 0$ and for $x^2 + y^2 = \frac{1}{3}$, we have $\dot{V}(x, y) = 2y^2(\frac{2}{3} - y^2) \ge 0$. This shows the positive invariance of M. Note that the only equilibrium is clearly given by $(0, 0) \notin M$. We apply the corollary to the Poincaré–Bendixson theorem (Corollary 4.35) and conclude that the positively invariant set M contains a periodic orbit.