Imperial College London

MATH50001

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Analysis 2

Date: Thursday, 6 May 2021

Time: 09:00 to 12:00

Time Allowed: 3 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. Define the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$f(x,y) = \Big(e^x \cos(y), x^2 y + \sin(y)\Big).$$

(a) Show that f is *differentiable* at every $(x,y) \in \mathbb{R}^2$. (4 marks)

(b) What is the *differential* of f at each $(x,y) \in \mathbb{R}^2$? (2 marks)

(c) Show that there is $\delta > 0$ such that for all real numbers a and b satisfying $|a-e| < \delta$ and $|b| < \delta$, the nonlinear system of equations

$$\begin{cases} e^x \cos(y) = a \\ x^2 y + \sin(y) = b \end{cases}$$

can be solved for x and y. (8 marks)

(d) Let $U \subset \mathbb{R}^2$ be an open set, and let $f : U \to \mathbb{R}$. Assume that the partial derivatives $D_1 f(x,y)$ and $D_2 f(x,y)$ exist at all $(x,y) \in U$, and there is a real number $M > 0$ such that

$$|D_1 f(x,y)| \leq M \quad \text{and} \quad |D_2 f(x,y)| \leq M, \qquad \text{for all } (x,y) \in U.$$

Show that $f : U \to \mathbb{R}$ is continuous. (6 marks)

(Total: 20 marks)

2. For $n \in \mathbb{N}$, let M_n denote the set of all $n \times n$ matrices with real entries. For $A \in M_n$, define

$$|||A||| = \sup\Big\{\|Av\| \,:\, v \in \mathbb{R}^n, \|v\| = 1\Big\},$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$.

(a) Show that for every $A \in M_n$, $|||A|||$ is a (finite) real number. (3 marks)

(b) Show that $|||\cdot|||$ is a norm on M_n. (7 marks)

Let d_M denote the metric induced on M_n from the norm $|||\cdot|||$.

(c) Show that the determinant $\det : (M_n, d_M) \to (\mathbb{R}, d_1)$ is a continuous function, where d_1 denotes the Euclidean metric on $\mathbb{R}^1$. (4 marks)

(d) Show that the set $\Omega = \{A \in M_n \,:\, A \text{ is invertible}\}$ is an open set in (M_n, d_M). (4 marks)

(e) Show that the set Ω is not connected in (M_n, d_M). (2 marks)

(Total: 20 marks)

3. (a) Let $E \subset \mathbb{R}^2$ be the set

$$E = \Big\{(x,y) \in \mathbb{R}^2 \mid x > 0, |y| < x\Big\} \cup \Big\{(x,y) \in \mathbb{R}^2 \mid x < 0, |y| \le -x\Big\}.$$

What are E°, ∂E, $\overline{E}$, and the set of isolated points of E? You do not need to justify your answers. (4 marks)

(b) Let $n \in \mathbb{N}$, and assume that $K \subset \mathbb{R}^n$ is a compact set such that $(0,0,\ldots,0) \notin K$. Show that there is $r > 0$ such that

$$K \subseteq \{(x^1, x^2, \ldots, x^n) \in \mathbb{R}^n \mid r \le \|(x^1, x^2, \ldots, x^n)\| \le 1/r\}.$$

(6 marks)

Let $C([0,1])$ denote the set of all continuous functions $f : [0,1] \to \mathbb{R}$, and let d_2 and d_∞ denote the standard metrics on the function space $C([0,1])$. Define the map $\Phi : C([0,1]) \to \mathbb{R}$ as

$$\Phi(f) = \int_0^1 f(t)dt.$$

Also, let d_1 denote the Euclidean metric on $\mathbb{R}^1$.

(c) Is the map $\Phi : (C([0,1]), d_\infty) \to (\mathbb{R}, d_1)$ continuous? Justify your answer. (5 marks)

(d) Is the map $\Phi : (C([0,1]), d_2) \to (\mathbb{R}, d_1)$ continuous? Justify your answer. (5 marks)

(Total: 20 marks)

4. (a) Is the function e^{iz^2} holomorphic in $\mathbb{C}$? Explain why. (2 marks)

   (b) What are the Cauchy-Riemann equations for e^{iz^2}? (4 marks)

   (c) Suppose $f(z)$ is holomorphic in an open connected set Ω. Can $g(z) = \overline{f(z)}$ be holomorphic in Ω? Explain why. (5 marks)

   (d) Verify that $u(x, y) = e^{x^2-y^2} \cos(2xy)$ is harmonic in $\mathbb{R}^2$. Find the harmonic conjugates v of u, and find a holomorphic function $f(z) = u + iv$ satisfying the condition $f(0) = 1$. (9 marks)

(Total: 20 marks)

5. (a) Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ in the annulus $0 < |z - z_0| < R$. Give the definition of z_0 being a removable singularity. (2 marks)

   (b) Verify if $z_0 = 0$ is a removable singularity for

   (i) $\dfrac{\sin z^3}{z^2}$, (ii) $z^5 e^{1/z}$, (iii) $\dfrac{\sinh z}{z}$.

   Explanations are not required. (3 marks)

   (c) Find the Laurent series for

$$\frac{z + 1}{z(z - 4)^3}, \qquad \text{in} \quad 0 < |z - 4| < 4.$$

(4 marks)

   (d) Compute the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 - 2x + 10}\, dx.$$

(6 marks)

   (e) Let $P(z) = z^5 - 12z^2 + 14$. Find the number of roots of $P(z)$ (solutions of the equation $P(z) = 0$) in

$$\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}.$$

(5 marks)

(Total: 20 marks)

6. (a) Let $f \not\equiv const$ be holomorphic in $D = \{z \in \mathbb{C} : |z| \leq R\}$. Show that the function $M(r) = \max_{z:\,|z|=r} |f(z)|$ is strictly increasing on the interval $(0, R)$. (2 marks)

(b) Let $f \not\equiv const$ be holomorphic in $\Omega_R = \{z \in \mathbb{C} : |z| \geq R\}$, $R > 0$, and $|f(z)| \to 0$, as $|z| \to \infty$. Prove that the maximum of $|f|$ is achieved on the boundary $\partial\Omega_R$ and the function $M(r) = \max_{z:|z|=r} |f(z)|$ is strictly decreasing on $[R, \infty)$. (4 marks)

(c) Find the Möbius transform f such that

$$f : (z_1, z_2, z_3) \to (w_1, w_2, w_3),$$

where $(z_1, z_2, z_3) = (-1, i, 1 + i)$ and $(w_1, w_2, w_3) = (i, \infty, 1)$. Find the image of

$$D = \left\{ z \in \mathbb{C} : \left| z - \frac{1-i}{2} \right| > \sqrt{\frac{5}{2}} \right\}.$$

(5 marks)

(d) Let $f(z)$ be holomorphic in $D_R = \{z \in \mathbb{C} : |z - z_0| < R\}$. Show that for any r, such that $0 < r < R$, we have

$$f'(z_0) = \frac{1}{\pi r} \int_0^{2\pi} P(\vartheta) e^{-i\vartheta}\, d\vartheta,$$

where $P(\vartheta) = \mathrm{Re}\left(f\left(z_0 + re^{i\vartheta}\right)\right)$. (9 marks)

(Total: 20 marks)

BSc and MSci EXAMINATIONS (MATHEMATICS)

May 2021

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MATH50001-Analysis II

MATH50001-Analysis II (Solutions)

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1. (a) Since $\exp$, $\sin$ and $\cos$ are differentiable (in one variable), and the sum/product of differentiable functions is differentiable, the components of f are differentiable in x and y. The partial derivatives of f are

sim. seen ⇓

$$D_1 f(x,y) = (e^x \cos(y), 2xy), \qquad D_2 f(x,y) = \Big(-e^x \sin(y), x^2 + \cos y\Big).$$

These are continuous in x and y. By a theorem in the lectures, if the partial derivatives exist and are continuous on an open set ($\mathbb{R}^2$ here), the map is differentiable. [2pt for the correct derivatives, 2pt for employing the correct result]

4, A

(b) By a theorem in the lectures the matrix of the derivative of f at (x,y) is

unseen ⇓

$$Df(x,y) = \begin{pmatrix} e^x \cos(y) & -e^x \sin y \\ 2xy & x^2 + \cos y \end{pmatrix}.$$

2, A

(c) [The application of the Inverse Function Theorem to solving systems of non-linear equations has been mentioned in the class, without an explicit example]. We solve the system of equations for $a = e$ and $b = 0$, and see that $(x,y) = (1,0)$ satisfies. The derivative of f at $(1,0)$, is

sim. seen ⇓

$$Df(x,y) = \begin{pmatrix} e & 0 \\ 0 & 2 \end{pmatrix}$$

which has a non-zero determinant, and hence it is invertible. Since, f is differentiable on $\mathbb{R}^2$, its derivative is continuous, and invertible at $(1,0)$, by the Inverse Function Theorem, there are open sets U and V in $\mathbb{R}^2$ with $(1,0) \in U$ and $f(1,0) = (e,0) \in V$ and $f : U \to V$ is a bijection. There is $\delta > 0$ such that all real numbers a and b satisfying $|a - e| < \delta$ and $|b - 0| < \delta$ are contained in V. Then, $(x,y) = f^{-1}(a,b)$ is a solution of the system. [2pt for finding any solution for $a = e$ and $b = 0$, 4pt for the correct use of IFT and verifying its assumptions, 2pt for correct introduction of δ]

8, B

(d) Fix an arbitrary $(x_0, y_0) \in U$. As U is open, there is $r > 0$ such that $B_r(x_0,y_0) \subset U$. For $(x,y) \in B_r(x_0,y_0)$, we write

meth seen ⇓

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| &= |f(x,y) - f(x,y_0) + f(x,y_0) - f(x_0,y_0)|, \\ &\le |f(x,y) - f(x,y_0)| + |f(x,y_0) - f(x_0,y_0)|. \end{aligned}$$

We apply the intermediate value theorem to $t \mapsto f(x,t)$ for $t \in [y,y_0]$, and $s \mapsto f(s,y_0)$ for $s \in [x,x_0]$, and obtain $y' \in [y,y_0]$ and $x' \in [x,x_0]$ such that

$$f(x,y) - f(x,y_0) = D_2 f(x,y')(y - y_0),$$

$$f(x,y_0) - f(x_0,y_0) = D_1 f(x',y_0)(x - x_0).$$

Therefore,

$$\begin{aligned} |f(x,y) - f(x_0,y_0)| &\le |D_2 f(x,y')||y - y_0| + |D_1 f(x',y_0)||x - x_0| \\ &\le M(|x - x_0| + |y - y_0|). \end{aligned}$$

This shows that f is continuous at (x_0,y_0). [2pt for reducing the problem to 1-D, 2pt for using IVT, 2pt for completing the proof.]

6, D

2. (a) Since the map $x \mapsto \|x\|$ is continuous (lectures), and the set $\|v\| = 1$ is compact (closed and bounded), the function $x \mapsto \|x\|$ realises its maximum at some point. Thus, the supremum is finite, and gives a real number. sim. seen ⇓ 3, A

(b) Evidently, we can add any two matrices (entry by entry), and also multiple any matrix by any scalar. Thus M_n is a vector space on $\mathbb{R}$. We need to verify the three properties for the norm function. sim. seen ⇓

(i) For every $A \in M_n$ and every $v \in \mathbb{R}^n$ satisfying $\|v\| = 1$, we have $\|Av\| \geq 0$. This shows that the supremum/maximum is a non-negative number. On the other hand, if $\|A\|_m = 0$, then for every $v \in \mathbb{R}^n$ satisfying $\|v\| = 1$, we must have $\|Av\| = 0$. By the properties of the norm on $\mathbb{R}^n$, this implies that for every $v \in \mathbb{R}^n$ satisfying $\|v\| = 1$, $Av = \bar{0}$. This implies that A is the zero matrix.

(ii) Let $A \in M_n$ and $\lambda \in \mathbb{R}$. Using the properties of the norm on $\mathbb{R}^n$,

$$\begin{aligned}
\|(\lambda A)\|_m &= \sup\{\|(\lambda A)v\| \mid v \in \mathbb{R}^n, \|v\| = 1\} \\
&= \sup\{\|\lambda(Av)\| \mid v \in \mathbb{R}^n, \|v\| = 1\} \\
&= \sup\{|\lambda|\|Av\| \mid v \in \mathbb{R}^n, \|v\| = 1\} \\
&= |\lambda| \sup\{\|Av\| \mid v \in \mathbb{R}^n, \|v\| = 1\} = |\lambda|\|A\|_m.
\end{aligned}$$

(iii) Let A and B be arbitrary elements in M_n. Using $\|x + y\| \leq \|x\| + \|y\|$, for all x and y in $\mathbb{R}^n$,

$$\begin{aligned}
\|A + B\|_m &= \sup\{\|(A + B)v\| \mid v \in \mathbb{R}^n, \|v\| = 1\} \\
&= \sup\{\|Av + Bv\| \mid v \in \mathbb{R}^n, \|v\| = 1\} \\
&\leq \sup\{\|Av\| + \|Bv\| \mid v \in \mathbb{R}^n, \|v\| = 1\} \\
&\leq \sup\{\|Av\| \mid v \in \mathbb{R}^n, \|v\| = 1\} + \sup\{\|Bv\| \mid v \in \mathbb{R}^n, \|v\| = 1\} \\
&= \|A\|_m + \|B\|_m.
\end{aligned}$$

[1pt for saying M_n is a vector space, 2pt for each of (i), (ii), and (iii)] 7, A

(c) The map $A \to \det A$ is a continuous function of the entries of A, since it is a polynomial in the entries of A. Thus, it is enough to show that if $\|A - B\|_m$ is small, then the corresponding entries of A and B are close. Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0)$, $\ldots$, $e_n = (0, 0, \ldots, 1)$ be the unit vectors in $\mathbb{R}^n$. By the definition of the norm $\|\cdot\|_m$, we have $\|(A - B)e_j\| = \|Ae_j - Be_j\|$ is small. This shows that the j-th column of A is close to the j-th column of B, when seen as vectors in $\mathbb{R}^n$. This implies that the corresponding entries of A are close to the corresponding entries of B. unseen ⇓ 4, B

(d) We note that a matrix is invertible iff its determinant is non-zero. By the previous part, $\det : M_n \to \mathbb{R}$ is continuous. Thus, since $(-\infty, 0) \cup (0, +\infty)$ is open in $\mathbb{R}$, its pre-image under $\det$ is an open set in M_n. meth seen ⇓ 4, C

(e) The function $L : (-\infty, 0) \cup (0, +\infty) \to \{-1, +1\}$, defined as $L(x) = x/|x|$ is continuous. Thus the function $L \circ \det$ on Ω is continuous and takes ± 1 values. By a theorem in the lectures, this implies that Ω is not connected. meth seen ⇓ 2, D

3. (a) We have sim. seen ⇓

$$E^\circ = \Big\{(x,y) \in \mathbb{R}^2 \mid x > 0, |y| < x\Big\} \cup \Big\{(x,y) \in \mathbb{R}^2 \mid x < 0, |y| < -x\Big\},$$

$$\partial E = \Big\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, |y| = x\Big\} \cup \Big\{(x,y) \in \mathbb{R}^2 \mid x \leq 0, |y| = -x\Big\},$$

$$\overline{E} = \Big\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, |y| \leq x\Big\} \cup \Big\{(x,y) \in \mathbb{R}^2 \mid x \leq 0, |y| \leq -x\Big\}.$$

The set of isolated points of E is empty. 4, A

(b) For $n \in \mathbb{N}$, consider sim. seen ⇓

$$A_n = \Big\{(x^1, x^2, \dots, x^n) \in \mathbb{R}^n \mid 1/n < \|(x^1, x^2, \dots, x^n)\| < n\Big\}.$$

Since $x \mapsto \|x\|$ is continuous, every A_n is an open subset of $\mathbb{R}^n$. By the hypothesis, $K \subset \cup_{n\in\mathbb{N}} A_n$, so the collection $\{A_n\}_{n\in\mathbb{N}}$ is an open cover for K. Since K is compact, there is a finite sub-collection of this collection which covers K. Let n be the largest index in that finite sub-collection. It follows that $K \subset A_n$.

[There are few other ways to do this problem, Using, Heine-Borel for the upper end, and extreme value property for the lower end (or both ends). 2pt for introducing a correct collection, 2pt for using compactness criterion, 2pt for completing the proof.] 6, A

(c) Yes. For f and g in $C([0,1])$, we have unseen ⇓

$$\begin{aligned}
\left|\int_0^1 f(t)\,dt - \int_0^1 g(t)\,dt\right| &= \left|\int_0^1 f(t) - g(t)\,dt\right| \\
&\leq \int_0^1 |f(t) - g(t)|\,dt \\
&\leq (1-0) \sup_{t\in[0,1]} |f(t) - g(t)| = d_\infty(f,g).
\end{aligned}$$

Thus, for every $\epsilon > 0$ we can let $\delta = \epsilon$. 5, B

(d) Yes. For f and g in $C([0,1])$, by Cauchy-Schwarz inequality, we have unseen ⇓

$$\begin{aligned}
\left|\int_0^1 f(t)\,dt - \int_0^1 g(t)\,dt\right|^2 &= \left|\int_0^1 (f(t) - g(t)) \cdot 1\,dt\right|^2 \\
&\leq \int_0^1 |f(t - g(t)|^2\,dt \cdot \int_0^1 1^2\,dt \\
&= (d_2(f,g))^2.
\end{aligned}$$

Thus, for every $\epsilon > 0$ we can let $\delta = \epsilon$. 5, C

4. (a) The function e^{iz^2} is holomorphic in $\mathbb{C}$, because it is a composition of two holomorphic functions e^{iw} and $w = z^2$. sim. seen ⇓

2, A

(b) If $z = x + iy$ we have sim. seen ⇓

$$e^{iz^2} = e^{i(x^2-y^2)-2xy} = e^{-2xy}(\cos(x^2 - y^2) + i\sin(x^2 - y^2)).$$

Denoting $u(x, y) = e^{-2xy}\cos(x^2 - y^2)$ and $v(x, y) = e^{-2xy}\sin(x^2 - y^2)$ we find

$$u'_x = e^{-2xy}(-2y\cos(x^2 - y^2) - 2x\sin(x^2 - y^2)) = v'_y \quad \text{and}$$
$$u'_y = e^{-2xy}(-2x\cos(x^2 - y^2) + 2y\sin(x^2 - y^2)) = -v'_x.$$

4, A

(c) Let $f(z) = u(x, y) + iv(x, y)$, where $z = x + iy$, be holomorphic in Ω. Then $u'_x = v'_y$ and $u'_y = -v'_x$. unseen ⇓

Consider $g(z) = \overline{f(z)} = u(x, y) - iv(x, y)$ and assume that g is holomorphic. Then $u'_x = -v'_y$ and $u'_y = v'_x$ which implies $u'_x = u'_y = v'_x = v'_x = 0$. Thus $f(z) \equiv \mathrm{const}$.

5, A

(d) *Step 1.* The function u is harmonic. Indeed, sim. seen ⇓

$$u'_x = e^{x^2-y^2}(2x\cos(2xy) - 2y\sin(2xy)),$$

$$u''_{xx} = e^{x^2-y^2}\left((4x^2 + 2 - 4y^2)\cos(2xy) - 8xy\sin(2xy)\right).$$

and respectively

$$u'_y = e^{x^2-y^2}(-2y\cos(2xy) - 2x\sin(2xy))$$

$$u''_{yy} = e^{x^2-y^2}\left((4y^2 - 2 - 4x^2)\cos(2xy) + 8xy\sin(2xy)\right).$$

Therefore $\Delta u = u''_{xx} + u''_{yy} = 0$. (4 marks)

Step 2. Using the C-R equation $u'_x = v'_y$ we find

$$v(x, y) = \int v'_y(x, y)\,dy = \int e^{x^2-y^2}(2x\cos(2xy) - 2y\sin(2xy))\,dy$$
$$= e^{x^2-y^2}\sin(2xy) + C(x).$$

(3 marks)

Derivating v w.r.t. x we find

$$v'_x = e^{x^2-y^2}(2x\sin(2xy) + 2y\cos(2xy)) + C'(x)$$
$$= -u'_y = e^{x^2-y^2}(2y\cos(2xy) + 2x\sin(2xy)).$$

Therefore $C(x) = \mathrm{const}$ and we finally obtain

$$f(z) = e^{x^2-y^2}(\cos(2xy) + i\sin(2xy)) + iC = e^{z^2} + iC.$$

Since $f(0) = 1$ we conclude $C = 0$.

(2 marks) 9, B

5. (a) The point z_0 is a removable singularity if in the Laurent expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ the coefficients $a_n = 0$ for all $n < 0$. seen ⇓ 2, A

(b) (one mark for each correct answer) (i) yes; (ii) no; (iii) yes. unseen ⇓ 3, A

(c) sim. seen ⇓

$$\frac{z+1}{z(z-4)^3} = \frac{1}{(z-4)^3}\,\frac{(z-4)+5}{4+z-4} = \frac{1}{4(z-4)^3}\,\frac{(z-4)+5}{1+\frac{z-4}{4}}$$
$$= \frac{1}{4(z-4)^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}(z-4)^n + \frac{5}{4(z-4)^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}(z-4)^n$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(z-4)^{n-2} + \sum_{n=0}^{\infty}\frac{5(-1)^n}{4^{n+1}}(z-4)^{n-3}$$
$$= \sum_{n=-2}^{\infty}\frac{(-1)^n}{4^{n+3}}(z-4)^n + \sum_{n=-3}^{\infty}\frac{5(-1)^{n+1}}{4^{n+4}}(z-4)^n.$$

4, A

(d) Let sim. seen ⇓

$$f(z) = \frac{e^{iz}}{z^2-2z+10}$$

and let

$$\gamma = \gamma_1 \cup \gamma_2, \quad \text{where} \quad \gamma_1 = \{z = x+iy : -R \le x \le R,\ y = 0\},$$
$$\gamma_2 = \{z = R\,e^{i\vartheta} : 0 \le \vartheta \le \pi\}, \qquad R > 4.$$

The function $f(z)$ has one pole in the upper half plane $z_0 = 1+3i$. Then

$$\oint_\gamma f(z)\,dz = \oint_\gamma \frac{e^{iz}}{z^2-2z+10}\,dz = 2\pi i\,\mathrm{Res}[f,z_0]$$
$$= 2\pi i \lim_{z\to 1+3i}\frac{e^{iz}}{z^2-2z+10} = 2\pi i \left.\frac{e^{iz}}{2z-2}\right|_{z=1+3i}$$
$$= \pi\frac{e^{i(1+3i)}}{3} = \pi\frac{e^{-3}}{3}(\cos 1 + i\sin 1).$$

(3 marks)

Next

$$\oint_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz.$$

Note that

$$\lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz = \int_{-\infty}^{\infty}\frac{\cos x + i\sin x}{x^2-2x+10}\,dx.$$

Moreover, using the ML-inequality we find

$$\left|\int_{\gamma_2} f(z)\,dz\right| = \left|\int_0^\pi \frac{e^{iR(\cos\vartheta + i\sin\vartheta)}}{R^2e^{2i\vartheta}-2Re^{i\vartheta}+10}\,R\,e^{i\vartheta}\,d\vartheta\right|$$
$$\le \pi R\,\frac{e^{-R\sin\vartheta}}{R^2-2R-10} \to 0, \quad \text{as} \quad R\to\infty.$$

Finally we obtain

$$\int_{-\infty}^{\infty}\frac{\cos x}{x^2-2x+10}\,dx = \mathrm{Re}\oint_\gamma f(z)\,dz = \pi\frac{e^{-3}}{3}\cos 1.$$

(3 marks) 6, C

(e) Consider $\{z : |z| = 2\}$ and let $P(z) = f(z) + g(z) = z^5 - 12z^2 + 14$, where $f(z) = -12z^2$ and $g(z) = z^5 + 14$. Then

sim. seen $\Downarrow$

$$|f(z)| = |-12z^2| = 48 \quad \text{and} \quad |g(z)| \leq |z^5| + 14 = 46.$$

This implies that if $|z| = 2$, then $|f(z)| > |g(z)|$. Since the equation $f(z) = -12z^2 = 0$ has two roots in $\{z : |z| < 2\}$, applying Rouche's theorem we find that $P(z)$ has two roots in $\{z : |z| < 2\}$.

(3 marks)

Let now $\{z : |z| = 1\}$, $f(z) = 14$ and $g(z) = z^5 - 12z^2$. Then

$$|g(z)| \leq |z^5| + |-12z^2| = 13 < 14.$$

This implies that $P(z)$ has no root in $\{z : |z| \leq 1\}$.

Answer: P has two roots in $\Omega = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

(2 marks) 5, C

6. (a) From the maximum modulus principle it follows that $\max_{z:|z|\le r}|f(z)|$ is achieved on the boundary and therefore equals $M(r)$. unseen ⇓

2, A

(b) Let us introduce $\varphi(\eta) = f(1/\eta)$. Then φ is holomorphic in $\{\eta : |\eta| \le 1/R\}$. It follows from **(a)** that the function $m(\rho) = \max_{\eta:|\eta|\le\rho}|\varphi(\eta)|$ is strictly increasing on $(0, 1/R]$. This implies that $M(r)$ is strictly decreasing. unseen ⇓

4, B

(c) By using the Cross-Ratio formula we have sim. seen ⇓

$$\frac{z+1}{z-i}\,\frac{1+i-i}{1+i+1} = \frac{w-i}{1-i}.$$

This implies

$$w = \frac{(1+2i)z+6-3i}{5(z-i)} = u + iv.$$

(3 marks)

The points $(z_1, z_2, z_3) = (-1, i, 1+i)$ belong to the circle

$$\partial D = \left\{ z \in \mathbb{Z} : \left| z - \frac{1-i}{2} \right| = \sqrt{\frac{5}{2}} \right\}.$$

The image $f(\partial D)$ is the straight line $\gamma = \{(u,v) \in \mathbb{R}^2 : u + v = 1\}$. Moving along the images $f(z_1) \to f(z_2) \to f(z_2)$ we find the orientation of γ. From this we conclude that the disc D is mapped to $\{(u,v) \in \mathbb{R}^2 : u + v > 1\}$.

(2 marks) 5, D

(d) Since f is holomorphic in D_R, unseen ⇓

$$0 = \oint_{|z-z_0|=r} f(z)\,dz = \oint_{|z|=r} f(z+z_0)\,dz = ir\int_0^{2\pi} f\left(z_0 + re^{i\vartheta}\right)\,e^{\vartheta}\,d\vartheta.$$

If we divide both sides by $\frac{i}{2\pi i r^2}$ and take the complex conjugate we obtain

$$0 = \frac{1}{2\pi r}\int_0^{2\pi} \overline{f\left(z_0 + re^{i\vartheta}\right)}\,e^{-i\vartheta}\,d\vartheta. \qquad (*)$$

(3 marks)

Using the generalised Cauchy's integral formula for the derivative $f'(z_0)$ we have

$$\begin{aligned} f'(z_0) &= \frac{1}{2\pi i}\oint_{|z-z_0|=r}\frac{f(z)}{(z-z_0)^2}\,dz = \frac{1}{2\pi i}\oint_{|z|=r}\frac{f(z_0+z)}{z^2}\,dz \\ &= \frac{1}{2\pi i}\int_0^{2\pi}\frac{f\left(z_0+re^{i\vartheta}\right)}{r^2\,e^{2i\vartheta}}\,ri\,e^{i\vartheta}\,d\vartheta = \frac{1}{2\pi r}\int_0^{2\pi} f\left(z_0+re^{i\vartheta}\right)\,e^{-i\vartheta}\,d\vartheta. \qquad (**) \end{aligned}$$

(3 marks)

Adding $(*)$ and $(**)$ we find

$$\begin{aligned} 0 + f'(z_0) &= \frac{1}{2\pi r}\int_0^{2\pi}\left(\overline{f\left(z_0+re^{i\vartheta}\right)} + f\left(z_0+re^{i\vartheta}\right)\right)\,e^{-i\vartheta}\,d\vartheta \\ &= \frac{1}{2\pi r}\int_0^{2\pi} 2\,\mathrm{Re}\left(f\left(z_0+re^{i\vartheta}\right)\right)\,e^{-i\vartheta}\,d\vartheta = \frac{1}{\pi r}\int_0^{2\pi} P(\vartheta)e^{-i\vartheta}\,d\vartheta. \end{aligned}$$

(3 marks) 9, D

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
Math50001	1	a) Most students are comfortable with this, but a few students forgot to mention that continuous partial derivatives on "an open set", implies that the map is differentiable. Part b) is straigthforward, in part b, many students forgot to verify the assumption of "continuously differentiable" in the inverse function theorem, which is crucial. Part b, was mostly treated well, except that most students applied the Mean Value Theorem on a potentially disconnected interval.
Math50001	2	Part (a) could be solved either by compactness arguments or by bounding the norm of Av directly. For the latter approach, which most students chose, the bound derived needs to be uniform in v. Item (b) also required verifying that M_n is a vector space, which many students forgot. In part (c) some care is needed since continuity is meant with respect to the norm d_M introduced in the problem. Parts (d) and (e) were solved very well by most.
Math50001	3	Ex a), b) and c) went overall quite well. In Ex a), a common mistake was to write the boundary only for the region of E with x<0. Recall that the boundary of a topological subspace is not necessarily contained in the subspace itself. Ex. d) was harder. Many students wrote that the function was not continuous or they did not justify their answer.
Math50001	4	Most of the students were able to solve Q4. A small miniroty had some misunderstandings when solving Q4 C).

Math50001	5	This question seemed relatively straightforward and most people got a high mark on it. The most difficult part was part d): lots of students didn't think of using the complex exponential and seeing the cos as its real part ending up then with an imaginary result. Part e): everyone thought of using Rouché, there were quite a few mistakes computing how many roots were in each part and more careful explanations on the method would have been welcome.
Math50001	6	General level was a bit low.