Imperial College London
MATH 50004 Multivariable Calculus
January Examination Date: 11th January 2021

Question One

A surface S is described parametrically by

$$x = r\cos\theta,\ \ y = r\sin\theta,\ \ z = r^2,\ \ (0 \le r \le a,\ \ 0 \le \theta \le 2\pi),$$

where a is a fixed positive constant, and the vector field $\mathbf{A}$ is given by

$$\mathbf{A} = y\,\mathbf{i} - z\,\mathbf{j} + x\,\mathbf{k}.$$

(a) [**3 marks**] What shape is the surface S? Is it open or closed?

(b) [**3 marks**] Find the unit normal $\widehat{\mathbf{n}}$ to S which has $\widehat{\mathbf{n}} \cdot \mathbf{k} < 0$.

(c) [**3 marks**] Calculate $\operatorname{div}\mathbf{A}$ and $\operatorname{curl}\mathbf{A}$. What type of vector field is $\mathbf{A}$?

(d) Evaluate

$$I = \int_S (\operatorname{curl}\mathbf{A}) \cdot \widehat{\mathbf{n}}\, dS$$

(i) [**4 marks**] by using the given parameterization of S and an appropriate Jacobian;

(ii) [**4 marks**] by using Stokes theorem and converting to an equivalent path integral;

(iii) [**3 marks**] by using the divergence theorem applied to a suitably chosen closed surface.

Question Two

(a) [**5 marks**] Show that the extremal curves $y = y(x)$ of the integral

$$I = \int_0^\pi \{r(x)(y'(x))^2 - q(x)(y(x))^2\}\, dx$$

satisfying the end conditions

$$y(0) = y(\pi) = 0$$

and the constraint

$$J = \int_0^\pi p(x)(y(x))^2\, dx = 1$$

are solutions of the equation

$$(r(x)y'(x))' + (q(x) - \lambda p(x))y = 0, \tag{1}$$

where $'$ denotes d/dx and λ is the Lagrange multiplier.

(b) [**5 marks**] By multiplying (1) by y and integrating from 0 to π, show that

$$\lambda + I = 0. \tag{2}$$

(c) [**10 marks**] Determine the extremal curves and stationary values of I for the special case

$$p(x) = q(x) = r(x) = 1.$$

Show that there are an infinite number of possible values for λ and verify that your solutions satisfy relation (2).

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SOLUTIONS

Question One Solution

(a) The surface is a paraboloid of circular cross-section. It is open with boundary curve a circle of radius a. [**3 marks**]

(b) The surface S is given by $z = x^2 + y^2$. Let $\phi = z - x^2 - y^2$ so that $\phi = 0$ on S [**1 mark**]. Then the unit normal to S is

$$\pm\nabla\phi/\left|\nabla\phi\right| = \pm(-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k})/\sqrt{4x^2 + 4y^2 + 1}. \text{ [\textbf{1 mark}]}$$

We then take the $-$ sign so that $\widehat{\mathbf{n}} \cdot \mathbf{k} < 0$. This gives

$$\widehat{\mathbf{n}} = (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k})/\sqrt{4x^2 + 4y^2 + 1}. \text{ [\textbf{1 mark}]}$$

(c)

$$\operatorname{div}\mathbf{A} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-z) + \frac{\partial}{\partial z}(x) = 0. \text{ [\textbf{1 mark}]}$$

$$\operatorname{curl}\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -z & x \end{vmatrix} = \mathbf{i} - \mathbf{j} - \mathbf{k}. \text{ [\textbf{1 mark}]}$$

We have $\operatorname{div}\mathbf{A} = 0$ so $\mathbf{A}$ is a solenoidal vector field [**1 mark**].

(d) (i) Firstly

$$(\operatorname{curl}\mathbf{A}) \cdot \widehat{\mathbf{n}} = \frac{2x - 2y + 1}{\sqrt{4x^2 + 4y^2 + 1}} = \frac{2x - 2y + 1}{\sqrt{4r^2 + 1}}. \text{ [\textbf{1 mark}]}$$

The appropriate Jacobian is

$$\mathbf{J} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial x/\partial r & \partial y/\partial r & \partial z/\partial r \\ \partial x/\partial\theta & \partial y/\partial\theta & \partial z/\partial\theta \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\mathbf{i} - (2r^2\sin\theta)\mathbf{j} + r\mathbf{k}$$

Hence

$$|\mathbf{J}| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{1 + 4r^2}. \text{ [\textbf{2 marks}]}$$

Then

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^a (\operatorname{curl}\mathbf{A}) \cdot \widehat{\mathbf{n}}\ |\mathbf{J}|\ dr\,d\theta \\ &= \int_0^{2\pi}\int_0^a (2x - 2y + 1)\,r\,dr\,d\theta \\ &= \int_0^{2\pi}\int_0^a (2r^2\cos\theta - 2r^2\sin\theta + r)\,dr\,d\theta \end{aligned}$$

The first two terms integrate to zero, leaving

$$I = 2\pi\left[r^2/2\right]_0^a = \pi a^2. \text{ [\textbf{1 mark}]}$$

(d)(ii) Using Stokes theorem we have

$$I = \oint_\gamma \mathbf{A} \cdot d\mathbf{r} \;\; [\mathbf{1\ mark}]$$

where γ is the boundary of S, i.e. a circle of radius a located at $z = a^2$. Since the normal to S has $\widehat{\mathbf{n}} \cdot \mathbf{k} < 0$, by the right hand rule we should traverse γ clockwise [**1 mark**]. Now on γ :

$$\begin{aligned}\mathbf{A} \cdot d\mathbf{r} &= y\,dx - z\,dy + x\,dz \\ &= a \sin\theta\, d(a\cos\theta) - a^2\, d(a\sin\theta) + 0 \;\; [\mathbf{1\ mark}]\end{aligned}$$

since $dz = 0$ on γ. We therefore have

$$I = \int_{2\pi}^{0} (-a^2 \sin^2\theta - a^3 \cos\theta)\; d\theta.$$

The second term integrates to zero, leaving

$$I = \int_0^{2\pi} a^2 \sin^2\theta\, d\theta = \pi a^2, \;\; [\mathbf{1\ mark}]$$

where the result $\int_0^{2\pi} \sin^2\theta\, d\theta = \pi$ can just be quoted.

(d)(iii) First we need to close the paraboloid by putting a circular lid on it at $z = a^2$. Call the lid S_L. Then by the divergence theorem

$$I + \int_{S_L} (\mathrm{curl}\, A) \cdot \widehat{\mathbf{n}}\, dS = \int_V \mathrm{div}\,(\mathrm{curl}\, A)\, dV = 0, \;\; [\mathbf{2\ marks}]$$

where V is the volume enclosed by the (now) closed surface. The outward normal to S_L is $\widehat{\mathbf{n}} = \mathbf{k}$, and so

$$I = -\int_{S_L} (\mathrm{curl}\, \mathbf{A}) \cdot \mathbf{k}\; dS = -\int_{\theta=0}^{2\pi} \int_{r=0}^{a} (-1)\, r\, dr\, d\theta = \pi a^2. \;\; [\mathbf{1\ mark}]$$

Question Two Solution

(a) We take

$$L = r(y')^2 - qy^2, \quad g = py^2 \;\; \textbf{[1 mark]}$$

and consider the Euler-Lagrange equation applied to $L + \lambda g$ where λ is a Lagrange multiplier to be found as part of the solution. We therefore have

$$\frac{\partial}{\partial y}(r(y')^2 - qy^2 + \lambda py^2) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}(r(y')^2 - qy^2 + \lambda py^2)\right) = 0 \;\; \textbf{[1 mark]}$$

$$\implies 2(\lambda p - q)y - \frac{d}{dx}(2ry') = 0, \;\; \textbf{[2 marks]}$$

which (after dividing by two) is the equation given in the question [**1 mark**].

(b) Multiplying by y and integrating:

$$\int_0^\pi (\lambda p - q)y^2\, dx = \int_0^\pi y\frac{d}{dx}(ry')\, dx = \left[yry'\right]_0^\pi - \int_0^\pi r(y')^2 dx, \;\; \textbf{[2 marks]}$$

after integrating by parts on the right hand side. The integrated term vanishes since $y(0) = y(\pi) = 0$ and so

$$\lambda \int_0^\pi py^2 dx = \int_0^\pi qy^2 - r(y')^2\, dx. \;\; \textbf{[1 mark]}$$

The integral on the left hand side is just the constraint J which has value unity (from the question) [**1 mark**], while the integral on the right hand side is $-I$. We therefore have

$$\lambda = -I, \;\; \textbf{[1 mark]}$$

as required.

(c) Now set $p = q = r = 1$ so that the ODE calculated above reduces to

$$y'' + (1 - \lambda)y = 0. \;\; \textbf{[1 mark]}$$

If we let $\beta^2 = 1 - \lambda$ then we can write the solutions as

$$y = A\cos\beta x + B\sin\beta x. \;\; \textbf{[2 marks]}$$

Applying the end conditions $y(0) = y(\pi) = 0$ we see that

$$A = 0, \;\; \beta = \pm1, \pm2, \pm3, \ldots \;\; \textbf{[2 marks]}$$

and hence we have an infinite number of possible values for λ :

$$\lambda = 1 - \beta^2 = 0, -3, -8, \ldots. \;\; \textbf{[1 mark]}$$

To find B we need to substitute into the integral constraint

$$\int_0^\pi y^2\, dx = 1,$$

which gives

$$\int_0^\pi B^2 \sin^2\beta x\, dx = B^2\frac{\pi}{2} = 1,$$

since β is an integer. Thus $B = \sqrt{2/\pi}$ and the extremal curves are

$$y = \sqrt{\frac{2}{\pi}}\sin\beta x, \;\; \beta = \pm1, \pm2, \pm3, \ldots. \;\; \textbf{[1 mark]}$$

The corresponding stationary values of I are

$$I = \int_0^\pi [(y')^2 - y^2]\, dx = \frac{2}{\pi}\int_0^\pi (\beta^2\cos^2\beta x - \sin^2\beta x)\, dx = \frac{2}{\pi}\left(\frac{\pi}{2}\beta^2 - \frac{\pi}{2}\right) = \beta^2 - 1 = -\lambda, \;\; \textbf{[2 marks]}$$

and so indeed $I + \lambda = 0$ as found earlier in the more general setting [**1 mark**].