Linear Algebra, Term 1, 2021-22 **Practice Questions for Exam**

This document is intended to help you with practice questions for the summer exam in Linear Algebra (the first part in Term 1). Last year 2020-21 was the first time this course has been given, so there is only one past paper. In previous years, there was a 2nd year course called Algebra 2, M2PM2, containing the material from Chapters 1-11 of this year's course. Chapters 12-16 contain new material:

- Cyclic Decomposition and Rational Canonical Forms (Chapter 12)
- Dual Spaces (Chapter 13)
- Inner Product Spaces and Linear Maps (Chapters 14,15)
- Bilinear and Quadratic Forms (Chapter 16).

Practice Questions on Chapters 1-11

For practice questions on Chapters 1-11, you can of course use last year's paper, but you can also use many of the questions on the past exam papers for M2PM2 Algebra 2, which can be found on Blackboard. Below you will find a list of relevant questions. The standard bookwork parts are not suitable for an open-book exam. But in fact many of the questions have rather small amounts of bookwork and are fine for you to practice on. I have marked with an asterisk those questions that are particularly suitable for open-book practice.

2020, Q3

2019, Q1(b), 3

2018, Q1(b), 3*

2017, Q3, 4

2016, Q3, 4

2015, Q4

2014, Q4*

2013, Q3, 4*

2012, Q4*

2011, Q3, 4*

2010, Q3, 4

2009, Q3*, 4*

2008, Q3*, 4*

Practice Questions on Chapters 12-16

There are of course many problems on these topics on Sheets 6-10, and many of these can be treated as suitable practice questions (unless you have already worked through all of them and can perfectly remember the solutions!).

For further practice, I have written a few extra suitable questions for each topic on the next few pages.

Bilinear and Quadratic Forms

1. Let p be an odd prime number, let $\mathbb{F}_p$ be the field of order p, and let V be the vector space $\mathbb{F}_p^2$. For $x, y \in V$ define

$$(x, y) = x_1y_1 + 4x_1y_2 + 4x_2y_1 + x_2y_2,$$

where $x = (x_1\ x_2)^T$, $y = (y_1\ y_2)^T$.

(a) Show that $(\ ,\ )$ is a symmetric bilinear form on V.

(b) For which values of p is the form $(\ ,\ )$ non-degenerate?

(c) Supposing that $(\ ,\ )$ is non-degenerate, find an orthogonal basis of V.

(d) Let $Q : V \mapsto \mathbb{F}_p$ be the quadratic form defined by $Q(x) = (x, x)$ for all $x \in V$.

  (i) In the cases where $(\ ,\ )$ is degenerate, show that Q is not surjective.

  (ii) In the cases where $(\ ,\ )$ is non-degenerate, show that Q is surjective.

  (You may assume that the set of nonzero squares $\{\alpha^2 : \alpha \in \mathbb{F}_p \setminus 0\}$ has size $\frac{1}{2}(p-1)$.)

2. Let V be an n-dimensional vector space over $\mathbb{R}$.

(a) Define what is meant by a skew-symmetric bilinear form $(\ ,\ )$ on V.

(b) Show that if $v_1, \ldots, v_n$ is a basis of V, and $a_{ij} = (v_i, v_j)$, then the matrix $A = (a_{ij})$ satisfies $A^T = -A$.

(c) Let B be the set of all skew-symmetric bilinear forms on V. You are given that B is a vector space over $\mathbb{R}$, under the natural operations of scalar multiplication and addition of bilinear forms. Find an expression for $\dim B$ in terms of n (where $n = \dim V$).

(d) Now let $V = \mathbb{R}^3$. For a vector $u \in V$, define a bilinear form $(\ ,\ )_u$ on V by

$$(x, y)_u = u.(x \times y) \qquad \text{for all } x, y \in V,$$

where $\times$ is the usual vector product. Show that $(\ ,\ )_u$ is skew-symmetric.

(e) Show that every skew-symmetric bilinear form on $V = \mathbb{R}^3$ is of the form $(\ ,\ )_u$ for some vector $u \in V$.

Inner Product Spaces and Linear Maps

3. Let V be a finite-dimensional inner product space over $F = \mathbb{R}$ or $\mathbb{C}$, with inner product $(\ ,\ )$, and let $T : V \mapsto V$ be a linear map.

(a) Define what is meant by the statement that T is self-adjoint, and state the spectral theorem for self-adjoint linear maps.

(b) Suppose that T is self-adjoint.

(i) Show that $(T(v), v) \in \mathbb{R}$ for all $v \in V$.

(ii) Show that $(T(v), v) \geq 0$ for all $v \in V$ if and only if all eigenvalues of T are non-negative.

(c) Now let V be the vector space of all polynomials over $\mathbb{R}$ of degree at most 1, with inner product

$$(f, g) = \int_0^1 f(x)g(x)\,dx \qquad \text{for all } f, g \in V.$$

For $a, b \in \mathbb{R}$ define $T_{ab} : V \mapsto V$ by

$$T_{ab}(f(x)) = af(x) + bf'(x) \qquad \text{for all } f \in V.$$

(i) For which values of a, b is T_{ab} self-adjoint?

(ii) For which values of a, b does T_{ab} satisfy $(T_{ab}f, f) \geq 0$ for all $f \in V$?

4. Let V be a finite-dimensional inner product space over $\mathbb{C}$, with inner product $(\ ,\ )$ and length function $||v|| = \sqrt{(v, v)}$ for $v \in V$.

(a) State the Cauchy-Schwarz and Triangle Inequalities.

(b) Let $v_1, \ldots, v_k \in V$ and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. Prove the following:

(i) $||\sum_{i=1}^k \alpha_i v_i|| \leq \sum_{i=1}^k |\alpha_i|\,||v_i||$,

(ii) $||\sum_{i=1}^k \alpha_i v_i||^2 \leq \left(\sum_{i=1}^k |\alpha_i|^2\right)\left(\sum_{i=1}^k ||v_i||^2\right)$.

(c) Let $e_1, \ldots, e_n$ be an orthonormal basis of V, and suppose that $v_1, \ldots, v_n$ are vectors in V such that

$$||e_i - v_i|| < \frac{1}{\sqrt{n}} \qquad \text{for } i = 1, \ldots, n.$$

Prove that $v_1, \ldots, v_n$ is a basis of V.

5. (a) Let V be a finite-dimensional inner product space, with inner product $(\ ,\ )$, length function $||v|| = \sqrt{(v, v)}$, and distance function $d(u, v) = ||u - v||$. Let W be a subspace of V, and let $w_1, \ldots, w_r$ be an orthonormal basis of W. For $v \in V$, define

$$v_W = \sum_{i=1}^r (v, w_i)\, w_i.$$

Prove that v_W is the unique vector in W that is closest to v in distance.

(b) Now let $V = \mathbb{R}^3$, and for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in V$ define

$$(x, y) = 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_1y_2 + x_2y_1 - x_1y_3 - x_3y_1.$$

(i) Show that this is an inner product on V.

(ii) Let $W = \{x \in V : x_1 + x_2 + x_3 = 0\}$. Find the vector in W that is closest to $(1, 1, 1)$ with respect to the distance function given by this inner product.

Dual Spaces

6. Let V be a finite-dimensional vector space, and let V^* be the dual space of V.

(a) Let $v_1, \ldots, v_n$ be a basis of V, and let $f_1, \ldots, f_n$ be the dual basis of V^*. Show that for $v \in V$ and $f \in V^*$, the following two equations hold:

(i) $v = \sum_{i=1}^{n} f_i(v)\, v_i$,

(ii) $f = \sum_{i=1}^{n} f(v_i)\, f_i$.

(b) Which of the following statements are true and which are false? For each, give either a proof or a counterexample.

(i) If $f, g \in V^*$ are such that $f(v) = 0 \Rightarrow g(v) = 0$ for $v \in V$, then there is a scalar λ such that $g = \lambda f$.

(ii) If U is a subspace of V with $U \neq V$, then there exists $f \in V^*$ such that $f(u) \neq 0$ for all $u \in U \setminus 0$.

(iii) If U is a subspace of V with $U \neq V$, then there exists $f \in V^* \setminus 0$ such that $f(u) = 0$ for all $u \in U$.

(iv) If $v_1, v_2 \in V$ with $v_1 \neq v_2$, then there exists $f \in V^*$ such that $f(v_1) \neq f(v_2)$.

Rational Canonical Form

7. (a) Write down all possible Rational Canonical Forms of 11×11 matrices over $\mathbb{Q}$ with minimal polynomial $(x+1)^2(x^2+x+1)^2(x^3+x+1)$.

(b) Does there exist a 4×4 matrix over $\mathbb{F}_5$ that has minimal polynomial $x^3 + x + 3$? Justify your answer.

(c) Determine whether the two matrices A, B below are similar to each other over $\mathbb{Q}$:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 \\ 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ -3 & -4 & -1 & -2 \end{pmatrix}.$$