Linear Algebra **January Test 2022**

1. (a) Let V be a finite-dimensional vector space over $\mathbb{C}$, and suppose $S : V \mapsto V$ and $T : V \mapsto V$ are linear maps that commute with each other (in other words, $ST = TS$).

(i) Let λ be an eigenvalue of T, and let the corresponding eigenspace be

$$V_\lambda = \{v \in V : T(v) = \lambda v\}.$$

Prove that V_λ is S-invariant.

(ii) Hence show that S and T have a common eigenvector (i.e. a vector v that is an eigenvector of both S and T).

(iii) Prove that there is a basis B of V such that both matrices $[S]_B$ and $[T]_B$ are upper triangular.

(b) Let $V = M_n(\mathbb{C})$, the vector space of $n \times n$ matrices over $\mathbb{C}$. Let B be a fixed matrix in V, and define linear maps $S : V \mapsto V$ and $T : V \mapsto V$ by

$$\begin{aligned} S(A) &= AB^T + BA \quad \text{for all } A \in V, \\ T(A) &= AB^T - BA \quad \text{for all } A \in V \end{aligned}$$

(where as usual B^T is the transpose of B).

(i) Show that S and T commute with each other.

(ii) In the case where $n = 2$ and $B = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$, find a common eigenvector in V for S and T.

Marks for part (a): 3,3,5; for part (b): 3,6

2. (a) For a field F, let $M_n(F)$ be the set of all $n \times n$ matrices over F. Recall that the relation defined on $M_n(F)$ by

$$A \sim B \Leftrightarrow A \text{ is similar to } B$$

is an equivalence relation, and we call the equivalence classes of this relation the *similarity classes* in $M_n(F)$.

(i) Giving your reasoning, calculate the number of similarity classes of matrices in $M_8(\mathbb{C})$ that have minimal polynomial $(x^3 + x + 1)^2$.

(ii) Calculate the number of similarity classes of matrices in $M_8(\mathbb{R})$ that have minimal polynomial $(x^3 + x + 1)^2$.

(iii) Calculate the number of similarity classes of matrices in $M_8(\mathbb{F}_2)$ that have minimal polynomial $(x^3 + x + 1)^2$.

(b) Let A be a matrix in $M_4(\mathbb{C})$ such that A is similar to A^2.

(i) Find all the possible Jordan Canonical Forms for A if its characteristic polynomial is x^4.

(ii) Find all the possible Jordan Canonical Forms for A if its characteristic polynomial is $(x-1)^4$.

Marks for part (a): 4,4,4; for part (b): 4,4

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1. (a) (i) Let $v \in V_\lambda$. Then $TS(v) = ST(v) = S(\lambda v) = \lambda S(v)$, so $S(v) \in V_\lambda$. Hence V_λ is S-invariant. **(3 marks)**

(ii) By (i), V_λ is S-invariant. As we are over $\mathbb{C}$, the restriction S_{V_λ} has an eigenvector. This is a common eigenvector of S and T. **(3 marks)**

(iii) This just follows the proof of the Triangularisation Theorem in the lectures. Use induction on $n = \dim V$. The result is obvious if $n = 1$. Assume $n \geq 2$. By (ii) there is a common evector v_1 of S and T, so the subspace $W = \mathrm{Sp}(v_1)$ is both S- and T-invariant. Therfore we have quotient maps $\bar{S}$ and $\bar{T}$ from $V/W \mapsto V/W$. By induction, there is a basis $W + v_2, \dots W + v_n$ of V/W with respect to which the matrices of $\bar{S}$ and $\bar{T}$ are both upper triangular. Then $B = \{v_1, v_2, \dots, v_n\}$ is a basis of V such that $[S]_B$ and $[T]_B$ are both upper triangular. **(5 marks)**

(b) (i) Check that ST and TS both send $A \mapsto A(B^T)^2 - B^2 A$. **(3 marks)**

(ii) This can be done by a computation finding the matrices of S and T with respect to the standard basis of $M_2(\mathbb{C})$, then computing an eigenspace of T and finding an eigenvector of S inside that. But here is a more cunning method: let v be an eigenvector of B with $Bv = \lambda v$, and let $A = vv^T$, a 2×2 matrix. Then

$$S(A) = vv^T B^T + Bvv^T = \lambda vv^T = \lambda A, \quad T(A) = vv^T B^T - Bvv^T = 0.$$

Hence $A = vv^T$ is a common evector of S, T. Compute that $v = (1, -1)^T$ is an evector of B. Hence $A = vv^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$ is a common evector of S and T. **(6 marks)**

TOTAL: 20

2. (a) (i) Over $\mathbb{C}$, $(x^3 + x + 1)^2$ factorizes as a product $(x - \alpha)^2 (x - \beta)^2 (x - \gamma)^2$, where $\alpha, \beta, \gamma \in \mathbb{C}$ are distinct. The possible JCFs in $M_8(\mathbb{C})$ with this min poly have blocks $J_2(\alpha)$, $J_2(\beta)$, $J_2(\gamma)$, and either one more block of size 2 (three possibilities), or two more blocks of size 1 (six possibs – three with identical size 1 blocks, three with different size 1 blocks). So the total number of JCFs is 9. **(4 marks)**

(ii) Over $\mathbb{R}$, $x^3 + x + 1$ has one real root and two complex conjugate ones, hence factorizes as $(x - \delta)(x^2 + ax + b) = f_1 f_2$. The possible RCFs in $M_8(\mathbb{R})$ with min poly $(x^3 + x + 1)^2 = f_1^2 f_2^2$ have blocks $C(f_1^2)$ and $C(f_2^2)$, and either one more block $C(f_2)$ or $C(f_1^2)$ of size 2, or two more blocks $C(f_1)^2$. Hence the total number of RCFs is 3. **(4 marks)**

(iii) Over $\mathbb{F}_2$, $f = x^3 + x + 1$ is irreducible. The RCF of a matrix in $M_8(\mathbb{F}_2)$ with min pol f^2 must have a block $C(f^2)$, which is of size 6. Any further block must be $C(f^i)$ for some i, so have size at least 3. This is not possible, so there are no such matrices in $M_8(\mathbb{F}_2)$. **(4 marks)**

(b) (i) If the char poly of A is x^4, its JCF has blocks $J = J_r(0)$. If $r \geq 2$, then J^2 has rank less than J. As A is similar to A^2, they have the same JCF, so it follows that the only possible Jordan blocks are $J_1(0)$. So the only possible JCF is $J_1(0)^4$, which is the zero matrix. **(4 marks)**

(ii) If the char poly of A is $(x - 1)^4$, its JCF has blocks $J = J_r(1)$. Now J^2 has 1's on the diagonal and 2's above the diagonal, so $\mathrm{rank}(J^2 - I) = r - 1$, and so the JCF of J^2 is also $J_r(1)$. Hence $J^2 \sim J$ for all Jordan blocks $J = J_r(1)$, and so all the 4×4 JCFs with evalue 1 are possible for A. The number of such JCFs is the number of partitions of 4 as a sum of positive integers, which is 5. **(4 marks)**

TOTAL: 20