Imperial College London

MATH50003

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2022

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science

Linear Algebra and Numerical Analysis

Date: 09 May 2022

Time: 09:00 – 12:00 (BST)

Time Allowed: 3:00 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS AS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Let n be a positive integer, and let $\lambda \in \mathbb{C}$. Define the *Jordan block* $J_n(\lambda)$.
State the Jordan Canonical Form (JCF) Theorem. (3 marks)

(b) Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}.$$

(i) Find the JCF of A.

(ii) Find an invertible 3×3 matrix P such that

$$P^{-1}AP = J,$$

where J is the JCF of A. (6 marks)

(c) Recall that a matrix is *nilpotent* if some power of it is equal to the zero matrix. Show that there is a nilpotent $n \times n$ matrix J_0 such that

$$J_n(\lambda) = \lambda I_n + J_0.$$

(You should justify that J_0 is nilpotent.) (2 marks)

(d) Now let B be an $n \times n$ matrix over $\mathbb{C}$. Using the JCF Theorem (or otherwise), prove that there exists a diagonalisable $n \times n$ matrix D, and a nilpotent $n \times n$ matrix N, such that the following two equations hold:

$$B = D + N,$$
$$DN = ND.$$ (6 marks)

(e) Let A be the 3×3 matrix in part (b). Find matrices D, N such that $A = D + N$, where D, N satisfy the properties given in part (d). (3 marks)

(Total: 20 marks)

2. (a) State the Primary Decomposition Theorem. (2 marks)

(b) Let F be a field, and let V be a finite-dimensional vector space over F. Let $f_1(x), f_2(x) \in F[x]$ be irreducible polynomials, with $\deg(f_1) = 1$, $\deg(f_2) = 2$. Let $T : V \mapsto V$ be a linear map with characteristic polynomial

$$c_T(x) = f_1(x)^k f_2(x)^l,$$

and minimal polynomial

$$m_T(x) = f_1(x)^2 f_2(x)^{l-2},$$

where $k \geq 2$ and $l \geq 4$.

(i) Let $V = V_1 \oplus V_2$ be the primary decomposition of V with respect to T. Prove that the dimensions of V_1 and V_2 are k and $2l$. (You may use any results from the lectures that you require, provided you state them clearly.)

(ii) Calculate (in terms of k, l) the number of different possible Rational Canonical Forms for T. Justify your answer. (10 marks)

(c) Calculate the total number of different similarity classes of invertible 2×2 matrices over the field $\mathbb{F}_3$ of 3 elements. Give your reasoning. (8 marks)

(Total: 20 marks)

3. (a) Let V be a finite-dimensional inner product space, and let W be a subspace of V. As in the lectures, we define the *orthogonal projection* $\pi_W : V \mapsto W$ as follows: for $v \in V$, we can write $v = w + w'$ for unique $w \in W$, $w' \in W^\perp$, and we define

$$\pi_W(v) = w.$$

(i) Prove that $(\pi_W)^2 = \pi_W$.

(ii) Show that $||\pi_W(v)|| \leq ||v||$ for all $v \in V$. (5 marks)

(b) Let $V = \mathbb{R}^3$ with the usual inner product, and let $T : V \mapsto V$ be the linear map defined by $T(v) = Av$ for all $v \in V$, where

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}.$$

Given that T is an orthogonal projection, find a subspace W such that $T = \pi_W$. (3 marks)

(c) Now suppose that V is a 2-dimensional inner product space over $\mathbb{R}$, and let $P : V \mapsto V$ be a linear map satisfying the following properties:

$P^2 = P$,

$||P(v)|| \leq ||v||$ for all $v \in V$,

$P \neq O_V, I_V$ (the zero and identity maps on V).

(i) Prove that there is a basis $B = \{v_1, v_2\}$ of V such that

$$[P]_B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

(ii) Using the second property of P listed above, prove that $(v_1, v_2) = 0$.

(iii) Deduce that P is an orthogonal projection. (12 marks)

(Total: 20 marks)

4. (a) (i) Show that

$$\frac{1}{7} = (0.001001001001\ldots)_2$$

and deduce the bits of $1/7$ in half-precision (F_{16}) floating point arithmetic (rounded to the nearest floating point number). (3 marks)

(ii) For the floating point computation

$$(1 \oslash 7) \oplus 1 = \frac{1}{7} + 1 + \delta$$

find an explicit constant c such that $|\delta| \leq c\epsilon_{\rm m}$ holds, where $\epsilon_{\rm m}$ is machine epsilon. You may assume that all computations involve normal numbers. (4 marks)

(b) Use Taylor series to derive a bound for

$$\left| f'(x) - \frac{f(x+2h) - f(x-h)}{3h} \right|$$

in terms of $M = \sup_{x-h \leq \chi \leq x+2h} |f''(\chi)|$ (assuming exact arithmetic). (4 marks)

(c) Use dual numbers, $a + b\epsilon$ such that $\epsilon^2 = 0$, to deduce the derivative of $\exp(x^2)$ at the point $x = 1/2$. (2 marks)

(d) Consider a 2D dual number $a + b\epsilon_x + c\epsilon_y$ such that $\epsilon_x\epsilon_y = \epsilon_x^2 = \epsilon_y^2 = 0$.

(i) Derive the formula for writing the product of two 2D dual numbers $(a+a_x\epsilon_x+a_y\epsilon_y)(b+b_x\epsilon_x + b_y\epsilon_y)$ where $a, a_x, a_y, b, b_x, b_y \in \mathbb{R}$ as a 2D dual number. (2 marks)

(ii) Show for all 2D polynomials

$$p(x,y) = \sum_{k=0}^{n}\sum_{j=0}^{m} c_{kj}x^k y^j$$

that

$$p(x + a\epsilon_x, y + b\epsilon_y) = p(x,y) + a\frac{\partial p}{\partial x}\epsilon_x + b\frac{\partial p}{\partial y}\epsilon_y.$$

(5 marks)

(Total: 20 marks)

5. (a) Consider the QL decomposition for a square invertible matrix $A \in \mathbb{R}^{n\times n}$:

$$A = QL$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal and L is lower triangular. Show that it is unique, provided that the diagonal entries of L are positive. (4 marks)

(b) Compute by hand the Cholesky decomposition $A = LL^\top$ of

$$A = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}.$$

(5 marks)

(c) Consider an evenly spaced grid $t_k = (k-1)h$ for $k = 1, \ldots, n$ and $h = 1/(n-1)$ with a weighted averages approximation:

$$u(t_k) \approx u_k, \qquad \frac{2u'(t_{k+1}) + u'(t_k)}{3} \approx \frac{u_{k+1} - u_k}{h}.$$

(i) Use this approximation to deduce a *lower-bidiagonal* linear system to approximate the solution to

$$\begin{aligned} u(0) &= 1 \\ u'(t) &= u(t) + \exp t. \end{aligned}$$

(5 marks)

(ii) Show that your approximation converges:

$$\sup_{1\le k\le n} |u_k - u(t_k)| \to 0$$

as $n \to \infty$, for $k = 1, \ldots, n$. You may assume that u is twice-differentiable and $|u''(t)| \le C < \infty$ for $0 \le t \le 1$, and can use without proof that

$$\lim_{n\to\infty} \left(\frac{3n+1}{3n-2}\right)^n = \mathrm{e}$$

where $\mathrm{e} = \exp 1$. Hint: for *consistency* first show for $|\delta| \le 5C/6$ that

$$\frac{u(t_{k+1}) - u(t_k)}{h} = \frac{2u'(t_{k+1}) + u'(t_k)}{3} + \delta h.$$

(6 marks)

(Total: 20 marks)

6. (a) For the function $f(\theta) = 4/(4 - \mathrm{e}^{\mathrm{i}\theta})$ state explicit formulae for its Fourier coefficients

$$\hat{f}_k := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\mathrm{e}^{-\mathrm{i}k\theta}\mathrm{d}\theta$$

and their discrete approximation:

$$\hat{f}_k^n := \frac{1}{n}\sum_{j=0}^{n-1} f(\theta_j)\mathrm{e}^{-\mathrm{i}k\theta_j}.$$

for *all* integers k, $n = 1, 2, \ldots$ and $\theta_j = 2\pi j/n$. (6 marks)

(b) Consider orthogonal polynomials

$$W_0(x) = 1, \qquad W_n(x) = 2^n x^n + O(x^{n-1})$$

as $x \to \infty$ and $n = 1, 2, \ldots$, orthogonal with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)w(x)\mathrm{d}x, \qquad w(x) = \sqrt{\frac{1-x}{1+x}}.$$

(i) Construct $W_0(x)$, $W_1(x)$ and hence show that $W_2(x) = 4x^2 + 2x - 1$. You may use without proof the formulæ

$$\int_{-1}^{1} w(x)\mathrm{d}x = \pi, \quad \int_{-1}^{1} xw(x)\mathrm{d}x = -\frac{\pi}{2},$$
$$\int_{-1}^{1} x^2 w(x)\mathrm{d}x = \frac{\pi}{2} \quad \text{and} \quad \int_{-1}^{1} x^3 w(x)\mathrm{d}x = -\frac{3\pi}{8}.$$

(4 marks)

(ii) Show for all $n = 0, 1, 2, \ldots$ that

$$W_n(\cos\theta) = \frac{\sin(n + 1/2)\theta}{\sin\frac{\theta}{2}}.$$

Hint: use the trigonometric identities:

$$2\sin\theta\sin\phi = \cos(\theta - \phi) - \cos(\theta + \phi), \qquad 2\cos\theta\sin\phi = \sin(\theta + \phi) - \sin(\theta - \phi)$$

to show that the right-hand side satisfies a 3-term recurrence. (5 marks)

(c) Compute the 2-point Gauss quadrature rule for $w(x) = \sqrt{\frac{1-x}{1+x}}$, that exactly integrates all polynomials p up to degree 3:

$$\int_{-1}^{1} p(x)w(x)\mathrm{d}x = w_1 p(x_1) + w_2 p(x_2).$$

You may use the formula for $W_2(x)$ from part (b). (5 marks)

(Total: 20 marks)