The total marks for this test is 40 points: each problem has 20 points.

Problem 1. For each of the following sets in $\mathbb{R}^2$ state if the set is (a) connected and (b) path connected? Justify your answers.

(i) $A_1 = ([-1,+1] \times \{0\}) \cup (\{0\} \times [-1,+1])$, [5 pts]

(ii) $A_2 = B_1((-1,0)) \cup B_1((1,0))$, [5 pts]

(iii) $A_3 = \overline{B_1((-1,0))} \cup B_1((1,0))$, [5 pts]

(iv) $A_4 = \{(x,y) \in \mathbb{R}^2 \mid y = f(x)\}$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. [5 pts]

Problem 2. In the following two problems assume that (X, d_X) and (Y, d_Y) are metric spaces and $f : X \to Y$ is continuous.

(i) Prove that if X is sequentially compact, then $f(X)$ is sequentially compact. [10 pts]

(ii) Prove that if X is compact and $f : X \to Y$ is injective, then $f : X \to f(X)$ is a homeomorphism. [10 pts]

Solution of problem 1.

(i) The set A_1 is path connected. To see this, let a and b be arbitrary points in A_1. Consider the map $g : [0,1] \to A_1$ defined as

$$g(t) = \begin{cases} (1-2t)a & \text{if } t \in [0,1/2] \\ (2t-1)b & \text{if } t \in [1/2,1] \end{cases}$$

Then, g is continuous on $[0,1]$ with $g(0) = a$ and $g(1) = b$. [3 points for this part]

By a theorem in the lectures, any path connected set is connected. Therefore, A_1 is connected.[2 points for this part]

(ii) Let $U = B_1((-1,0))$ and $V = B_1((1,0))$. Any ball in a metric spaces is open, thus, U and V are open. Moreover, $U \cap V = \emptyset$, $A_2 \subset U \cap V$. This shows that A_2 is disconnected. [3 points for this part]

The set A_2 cannot be path connected, since otherwise it must be connected (same theorem in the lectures). [2 points for this part]

(iii) The set A_3 is path connected. Note that $(0,0)$ belongs to A_3. Let a and b be arbitrary points in A_1. We can consider a path from a to $(0,0)$ in the ball containing a and follow that path by a path from $(0,0)$ to b inside the ball which contains b. More precisely, consider the map $g : [0,1] \to A_1$ defined as

$$g(t) = \begin{cases} (1-2t)a & \text{if } t \in [0,1/2] \\ (2t-1)b & \text{if } t \in [1/2,1] \end{cases}$$

Then, g is continuous on $[0,1]$ with $g(0) = a$ and $g(1) = b$. We need to show that for every $t \in [0,1]$, $g(t) \in A_3$. However, since for every $a \in A_3$, the line segment between a and $(0,0)$ without its end points belongs to A_3, the image of g is contained in A_3. [3 points for this part]

By a theorem in the lectures, any path connected set is connected. Therefore, A_3 is connected.[2 points for this part]

(iv) The set A_4 is path connected. For arbitrary paints $(a, f(a))$ and $(b, f(b))$ in the set A_4 we may consider the continuous path $g : [0,1] \to A_4$ as

$$g(t) = \Big((1-t)a + tb, f((1-t)a + tb)\Big).$$

By the definition of g, for every $t \in [0,1]$, $g(t) \in A_4$. Since f is continuous, g is continuous. Also we have $g(0) = (f, f(a)$ and $g(1) = (b, f(b))$. Therefore, A_4 is path connected. [3 points for this part]

By a theorem in the lectures, any path connected set is connected. Therefore, A_4 is connected.[2 points for this part]

Solution of problem 2.

(i) *Method 1:* In the lectures we have proved that the compactness and sequential compactness are equivalent for metric spaces. If X is sequentially compact, then X is compact. By another theorem, the image of a compact set by a continuous map is compact. Thus, $f(X)$ is compact. This implies that $f(X)$ is sequentially compact. [10 points for this part, give 5 pts for stating the theorems correctly, and 5 pts for completing the proof using those theorems.]

Method 2: Let $(y_n)_{n\geq 1}$ be an arbitrary sequence in $f(X)$. Then, for each $n \in \mathbb{N}$, there is $x_n \in X$ such that $f(x_n) = y_n$. The point x_n exists, but it may not be unique. When there are more than one choices for x_n, we may choose any of the choices we wish. Since X is sequentially compact, there is a subsequence of $(x_n)_{n\geq 1}$, say $(x_{n_k})_{k\geq}$ which converges to some $x \in X$. Since f is continuous, the sequence $(y_{n_k})_{k\geq 1} = f(x_{n_k})_{k\geq}$ converges to $f(x) \in f(X)$. This completes the proof. [10 points, subtract partial points for minor errors proportionally.

(ii) Since $f : X \to f(X)$ is onto, there only remains to show that $f^{-1} : f(X) \to X$ is continuous. To show that this map is continuous, it is enough to show that the preimage of any closed set in X by f^{-1} is closed in $f(X)$. Equivalently, we need to show that for every closed set E in X, $f(E) = (f^{-1})^{-1}(E)$ is closed in $f(X)$. To show this, let E be an arbitrary closed set in X. Since X is compact, and E is closed, E is compact. Since f is continuous, and the image of any compact set by a continuous map is compact, we conclude that $f(E)$ is compact in Y. However, since every compact set is closed, $f(E)$ is closed as well. [10 points