Imperial College London

MATH50003

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Linear Algebra and Numerical Analysis

Date: Wednesday, 12 May 2021

Time: 09:00 to 12:00

Time Allowed: 3 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let F be a field, and let A be the following 4×4 matrix over F:

$$A = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

(i) Find the characteristic polynomial of A. (1 mark)

(ii) Find the minimal polynomial of A. (3 marks)

(iii) For which of the following fields F is A diagonalisable over F:

(α) $F = \mathbb{C}$?

(β) $F = \mathbb{R}$?

(γ) $F = \mathbb{F}_3$, the field of 3 elements?

Give your reasoning. (6 marks)

(b) Let B be an $n \times n$ matrix over a field F, and suppose that B has minimal polynomial $x^2 + x + 1$.

(i) Show that B is invertible. (1 mark)

(ii) Prove that the minimal polynomial of B^{-1} is also $x^2 + x + 1$. (3 marks)

(iii) Prove that if $F = \mathbb{C}$ and n is odd, then B is not similar to B^{-1}. (3 marks)

(iv) Prove that if $F = \mathbb{R}$ and n is even, then B is similar to B^{-1}. (3 marks)

(You may use any results from the course that you require provided you state them clearly.)

(Total: 20 marks)

2. (a) Let p be a prime number, let $\mathbb{F}_p$ denote the field of p elements, and let A be the following 3×3 matrix over $\mathbb{F}_p$:

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

(i) Show that the characteristic polynomial of A factorizes as a product of linear factors.

(ii) Find the Jordan Canonical Form of A (the answer will depend on p).

(iii) Let $p = 2$, let V be the vector space $(\mathbb{F}_2)^3$, and let $T : V \mapsto V$ be the linear map defined by $T(v) = Av$ ($v \in V$). Find a Jordan basis for T. (7 marks)

(b) Let V be a vector space, and let $T : V \mapsto V$ be a linear map with the following properties:

(1) the characteristic polynomial $c_T(x) = x^5(x-1)^6$,

(2) rank$(T) = 8$ and rank$(T^2) = 7$,

(3) rank$(T - I) = 9$ and rank$(T - I)^3 = 6$.

Find the Jordan Canonical Form of T. (6 marks)

(c) Let F be a field, let n be a positive integer, and let $c(x) \in F[x]$ be a polynomial of degree n. Let $m(x) \in F[x]$ be a polynomial satisfying the following two conditions:

(1) $m(x)$ divides $c(x)$,

(2) if $p(x) \in F[x]$ is an irreducible factor of $c(x)$, then $p(x)$ divides $m(x)$.

Decide whether the following statement is true or false: there exists an $n \times n$ matrix over F that has characteristic polynomial $c(x)$ and minimal polynomial $m(x)$. If you think it is true, give a proof, and if you think it is false, give a counterexample. (7 marks)

(You may use any results from the course that you require provided you state them clearly.)

(Total: 20 marks)

3. (a) Let V be a finite-dimensional inner product space over $\mathbb{C}$, and let S, T be linear maps $V \mapsto V$.

(i) Define the adjoint map $T^* : V \mapsto V$.

(ii) Show that $(ST)^* = T^* S^*$.

(iii) Prove that if T is invertible, then T^* is also invertible and $(T^*)^{-1} = (T^{-1})^*$. (5 marks)

(b) Let $V = \mathbb{C}^2$, and for $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in V$, define

$$(u, v) = u_1 \bar{v}_1 + i u_1 \bar{v}_2 - i u_2 \bar{v}_1 + 2 u_2 \bar{v}_2.$$

(i) Show that this is an inner product on V.

(ii) Find an orthonormal basis of V with respect to this inner product.

(iii) Define a linear map $T : V \mapsto V$ by

$$T(v) = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} v \quad (v \in V).$$

Find a complex 2×2 matrix A such that $T^*(v) = Av$ for all $v \in V$, where T^* is the adjoint of T with respect to the above inner product. (8 marks)

(c) Let V be a finite-dimensional inner product space over $\mathbb{C}$ with inner product $(\, , \,)$, and let $a, b \in V \setminus 0$. Define $T : V \mapsto V$ by

$$T(v) = (v, a)\, b \qquad \text{for all } v \in V.$$

(i) Show that T is a linear map.

(ii) For $v \in V$, find $T^*(v)$ in terms of v, a, b.

(iii) Prove that if $T = T^*$, then $b = \lambda a$ for some $\lambda \in \mathbb{R}$. (7 marks)

(Total: 20 marks)

4. (a) Consider the overdetermined linear system $\mathsf{A}\mathbf{x} = \mathbf{b}$ with $\mathsf{A} \in \mathbb{R}^{n\times 1}$ and $\mathbf{b} \in \mathbb{R}^n$ of n equations with one unknown, where

$$\mathsf{A} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Solve the above system as a least squares problem by minimizing the residual. (8 marks)

(b) We try to fit a circle with center (c_1, c_2) and radius r in a least-squares sense to n given points $(x_i, y_i),\ i = 1, \ldots, n$ in the (x, y)-plane, where $n \geq 3$.

(i) Derive a system of equations for c_1, c_2 and r. (4 marks)

(ii) Use the substitution $c_1 = \alpha/2, c_2 = \beta/2$ and $r^2 = \gamma + c_1^2 + c_2^2$ to bring the system from Part (i) into a form $\mathsf{A}\mathbf{x} = \mathbf{b}$. State the form of the matrix A and vector $\mathbf{b}$ and express the desired variables (c_1, c_2, r) in terms of the solution $\mathbf{x}$. (6 marks)

(iii) Comment on why the condition $n \geq 3$ is necessary. (2 marks)

(Total: 20 marks)

5. (a) Let $T_k(x)$ denote the Chebyshev polynomial of degree k. Express the composite polynomial $T_n(T_m(x))$ of degree $n \cdot m$ in terms of Chebyshev polynomials.

(5 marks)

(b) Consider the data $f(0) = 5, f(1) = 3, f(3) = 5, f(4) = 12.$

(i) Obtain the interpolating polynomial $p_3(x)$ in Newton form. (2 marks)

(ii) The data suggest that f has a minimum between $x = 1$ and $x = 3$. Find an approximate value for the location $x_{\min}$ of the minimum by considering the interpolating polynomial. (3 marks)

(c) Let $f(x) = (1 + a)^x, \quad |a| < 1$. Show that the Newton polynomial $p_n(x)$ with interpolation points $0, \ldots, n$ is the truncation of the binomial series for f to $n + 1$ terms, i.e.,

$$p_n(x) = \sum_{k=0}^{n} \binom{x}{k} a^k.$$

(10 marks)

(Total: 20 marks)

6. (a) Obtain the three-point Gauss-Hermite quadrature formula

$$\int_{-\infty}^{\infty} f(t)e^{-t^2}\ dt \approx w_0 f(t_0) + w_1 f(t_1) + w_2 f(t_2).$$

Derive analytic expressions for the sampling points $\{t_0, t_1, t_2\}$ and integration weights $\{w_0, w_1, w_2\}$. Use the identity $\int_0^\infty t^{2n}\exp(-t^2)\ dt = \sqrt{\pi}\frac{(2n)!}{2^{2n+1}n!}$. (13 marks)

(b) We wish to evaluate the integral $\int_{-1}^{1}\exp(-x)\ dx$ using Gauss quadrature (based on Legendre polynomials). Give an estimate of the number of sampling points until the integration error reaches machine precision $\epsilon \approx 10^{-14}$. Justify your answer. (7 marks)

(Total: 20 marks)

Linear Algebra and Numerical Analysis MATH 50003/50012/50016 Solutions

1. (a) (i) Characteristic poly is $x^4+2x^3+3x^2+2x+1=(x^2+x+1)^2$. (**1 mark, category A**)

(ii) Check that $A^2+A+I=0$, so the min poly $m_A(x)$ divides x^2+x+1. The min poly is obviously not linear as A is not a scalar multiple of I, hence $m_A(x)=x^2+x+1$. (**3 marks, A**)

(iii) (α) Over $\mathbb{C}$, $m_A(x)$ factorizes as $(x-\omega)(x-\omega^2)$, where $\omega=e^{2\pi i/3}$. This is a product of distinct linear factors, so by a result in lectures, A is diagonalisable. (**2 marks, B**)

(β) Over $\mathbb{R}$, $m_A(x)$ does not factorize as a product of distinct linear factors, so by the same result in lectures, A is not diagonalisable. (**2 marks, B**)

(γ) Over $\mathbb{F}_3$, $m_A(x)=(x-1)^2$, which again is not a product of distinct linear factors, so A is not diagonalisable. (**2 marks, B**)

(b) (i) As $B^2+B+I=0$ we have $B(-B-I)=I$, so B is invertible (with inverse $-B-I$). (**1 mark, A**)

(ii) Multiplying through by B^{-2} we get $I+B^{-1}+B^{-2}=0$, so B^{-1} satisfies x^2+x+1. The min poly of B^{-1} is not linear as B^{-1} is not a scalar mult of I. Hence the min poly is x^2+x+1. (**3 marks, A**)

(iii) Over $\mathbb{C}$, $m_B(x)$ factorizes as $(x-\omega)(x-\omega^{-1})$, where $\omega=e^{2\pi i/3}$. This is a product of distinct linear factors, so by a result in lectures, B is diagonalisable and is similar to a diagonal matrix of the form $\text{diag}(\omega I_r,\omega^{-1}I_s)$, where $r+s=n$. Then B^{-1} is similar to the inverse of this, which is $\text{diag}(\omega^{-1}I_r,\omega I_s)$. As n is odd we have $r\neq s$. Hence B and B^{-1} have different eigenvalue multiplicities, so are not similar. (**3 marks, C**)

(iv) Over $\mathbb{R}$, $m_B(x)=x^2+x+1$ is irreducible, so as the characteristic poly $c_B(x)$ has the same irreducible factors as $m_B(x)$, we have $c_B(x)=(x^2+x+1)^m$ for some m, and $n=2m$. This means that the Rational Canonical Form of B is $C\oplus\cdots\oplus C$, where C is the companion matrix of x^2+x+1, namely $C=\begin{pmatrix}0 & -1\\ 1 & -1\end{pmatrix}$. As B^{-1} also has min poly x^2+x+1, it has the same RCF as B. Hence by the RCF theorem, B is similar to B^{-1}. (**3 marks, D**)

2. (a) (i) The char poly $c_A(x) = x^3 - x^2 - x + 1 = (x-1)^2(x+1)$. (**1 mark, A**)

(ii) The min poly of A is also $(x-1)^2(x+1)$. So if $p \neq 2$, the JCF is $J_2(1) \oplus J_1(-1)$; and if $p = 2$ the JCF is $J_3(1)$. (**3 marks, A**)

(iii) For $p = 2$, T has min poly $(x-1)^3$. A basis for $(T-I)^2V$ is $v_1 = e_1 + e_3$, and we have $(T-I)(e_1+e_2) = v_1$ and $(T-I)(e_1) = e_1 + e_2$. So a Jordan basis is

$$e_1 + e_3,\ e_1 + e_2,\ e_1.$$

(**3 marks, A**)

(b) By property (1) we have $\dim V = 11$, so by (2), the geometric mult of the eigenvalue 0 is $g(0) = 3$. Hence the JCF has three 0-blocks of sizes adding up to 5. As T^2 has rank 7, the squares of these blocks have total rank 1, so they are $J_3(0), J_1(0), J_1(0)$. By (3) we have $g(1) = 2$ so there are two 1-blocks of sizes adding to 6. As $(T-I)^3$ has rank 6, the cubes of these blocks have total rank 1, so they are $J_4(1), J_2(1)$. Hence the JCF is

$$J_3(0) \oplus J_1(0)^2 \oplus J_4(1) \oplus J_2(1).$$

(**6 marks, B**)

(c) This is true. By properties (1) and (2), the factorizations of $c(x)$ and $m(x)$ in $F[x]$ are

$$c(x) = p_1(x)^{a_1} \cdots p_r(x)^{a_r}, \quad m(x) = p_1(x)^{b_1} \cdots p_r(x)^{b_r},$$

where each $p_i(x)$ is irreducible, and $a_i \geq b_i > 0$ for $1 \leq i \leq r$. For each i, define the matrix

$$C_i = C(p_i(x)^{b_i}) \oplus C(p_i(x)) \oplus \cdots \oplus C(p_i(x)),$$

where $C(f(x))$ is the companion matrix of a polynomial $f(x)$, and there are $a_i - b_i$ copies of $C(p_i(x))$. Then by standard results from the lectures, C_i has min poly $p_i(x)^{b_i}$ and char poly $p_i(x)^{a_i}$. Hence the matrix

$$C = C_1 \oplus \cdots \oplus C_r$$

has min poly $\prod_1^r p_i(x)^{b_i} = m(x)$ and char poly $\prod_1^r p_i(x)^{a_i} = c(x)$. (**7 marks, D**)

3. (a) (i) $T^* : V \mapsto V$ is the map such that $(T(v), w) = (v, T^*(w))$ for all $v, w, \in V$. (**1 mark, A**)

(ii) $(ST(v), w) = (T(v), S^*(w)) = (v, T^*S^*(w)$ for all v, w. Hence $(v, (ST)^*w) = (v, T^*S^*(w)$, which by a standard result implies that $(ST)^*(w) = T^*S^*(w)$ for all $w \in V$. (**1 mark, A**)

(iii) Applying (ii) to the equation $TT^{-1} = I$ gives $(T^{-1})^*T^* = I^*$. Since $I^* = I$, this implies that T^* has inverse $(T^{-1})^*$. (**3 marks, B**)

(b) (i) The given map (,) is left-linear and satisfies $(v, u) = \overline{(u, v)}$. Also, for $u \neq 0$,

$$\begin{aligned} (u, u) &= u_1\bar{u}_1 + iu_1\bar{u}_2 - iu_2\bar{u}_1 + 2u_2\bar{u}_2 \\ &= (u_1 - iu_2)(\bar{u}_1 + i\bar{u}_2) + u_2\bar{u}_2 \\ &= |u_1 - iu_2|^2 + |u_2|^2 > 0. \end{aligned}$$

Hence (,) is an inner product. (**2 marks, A**)

(ii) To find an orthonormal basis, we apply Gram-Schmidt to the standard basis e_1, e_2. Observe that $||e_1|| = 1$. Now let $v_2 = e_2 - (e_2, e_1)\, e_1 = e_2 + ie_1 = (i, 1)^T$ (so that $(v_2, e_1) = 0$). Then $||v_2|| = 1$ also, so an orthonormal basis is

$$e_1,\ e_2 + ie_1.$$

(**2 marks, A**)

(iii) Let $E = \{e_1, e_2\}$ be the standard basis, and $B = \{e_1,\ e_2 + ie_1\}$ the above orthonormal basis, so the change of basis matrix is $P = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$. Then $[T]_E = A$, so

$$[T]_B = P^{-1}AP = \begin{pmatrix} 2 & i \\ 0 & 1 \end{pmatrix} := X.$$

By a standard result from lectures, $[T^*]_B = \bar{X}^T$, and hence

$$[T]_E = P\bar{X}^TP^{-1} = \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2i \\ -i & 0 \end{pmatrix}.$$

This is the required matrix A. (**4 marks, B**)

(c) (i) T is linear since the inner product (,) is left-linear. (**1 mark, A**)

(ii) For $v, w \in V$,

$$(T(v), w) = ((v, a)b,\ w) = (v, a)(b, w) = (v,\ \overline{(b, w)}a) = (v,\ (w, b)a).$$

This is equal to $(v, T^*(w))$, so $T^*(w) = (w, b)\, a$ for all w. Or to change the variable to v,

$$T^*(v) = (v, b)\, a \quad \forall v \in V.$$

(**3 marks, C**)

(iii) Suppose $T = T^*$. Then $(v, a)\, b = (v, b)\, a$ for all $v \in V$. Taking $v = a$ gives $b = \frac{(a,b)}{(a,a)}a$. So $b = \lambda a$, where $\lambda \in \mathbb{C}$, and so

$$T(v) = \lambda(v, a)\, a \ \ \forall v \in V.$$

Then $(T(a), a) = \lambda(a, a)^2$. This is equal to $(a, T^*(a)) = (a, T(a)) = \bar{\lambda}(a, a)^2$. So $\lambda \in \mathbb{R}$. (**3 marks, C**)

1. (a) The square residual is given as $\|\mathsf{A}\mathbf{x} - \mathbf{b}\|_2^2$ which results in

$$r(x_1) = \sum_{i=1}^{n}(x_1 - b_i)^2 = nx_1^2 - 2x_1\sum_{i=1} b_i + C. \tag{1}$$

Setting to zero the derivative of the residual with respect to x_1 yields $2nx_1 = 2\sum_{i=1}^{n} b_i$ which results in the solution for x_1 as $x_1 = \frac{1}{n}\sum_{i=1}^{n} b_i$, i.e. the average of the entries of $\mathbf{b}$. SEEN

A: 6 marks (minimum, final solution), B: 2 marks (residual) (8 marks)

(b) We are given the generally overdetermined system

$$(x_i - c_1)^2 + (y_i - c_2)^2 = r^2, \qquad i = 1, ..., n. \tag{2}$$

We reformulate the square residual by implementing the suggested substitution of variables. We obtain

$$\begin{aligned}
0 &= (x_i - c_1)^2 + (y_i - c_2)^2 - r^2, & (3)\\
&= \left(x_i - \frac{\alpha}{2}\right)^2 + \left(y_i - \frac{\beta}{2}\right)^2 - \gamma - \left(\frac{\alpha}{2}\right)^2 - \left(\frac{\beta}{2}\right)^2, & (4)\\
&= x_i^2 + y_i^2 - x_i\alpha - y_i\beta - \gamma. & (5)
\end{aligned}$$

The last line represents a linear system in the unknown coefficients α, β, γ. We can cast the linear system in α, β, γ into matrix form $\mathsf{A}\mathbf{x} = \mathbf{b}$ as

$$\mathsf{A} = \begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} x_1^2 + y_1^2 \\ x_2^2 + y_2^2 \\ \vdots \\ x_n^2 + y_n^2 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}. \tag{6}$$

Once we have solved for α, β, γ we can recover the circle parameters as

$$c_1 = \frac{\alpha}{2} \qquad c_2 = \frac{\beta}{2} \qquad r = \sqrt{\frac{\alpha^2}{4} + \frac{\beta^2}{4} + \gamma}. \tag{7}$$

The condition $n \geq 3$ ensures unique solutions to the least-squares problem. BOOK-WORK

A: 6 marks (nonlinear residual, final solution, comment on condition), C: 3 marks (matrix system), D: 3 marks (transform)

(12 marks)

(Total: 20 marks)

2. (a) We use the substitution $x = \cos\theta$. We then have

$$T_n(T_m(\cos\theta)) = T_n(\cos m\theta) = \cos(n \cdot m\theta) = T_{nm}(\cos\theta). \tag{8}$$

We therefore have $T_n(T_m(x)) = T_{nm}(x)$. BOOK-WORK

B: 5 marks (5 marks)

(b) We have the table of divided difference

$$\begin{array}{lllll} x & f(x) & & & \\ 0 & 5 & & & \\ 1 & 3 & -2 & & \\ 3 & 5 & 1 & 1 & \\ 4 & 12 & 7 & 2 & \frac{1}{4} \end{array} \tag{9}$$

from which we can read off the Newton form of the interpolating polynomial

$$p_3(x) = 5 - 2x + x(x-1) + \frac{1}{4}x(x-1)(x-3). \tag{10}$$

In canonical form, we have $p_3(x) = 5 - \frac{9}{4}x + \frac{1}{4}x^3$.

For the minimum we have $3x^2 = 9$ and therefore $x_{\min} = \sqrt{3}$ (for a positive second derivative $p_3''(x_{\min})$). SEEN

A: 3 marks (minimum), B: 2 marks (tableau, polynomial) (5 marks)

(c) We use induction on k to find that

$$f[i, i+1, ..., i+k] = \frac{a^k}{k!} f(i). \tag{11}$$

Therefore, using this formula for $i = 0$ we get

$$p_n(x) = \sum_{k=0}^{n} f[0, 1, ..., k] \prod_{j=0}^{k-1} (x - j) = \sum_{k=0}^{n} \frac{a^k}{k!} \binom{x}{k} k! = \sum_{k=0}^{n} \binom{x}{k} a^k. \tag{12}$$

UNSEEN

C: 5 marks (induction), D: 5 marks (final solution) (10 marks)

(Total: 20 marks)

3. (a) The third-order Hermite polynomial is $H_3(t) = 8t^3 - 12t$. The roots of this polynomial provides the sampling points. We have $t_1 = 0$ and $t_{0,2} = \pm\sqrt{\frac{3}{2}}$.

For the weights we use the exactness constraint for 1 and t^2 and use the symmetry of the weights, i.e., $w_0 = w_2$. Exactness for $f(t) = 1$ produces

$$w_1 + 2w_0 = \sqrt{\pi}. \tag{13}$$

For exactness for $f(t) = t^2$ we have

$$w_0 \frac{3}{2} + w_2 \frac{3}{2} = \frac{\sqrt{\pi}}{2}. \tag{14}$$

From these two equations we obtain the weights

$$w_0 = w_2 = \frac{\sqrt{\pi}}{6} \qquad w_1 = \frac{2}{3}\sqrt{\pi}. \tag{15}$$

SEEN, BOOK-WORK

A: 5 marks (final solution), B: 3 marks (roots), C: 1 mark (symmetry), D: 4 marks (weight system) (13 marks)

(b) We approximate the exponential by a Taylor series. The n-th term is

$$\frac{x^n}{n!}. \tag{16}$$

This term is approaching machine precision $\epsilon = 10^{-14}$ at about $n = 16$ with $16! \approx 2 \cdot 10^{13}$. In this case, the x^{16}-coefficient is $4.8 \cdot 10^{-14}$. Since the Gaussian quadrature is exact for polynomials up to degree $2n + 1$ we reach machine precision at $2n + 1 = 16$ or $n \approx 8$, i.e., with eight sampling points. SEEN

A: 4 marks (Taylor series), B: 3 marks (final value) (7 marks)

(Total: 20 marks)

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.

Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH50003	1	The marks on this question were rather low. Many candidates lost marks for the straightforward parts on calculating minimal polynomials, because they omitted to prove the minimality of the degree. Parts (b)(ii),(iii) were found difficult by most and not well answered.
MATH50003	2	The marks for this question were moderate. Part (a) was a standard JCF computation and was in general quite well done; part (b) was similar to some exercises and problems covered in the lectures and was well done; part (c) was misunderstood by many candidates.
MATH50003	3	The third question had 3 parts. Part (a) was quite easy; two routine questions about adjoints and the invertibility of the adjoint of an invertible linear transformation which has a simple one line proof. The typical error was an incorrect order of the identities used and a lack of an English explanation what the identities meant. Part (b) was about an explicit pairing given by a Hermitian matrix. Typical errors were insufficient or incorrect arguments why the pairing was positive definite, a sign error in the Gram-Schmidt process caused by confusing symmetric pairings with hermitian ones, and confusing the problem of finding an orthonormal basis with eigenvectors. Part (iii) could be done without using part (ii), but this is quite computational, and many students did not explain what their computations was about. Part (c) was more interesting. Here what was missing typically is using that T is self-adjoint in part (iii) to show that lambda is real, which cannot be deduced otherwise. A lot of computations had no explanations and were going nowhere.

MATH50003	4	The students have in general done well with the linear-algebra question. In part (a), the student often did not attempt to minimize the residual directly via differentiation, but via a QR-decomposition based on Gram-Schmidt or Householder. In part (b), when attempted, the students arrived at the correct results. A common mistake in (b) was the incorrect formulation of the residual at the beginning, but no points were subtracted, if continued correctly.
MATH50003	5	The question was well done overall. Part a was very easy if you knew what you were trying to show. Part b was an exercise in deriving a Newton polynomial. Several students having obtained an incorrect polynomial, magically wrote down the correct one. Presumably, they obtained it from a calculator or some online resource. This was permitted, but it is always a good idea to explain where this has come from, rather than give the impression it follows from the line above - otherwise you have made a (further) mathematical mistake. Part c was the most challenging. Some students found arguments which are more concise and elegant than the intended proof by induction, for example by invoking the uniqueness of the polynomial interpolant. Those who used induction often suffered by not knowing in advance what initial postulate was best to assume - this led to unstructured answers. But overall, the students did well on this question.
MATH50003	6	A mixture of stronger and weaker solutions to this question. Some common pitfalls in part (a): (1) many students wasted a lot of time calculating the third Hermite polynomial when the question didn't ask to do this, and the intention in open book mode was that it could just be retrieved from notes. (2) several students used the general formula for weights and consequently lost time and made frequent algebraic errors. It is a simpler calculation to use the fact that the integral should be exact for all polynomials of degree 0,1,2, so just make a system of equations where each equation comes from integrating 1,t,t^2. (3) several students used the integral for polynomials of higher degree than this, which can work if the degree is less than 2n+1, but can also result in non-independent systems if not careful. (4) several students missed that the provided integral formula had the lower limit as zero, not minus infinity, and this introduced errors into the calculation. In part (b) the main problem was students seeming to misunderstand what was being asked.