Linear Algebra **January Test 2021**

1. Let F be a field, and let A be an $n \times n$ matrix over F.

(a) Define the *minimal polynomial* $m_A(x)$ of A. Prove that if $p(x) \in F[x]$ is a polynomial such that $p(A) = 0$, then $p(x)$ is divisible by $m_A(x)$.

(b) Suppose $F = \mathbb{R}$, $n \geq 3$ and $A^3 = I$, the identity matrix. Write down all the possible minimal polynomials $m_A(x)$, and give examples of matrices A over $\mathbb{R}$ with each of these minimal polynomials.

(c) Find all the Rational Canonical Forms of 6×6 matrices A over $\mathbb{R}$ such that $A^3 = I$.

(d) Now let $F = \mathbb{F}_3$, the field of 3 elements. Giving your reasoning, calculate the number of different Jordan Canonical Forms of 6×6 matrices A over F such that $A^3 = I$.

(**Marks for parts a,b,c,d: 3,6,4,7**)

2. Let V be a finite-dimensional inner product space over F, where $F = \mathbb{R}$ or $\mathbb{C}$, and let $T : V \to V$ be a linear map.

(a) Define what is meant by the *adjoint* map $T^* : V \to V$.

(b) Show that $(T^*)^* = T$.

(c) Prove that T^*T is self-adjoint.

(d) Show that if λ is an eigenvalue of T^*T, then λ is real and $\lambda \geq 0$.

(e) Using the Spectral Theorem for T^*T (or otherwise), prove that if λ_{max} is the largest eigenvalue of T^*T, then

$$||T(v)||^2 \leq \lambda_{max}||v||^2 \qquad \text{for all } v \in V.$$

(f) Now let V be the vector space of polynomials over $\mathbb{R}$ of degree at most 1, with inner product

$$(f, g) = \int_{-1}^{1} f(x)g(x)\,dx \quad \text{for all } f, g \in V.$$

Let $T : V \to V$ be the linear map defined by

$$T(f(x)) = f(x) + f'(x) \qquad \text{for all } f(x) \in V.$$

Calculate the largest eigenvalue of T^*T.

(**Marks for parts a,b,c,d,e,f: 2,2,2,4,4,6**)

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1. (a) The min poly of A is the (unique) polynomial $m(x) \in F[x]$ such that $m(A) = 0$, $m(x)$ is monic, and $\deg(m(x))$ is as small as possible. (**1 mark**)

Suppose $p(x) \in F[x]$ and $p(A) = 0$. There are polys $q, r \in F[x]$ such that $\deg(r(x)) < \deg(m_A(x))$ and

$$p(x) = q(x)m_A(x) + r(x).$$

Then $0 = p(A) = q(A)m_A(A) + r(A) = r(A)$. As $\deg(r(x)) < \deg(m_A(x))$, this implies that $r = 0$, hence m_A divides p. (**2 marks, in lec notes**)

(b) By (a), $m_A(x)$ divides $x^3 - 1$. Over $\mathbb{R}$, we have $x^3 - 1 = (x-1)(x^2 + x + 1)$ and $x^2 + x + 1$ is irreducible. Hence the possibilities for $m_A(x)$ are

$$x - 1,\ x^2 + x + 1,\ x^3 - 1. \quad (\textbf{3 marks})$$

Examples of matrices with these respective minimal polys are

$$I, \quad \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}. \quad (\textbf{3 marks})$$

(c) The RCFs are block-diagonal sums of companion matrices $C_1 = (1)$ and $C_2 = C(x^2 + x + 1) = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. The 6×6 RCFs are therefore

$$I_6,\ \ I_4 \oplus C_2,\ \ I_2 \oplus C_2 \oplus C_2,\ \ C_2 \oplus C_2 \oplus C_2. \quad (\textbf{4 marks})$$

(d) Over $\mathbb{F}_3$, the poly $x^3 - 1$ factorizes as $(x-1)^3$. (**2 marks**)

So any matrix A over $\mathbb{F}_3$ satisfying $A^3 = I$ has min poly $(x-1)^a$ with $a \leq 3$, hence has a JCF with maximal block size at most 3. (**1 mark**)

Writing $J_i = J_i(1)$, the possible JCFs are

$$J_3^2,\ J_3 \oplus J_2 \oplus J_1,\ J_3 \oplus J_1^3,\ J_2^3,\ J_2^2 \oplus J_1^2,\ J_2 \oplus J_1^4,\ J_1^6.$$

There are 7 of these altogether. (**4 marks**)

TOTAL: 20

2. (a) The adjoint map $T^* : V \to V$ is the unique linear map such that

$$(T(v), w) = (v, T^*(w)) \qquad \text{for all } v, w \in V. \quad (\textbf{2 marks})$$

(b) From the above equation,

$$(v, (T^*)^* w) = (T^*(v), w) = \overline{(w, T^*(v))} = \overline{(T(w), v)} = (v, T(w)).$$

Hence $(T^*)^*(w) = T(w)$ for all $w \in V$, ie. $(T^*)^* = T$. (**2 marks, seen**)

(c) From the defining equation, for any $v, w \in V$,

$$(T^*T(v), w) = (T(v), T^{**}(w)) = (T(v), T(w)) = (v, T^*T(w)).$$

Hence T^*T is self-adjoint. (**2 marks**)

(d) Let $S = T^*T$, self-adjoint. If λ is an evalue of S, and v an evector such that $S(v) = \lambda v$, then

$$(S(v), v) = (v, S^*(v)) = (v, S(v)) \Rightarrow (\lambda v, v) = (v, \lambda v) \Rightarrow \lambda(v, v) = \bar{\lambda}(v, v) \Rightarrow \lambda \in \mathbb{R}.$$

(**2 marks, seen**)

Also

$$\lambda(v, v) = (S(v), v) = (T^*T(v), v) = (T(v), T(v)) \geq 0.$$

Hence $\lambda \geq 0$. (**2 marks**)

(e) By the Spectral Theorem, there is an orthonormal basis $v_1, \ldots, v_n$ of V consisting of eigenvectors for $S = T^*T$. Say $S(v_i) = \lambda_i v_i$ for $1 \leq i \leq n$. Then for $v = \sum_1^n \alpha_i v_i \in V$, we have $S(v) = \sum_1^n \lambda_i \alpha_i v_i$, so

$$\begin{aligned} ||T(v)||^2 &= (T(v), T(v)) \\ &= (T^*T(v), v) \\ &= (\textstyle\sum \lambda_i \alpha_i v_i, \sum \alpha_i v_i) \\ &= \textstyle\sum \lambda_i \alpha_i \bar{\alpha}_i \\ &\leq \lambda_{max} \textstyle\sum \alpha_i \bar{\alpha}_i \\ &= \lambda_{max} ||v||^2. \quad (\textbf{4 marks}) \end{aligned}$$

(f) First need to find an orthonormal basis of V. Check that $B = \{f_1, f_2\}$ is orthormal, where

$$f_1 = \frac{1}{\sqrt{2}}, \ f_2 = \sqrt{\frac{3}{2}} x. \quad (\textbf{2 marks})$$

Then T sends $f_1 \to f_1$, $f_2 \to f_2 + \sqrt{3} f_1$, so the matrix of T wrt B is

$$A = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}. \quad (\textbf{1 mark})$$

By a result in lectures, the matrix of T^* wrt B is A^T, so the matrix of T^*T is

$$A^T A = \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 4 \end{pmatrix}. \quad (\textbf{1 mark})$$

This has characteristic poly $x^2 - 5x + 1$, so the largest eigenvalue is $\frac{1}{2}(5 + \sqrt{21})$. (**2 marks**)