Imperial College
London

MATH50001/MATH50017

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
January 2022

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Analysis 2

Date: 11 January 2022

Time: 11:00 AM to 12:00 PM

Time Allowed: 1 hour

Upload Time Allowed: 30 minutes

This paper has 4 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS ONE PDF TO THE RELEVANT DROPBOX ON BLACKBOARD INCLUDING A COMPLETED COVERSHEET WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

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Question 1. Which of the following sets in the Euclidean metric space $(\mathbb{R}^2, d_2)$ is connected? Justify your answers.

(i) $A = \{(x, y) \in \mathbb{R}^2 \mid |y| < |x|\}$, [5 pts]

(ii) $B = \{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{Q} \text{ or } y \in \mathbb{Q}\}$. [5 pts]

Question 2. Let (X, d) be a connected metric space, and assume that $f : X \to \mathbb{R}$ is *locally constant*, which means that for every $x \in X$ there is $r > 0$ such that f is constant on the ball $B_r(x)$. Prove that f is constant on all of X. [10 pts]

Question 3.

- Is the set $\mathbb{Q} \cap [0, 1]$ with the metric $d_1(x, y) = |x - y|$ a compact metric space? [5 pts]
- Let Ω be the set of all continuous functions $f : [0, 1] \to \mathbb{R}$ such that $f(0) = f(1) = 0$. Consider the supremum metric

$$d_\infty(f, g) = \sup_{t \in [0,1]} |f(t) - g(t)|$$

on Ω. Is the metric space (Ω, d_∞) sequentially compact? Justify your answer. [5 pts]

Question 4. Let $Z = [0, 1]^{\mathbb{N}}$, i.e. the set of all sequences in $[0, 1]$, and consider the metric d on Z defined as

$$d((x^i)_{i\in\mathbb{N}}, (y^i)_{i\in\mathbb{N}}) = \sup_{i\in\mathbb{N}} \frac{|x^i - y^i|}{i}.$$

Is (Z, d) a complete metric space? Justify your answer. [10 pts]

Solution of Question 1. The set A is not connected. [1pt for the correct answer]

For example, consider the sets

$$U = \{(x, y) \in \mathbb{R}^2 \mid x > 0, |y| < |x|\}, \quad V = \{(x, y) \in \mathbb{R}^2 \mid x < 0, |y| < |x|\}.$$

[2pt for introducing these sets, or any other pair of sets which work]

Evidently, U and V are open sets in $\mathbb{R}^2$, $U \cap V = \emptyset$, $A \subseteq U \cup V$, $A \cap U \neq \emptyset$ and $A \cap V \neq \emptyset$. Thus, U and V disconnect the set A. [2pt for verifying/stating all the required properties]

The set B is connected. [1pt for the correct answer]

In order to show that B is connected, by a theorem in the lectures, it is enough to show that B is path-connected. [1pt for suggesting this approach, and correctly stating the relation between path connectedness and connectedness]

Let $b_1 = (x_1, y_1)$ and $b_2 = (x_2, y_2)$ be arbitrary points in B. We consider four possibilities below:

(i) If x_1 and x_2 are rational numbers, we may follow the vertical line from (x_1, y_1) to $(x_1, 0)$, and then follow the horizontal line from $(x_1, 0)$ to $(x_2, 0)$, and then follow the vertical line from $(x_2, 0)$ to (x_2, y_2). These line segments belong to the set B, and form a continuous path from (x_1, y_1) to (x_2, y_2).
(ii) If y_1 and y_2 are rational numbers, we follow the horizontal line from (x_1, y_1) to $(0, y_1)$, and then follow the vertical line from $(0, y_1)$ to $(0, y_2)$, and then follow the horizontal line from $(0, y_2)$ to (x_2, y_2).
(iii) If x_1 and y_2 are rational numbers, we may follow the line from (x_1, y_1) to (x_1, y_2), and then follow the horizontal line from (x_1, y_2) to (x_2, y_2).
(iv) if x_2 and y_1 are rational numbers, we may form a path similar to the one in item (iii).

[3pt for building the correct paths as in the above items]

Solution of Question 2. If X is the empty set, there is nothing to prove. Below assume that X is not the empty set. [1pt for considering this case]

Fix an arbitrary point $x \in X$ and let $f(x) = c \in \mathbb{R}$. Define the set

$$W = \{z \in X \mid f(z) = c\}.$$

The set W is open in X. To see this, fix an arbitrary $z \in W$. Since f is locally constant, there is $r > 0$ such that f is constant on $B_r(z)$. This implies that $B_r(z) \subseteq W$. [3pt for this part]

The set W is also closed in X. To see this, let $(z_i)_{i \geq 1}$ be an arbitrary sequence in W which converges to z in X. Because f is locally constant, there is $r > 0$ such that f is constant on $B_r(z)$. Since z_i converges to z, there is $n \in \mathbb{N}$ such that $z_n \in B_r(z)$. This implies that $f(z) = f(z_n) = c$. Thus $z \in W$. [3pt for this part]

Since X is connected, by a result in the lectures, the only subsets of X which are both open and closed are $\emptyset$ and X. Since $W \neq \emptyset$, we must have $W = X$. This completes the proof. [3pt for stating the result from the lectures, and employing it]

Solution of Question 3.

No, it is not compact. [1pt for the correct answer]

By a result in the lectures, any compact metric space must be closed. [1pt for stating this result]

However, the set $\mathbb{Q} \cap [0,1]$ is not closed with respect to the metric d_1. For example, there is a sequence of rational numbers in $[0,1]$ which converges to $\sqrt{2}/2$, with respect to d_1, but $\sqrt{2}/2$ does not belong to $\mathbb{Q} \cap [0,1]$. [3pt for showing that it is not closed]

There are other ways to show that this metric space is not compact. For example, by showing that it is not sequentially compact. Or by directly identifying an open cover which does not have any finite sub-cover.

It is not sequentially compact. [1pt for the correct answer]

For example, the sequence of maps $f_n(x) = nx(1-x)$ belongs to Ω. But it does not have any convergent subsequence. [2pt for presenting any correct example]

Assume in the contrary that there is a subsequence of this sequence, say f_{n_i} for $i \in \mathbb{N}$, which converges to a map $g : [0,1] \to \mathbb{R}$ with respect to d_∞. We must have

$$|g(1/2) - f_{n_i}(1/2)| \leq d_\infty(g, f_{n_i}) \to 0, \qquad \text{as } i \to \infty.$$

In particular, for large values of i we must have $|g(1/2) - f_{n_i(1/2)}| \leq 1$. But, $f_{n_i}(1/2) = n_i/4 \to \infty$ as $i \to \infty$. This shows that $g(1/2)$ cannot be any real number. [2pt for justifying the counter example works]

Solution of Question 4.

The metric space (Z, d) is complete. [1pt for the correct answer]

To see this, let x_n, for $n \geq 1$, be an arbitrary Cauchy sequence in (Z, d). Let

$$x_n = (x_n^1, x_n^2, x_n^3, \dots)$$

denote the coordinates of x_n, for each $n \geq 1$.

Fix an arbitrary $j \in \mathbb{N}$. We have

$$|x_n^j - x_m^j| = j \frac{x_n^j - x_m^j}{j} \leq j d(x_n, x_m).$$

This implies that for each fixed $j \in \mathbb{N}$, $(x_n^j)_{n \geq 1}$ is a Cauchy sequence in $(\mathbb{R}, d_1)$. Because, $(\mathbb{R}, d_1)$ is a complete metric space, $(x_n^j)_{n \geq 1}$ converges to some real number, say y^j. Moreover, since every x_n^j belongs to $[0, 1]$, we must have $y^j \in [0, 1]$. [3pt for reducing the problem to $\mathbb{R}^1$]

Let us define $y = (y^j)_{j \geq 1}$ in Z. [1pt for correctly identifying the limit]

We claim that, $(x_n)_{n \geq 1}$ converges to y in (Z, d). To see this, fix an arbitrary $\epsilon > 0$. There is $m \in \mathbb{N}$ such that $1/m < \epsilon$. Since x_n^j and y^j belong to $[0, 1]$, this implies that for all $j \geq m$ and all $n \geq 1$ we have

$$\frac{x_n^j - y^j}{j} \leq \frac{1}{j} \leq \frac{1}{m} < \epsilon.$$

Let $j \leq m$ be a positive integer. Since $x_n^j \to y^j$, as $n \to \infty$, there is $N_j \in \mathbb{N}$ such that for all $n \geq N_j$ we have $|x_n^j - y^j|/j < \epsilon$.

Combining the above relations, we conclude that for all $n \geq \max\{N_1, N_2, \dots, N_n\}$ we have

$$d(x_n, y) = \sup_{j \geq 1} \frac{x_n^j - y^j}{j} < \epsilon.$$

[5pt for completing the proof]