Solutions to Revision Questions

1. a) Form is $(x,y) = x^T \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix} y$. Symmetric, as matrix is symmetric.

b) Let $A = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$. By lectures, the form is non-degenerate iff A is invertible. Since $|A| = -15$, $|A|$ is 0 in $\mathbb{F}_p$ iff $p = 3$ or 5. So the form is non-degenerate iff $p \neq 3$ or 5.

c) Note $(e_1, e_1) = 1$, so take e_1 as the first vector.

Next $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in e_1^\perp$ iff $x_1 + 4x_2 = 0$.

Take $v_2 = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$. Then $(v_2, v_2) = -15$.

So provided $p \neq 3, 5$, $\{e_1, v_2\} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \end{pmatrix}\}$ is an orthogonal basis.

(d) We have $Q(x) = x_1^2 + 8x_1x_2 + x_2^2$.

Change variables to $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ where $x = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} y$. Then we get equivalent quadratic form $Q'(y) = y_1^2 - 15y_2^2$; and Q surjective $\iff$ Q' surjective.

For $p = 3$ or 5: $Q'(y) = y_1^2$, not surjective.

For $p \neq 3$ or 5 Use the trick of Sheet 10, Q8(ii): for $c \in \mathbb{F}_p$, let

$$\Delta_1 = \{y_1^2 - c : y_1 \in \mathbb{F}_p\}, \quad \Delta_2 = \{15y_2^2 : y_2 \in \mathbb{F}_p\}.$$

Then $|\Delta_1| = |\Delta_2| = \frac{1}{2}(p-1)+1$ (the $+1$ is for the $y_i = 0$ term), so $|\Delta_1| = |\Delta_2| > \frac{1}{2}|\mathbb{F}_p|$. Hence $\Delta_1 \cap \Delta_2 \neq \emptyset$, so $\exists\, y_1, y_2 \in \mathbb{F}_p$ s.t. $y_1^2 - c = 15y_2^2$, so $y_1^2 - 15y_2^2 = c$.

Hence $Q'(y) = c$, showing Q' (hence Q) surjective.

2. (a,b) ✓

(c) Let $f_1 = (\, , \,)$ ~~$(u,v) \mapsto f_1$~~

$f_2 = (\, , \,)$

be skew-symm. bilinear forms.

Define $(f_1+f_2)(u,v) = f_1(u,v) + f_2(u,v)$

$(\lambda f_1)(u,v) = \lambda f_1(u,v)$.

These operations make $\mathcal{B} = \{\text{all s.s.b.f.}\}$ a vector space.

~~Consider~~ f

Fix a basis $v_1, \dots, v_n$

~~Given~~ Have map $\phi : \mathcal{B} \longrightarrow S = \{ A : A^T = -A \}$.

sending $f \longrightarrow A_f$

where A_f has ij-entry $f(v_i, v_j)$.

Check ϕ is linear,

injective (as $\ker \phi = 0$)

surjective (as any $A \in S$ defines a s.s.b.f $(u,v) = [u]^T A [v]$)

Hence ϕ is an isomorphism, so

$$\underline{\dim \mathcal{B} = \dim S.}$$

Basis for S is

$$\left[\begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & 0 \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix} = E_{12} - E_{21}, \text{ etc} \right]$$

ie. $\{E_{ij} - E_{ji} : i < j\}$

Hence $\dim S = \binom{n}{2} = \frac{1}{2}n(n-1)$

(d) Let $f_u(x,y) = u.(x \times y)$

Then
$$f_u(y,x) = u.(y \times x) = -u.(x \times y)$$
$$= -f_u(x,y)$$

So f_u is skew-symm.

(e) Let S = space of all s.s. b.f. on $\mathbb{R}^3$.

Know by (c): $\dim S = \binom{3}{2} = 3$.

Define $\phi : \mathbb{R}^3 \to S$ by

$$u \longrightarrow f_u.$$

Then ϕ is linear,

injective (as $\ker \phi = 0$)

surjective (as $\dim \mathbb{R}^3 = 3 = \dim S$).

3. (a) ✓

(b) (i) $(Tv, v) = (v, T^*v) = (v, Tv) = \overline{(Tv, v)} \Rightarrow (Tv, v) \in \mathbb{R}$.

(ii) ($\Rightarrow$) Suppose $(Tv, v) \geq 0 \ \forall v$. If $\exists$ evalue $\lambda < 0$, with evector v s.t. $Tv = \lambda v$, then

$$(Tv, v) = (\lambda v, v) = \lambda (v, v) < 0 \ \#.$$

($\Leftarrow$) By Spectral Thm, $\exists$ orthonormal basis $v_1, \ldots, v_n$ of evectors for T. Let $\lambda_1, \ldots, \lambda_n$ be the corr. evalues, and suppose $\lambda_i \geq 0 \ \forall i$. For $v \in V$, $v = \sum_1^n \alpha_i v_i$, so

$$(Tv, v) = \left(\sum_1^n \lambda_i \alpha_i v_i, \sum \alpha_i v_i \right)$$

$$= \sum_1^n \lambda_i \alpha_i \bar{\alpha}_i \geq 0.$$

(c) (i) V has orthonormal basis u_1, u_2 where $u_1 = 1$, $u_2 = \sqrt{3}(1-2x)$

Let $B = \{u_1, u_2\}$. Then

$$T_{ab} : \begin{array}{l} u_1 \to a u_1 \\ u_2 \to a u_2 - 2\sqrt{3}\, b u_1 \end{array}$$

so $[T_{ab}]_B = \begin{pmatrix} a & -2\sqrt{3}\, b \\ 0 & a \end{pmatrix}$. This is symmetric iff $b = 0$,

so T_{ab} is self-adjoint iff $b = 0$.

(ii) Let $f \in V$, $[f]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$. Then

$$[T_{ab} f]_B = \begin{pmatrix} a & -2\sqrt{3}\, b \\ 0 & a \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} a\alpha_1 - 2\sqrt{3}\, b \alpha_2 \\ a \alpha_2 \end{pmatrix}$$

so

$$(T_{ab} f, f) = a\alpha_1^2 - 2\sqrt{3}\, b\, \alpha_1 \alpha_2 + a \alpha_2^2.$$

If $a \neq 0$ this is

$$a \left(\left(\alpha_1 - \frac{\sqrt{3}\, b}{a} \alpha_2 \right)^2 + \left(1 - \frac{3b^2}{a^2} \alpha_2^2 \right) \right).$$

So it is ≥ 0 iff $a > 0$ and $\frac{3b^2}{a^2} \leq 1$.

If $a = 0$, $(T_{ab}f, f)$ is ≥ 0 $\forall f$ only for $b = 0$.

So values of a, b are

$$\boxed{\begin{array}{l} \cdot\ a = b = 0 \\ \cdot\ a > 0, \quad b^2 \leq \frac{a^2}{3} \end{array}}$$

4. (a) ✓

(b) (i) Use Triangle inequality & induction on k:

$$\left\| \sum_1^k \alpha_i v_i \right\| = \left\| \sum_1^{k-1} \alpha_i v_i + \alpha_k v_k \right\|$$

$$\leq \left\| \sum_1^{k-1} \alpha_i v_i \right\| + |\alpha_k| \, \|v_k\| \quad (\Delta \text{ inequal.})$$

$$\leqslant \sum_1^{k-1} |\alpha_i| \, \|v_i\| + |\alpha_k| \, \|v_k\| \quad (\text{induction})$$

(ii) $$\left\| \sum_1^k \alpha_i v_i \right\|^2 \leqslant \left(\sum_1^k |\alpha_i| \, \|v_i\| \right)^2 \quad (\text{by (i)})$$

$$\leqslant \left(\sum_1^k |\alpha_i| \right)^2 \left(\sum_1^k \|v_i\|^2 \right) \quad (\text{by Cauchy-Schwarz}).$$

(c) It is enough to show $v_1, \ldots, v_n$ are linearly indep. Suppose

$$\sum_1^n \alpha_i v_i = 0,$$

where some $\alpha_i \neq 0$. Then

$$\left\| \sum_1^n \alpha_i (e_i - v_i) \right\|^2 = \left\| \sum_1^n \alpha_i e_i \right\|^2 = \sum_1^n |\alpha_i|^2. \quad (1)$$

Also by (b) (ii),

$$\left\| \sum_1^n \alpha_i (e_i - v_i) \right\|^2 \leqslant \left(\sum_1^n |\alpha_i|^2 \right) \left(\sum_1^n \|e_i - v_i\|^2 \right) \quad (2)$$

Now

$$\sum_1^n \|e_i - v_i\|^2 < n \cdot \left(\frac{1}{\sqrt{n}} \right)^2 = 1.$$

Hence combining (1) and (2),

$$\sum_1^n |\alpha_i|^2 < \sum_1^n |\alpha_i|^2 \quad \divideontimes.$$

Therefore $\alpha_i = 0 \ \forall i$, proving linear independence.

5. (a) is standard from lectures (proof was on a problem sheet).

(b) (i) Given $(x,y) = x^T \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} y$.

Then
$$(x,x) = 2x_1^2 + 2x_2^2 + x_3^2 + 2x_1x_2 - 2x_1x_3$$
$$= (x_1+x_2)^2 + (x_1-x_3)^2 + x_2^2.$$

This is ≥ 0, and is $= 0$ iff $x = 0$.

Hence $(\ ,\ )$ is an inner product.

[Another method is to show all evalues of $\begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$ are > 0].

(ii) Find an orthonormal basis of W:

$$w_1 = \frac{1}{\sqrt{2}}(1,-1,0), \quad w_2 = \frac{1}{\sqrt{3}}(0,1,-1).$$

Let $v = (1,1,1)$. By (a), closets vector in W to x is

$$v_W = (v,w_1)w_1 + (v,w_2)w_2$$
$$= -\frac{1}{\sqrt{2}} w_1 + \sqrt{3}\, w_2$$
$$= \left(-\frac{1}{2}, \frac{3}{2}, -1\right).$$

6(b)(i) True. Clear if $f = 0$ or $g = 0$, so assume $f, g \neq 0$.

Note $\underset{\text{dim } n}{V} \xrightarrow{f} \underset{\text{dim } 1}{F}$, so $\ker f$ has dim. $n-1$.

So condition $f(v) = 0 \Rightarrow g(v) = 0$

implies $\ker f \subseteq \ker g$

$\Rightarrow$ $\ker f = \ker g$.

Let $H = \ker f = \ker g$, dim $n-1$.

Pick $v_0 \in V \setminus H$.

Any $v \in V$ has form $v = h + \alpha v_0$ $(\alpha \in F)$

So

$$f(v) = f(h) + \alpha f(v_0) = \alpha f(v_0)$$

$$g(v) = \alpha g(v_0)$$

Hence

$$g(v) = f(v) \cdot \big(\underbrace{f(v_0)^{-1} g(v_0)}_{\lambda} \big).$$

(ii) False: Eg. Let $\dim V \geq 3$ and $\dim U \geq 2$.

Then for any $f \in V^*$, $\dim(\ker f) \geq n-1$

so

$$U \cap (\ker f) \neq 0.$$

True

(iii) Let $U \subsetneqq V$. Take a basis $u_1, \ldots, u_r$ of U,

extend to basis $u_1, \ldots, u_n$ of V.

For dual basis $f_1, \ldots, f_n$ of V^*, $f_n(u) = 0 \ \forall u \in U$.

(iv) True Let $w = v_1 - v_2 \neq 0$.

Extend to a basis $w, w_2, \ldots, w_n$ of V.

For dual basis $f_1, \ldots, f_n$, have

$$f_n(w) = 1$$

$$\Rightarrow f_n(v_1 - v_2) = 1$$

$$\Rightarrow \underline{f_n(v_1) = f_n(v_2) + 1}.$$

7. a) The RCFs are ~~not~~ made up of the following companion matrices:

$$A = C(x+1)^2 \qquad (2\times 2)$$

$$B = C(x+1) \qquad (1\times 1)$$

$$C = C(x^2+x+1)^2 \qquad (4\times 4)$$

$$D = C(x^2+x+1) \qquad (2\times 2)$$

$$E = C(x^3+x+1) \qquad (3\times 3)$$

(noting that x^2+x+1 and x^3+x+1 are irreducible over $\mathbb{Q}$).

We know from the minimal poly given that A, C & E must be present. Hence the possible 11×11 RCFs are

- $A \oplus A \oplus C \oplus E$
- $A \oplus B \oplus B \oplus C \oplus E$
- $A \oplus C \oplus D \oplus E$.

b) Yes: x^3+x+3 has a root $1 \in \mathbb{F}_5$. So the matrix

$$C(x^3+x+3) \oplus (1) = \left(\begin{array}{ccc|c} 0 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right)$$

has min poly x^3+x+1.

c) Compute that A and B both have char. poly $(x^2-2)^2$.

However $A^2-2I \neq 0$ whereas $B^2-2I = 0$.

Hence A, B are not similar.