Imperial College London
MATH 50004 Multivariable Calculus
January Examination Date: 10th January 2022

Question One

Let S be the closed surface defined by the region in the first octant bounded by the surfaces

$$x^2 + y^2 = 4, \;\; z = 0, \;\; z = 3, \;\; x = 0, \;\; y = 0,$$

and let $\widehat{\mathbf{n}}$ be the unit outward normal to S.

(a) [**2 marks**] Sketch the surface S.

Consider the vector field

$$\mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + z\,\mathbf{k}$$

where A_1, A_2 are constants.

(b) [**5 marks**] Let S_1 denote the curved part of S. By projecting onto the $x - z$ plane, show that

$$\int_{S_1} \mathbf{A} \cdot \widehat{\mathbf{n}}\; dS = 6A_1 + 6A_2.$$

(c) [**3 marks**] Demonstrate that the same answer is obtained by projecting onto the $y - z$ plane. Why can't we project onto the $x - y$ plane in this case?

(d) [**7 marks**] For $i = 2, 3, 4, 5$ calculate

$$\int_{S_i} \mathbf{A} \cdot \widehat{\mathbf{n}}\; dS$$

where the S_i are the other four faces of S.

(e) [**1 mark**] Hence calculate the flux of $\mathbf{A}$ across S.

(f) [**2 marks**] Check your answer by applying the divergence theorem and considering the appropriate volume integral.

Question Two

(a) [**6 marks**] Show that the extremal curve $y(x)$ of the integral

$$I = \int_0^{\pi} 4y^2 - (y')^2\, dx$$

subject to the end conditions

$$y(0) = 1, \;\; y'(\pi) = 0$$

is given by

$$y(x) = \cos 2x.$$

(b) [**14 marks**] If the integral constraint

$$\int_0^{\pi} y(x) \cos 2x\, dx = \pi$$

is added to the problem, find the new extremal curve of I.

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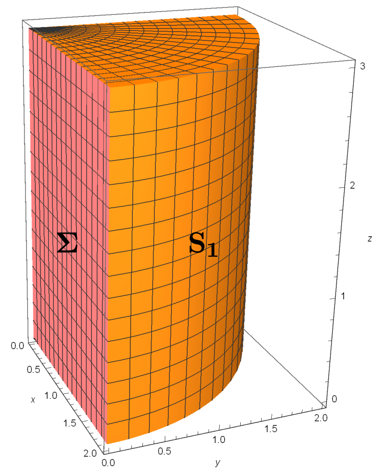
SOLUTIONS

Question One Solution



(a) sketch **[2 marks]**
(b) The normal to S_1 is

$$\nabla(x^2+y^2)/\left|\nabla(x^2+y^2)\right| = (2x\mathbf{i}+2y\mathbf{j})/(4x^2+4y^2)^{1/2} = (x\mathbf{i}+y\mathbf{j})/2, \quad \textbf{[1 mark]}$$

and so we have on S_1:

$$\mathbf{A}\cdot\widehat{\mathbf{n}} = \frac{1}{2}(xA_1+yA_2). \ \textbf{[1 mark]}$$

Projecting onto $y=0$:

$$dS = \frac{dx\,dz}{|\widehat{\mathbf{n}}\cdot\mathbf{j}|} = \frac{2dx\,dz}{y}.$$

Then by the projection theorem:

$$\int_{S_1}\mathbf{A}\cdot\widehat{\mathbf{n}}\,dS = \int_{\Sigma}\frac{1}{2}(xA_1+yA_2)\frac{2dx\,dz}{y}.$$

Substituting for $y=(4-x^2)^{1/2}$:

$$\begin{aligned}\int_{S_1}\mathbf{A}\cdot\widehat{\mathbf{n}}\,dS &= \int_{z=0}^{z=3}\int_{x=0}^{x=2}\left[\frac{xA_1}{(4-x^2)^{1/2}}+A_2\right]dx\,dz\\ &= 6A_2+3A_1\int_0^2 x(4-x^2)^{-1/2}dx\\ &= 6A_2-3A_1[(4-x^2)^{1/2}]_0^2\\ &= 6A_2+6A_1.\ \textbf{[3 marks]}\end{aligned}$$

(c) To project onto $x=0$ we have

$$dS = \frac{dy\,dz}{|\widehat{\mathbf{n}}\cdot\mathbf{i}|} = \frac{2dy\,dz}{x}.$$

Substituting $x = (4 - y^2)^{1/2}$:

$$\begin{aligned}\int_{S_1} \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS &= \int_{z=0}^{z=3} \int_{y=0}^{y=2} [\frac{yA_2}{(4-y^2)^{1/2}} + A_1]\, dy\, dz \\ &= 6A_2 + 6A_1\end{aligned}$$

as above [**2 marks**]. We cannot project onto $z = 0$ because $\widehat{\mathbf{n}} \cdot \mathbf{k} = 0$ (the surface S_1 does not cast a shadow in this direction) [**1 mark**].

(d) Turning now to the other four faces.

(i) The plane $y = 0$. Here $\widehat{\mathbf{n}} = -\mathbf{j} \Rightarrow \mathbf{A} \cdot \widehat{\mathbf{n}} = -A_2$. The integral is therefore

$$-\int_0^3 \int_0^2 A_2 dx\, dz = -6A_2. \text{ [\textbf{2 marks}]}$$

(ii) The plane $x = 0$. Here $\widehat{\mathbf{n}} = -\mathbf{i} \Rightarrow \mathbf{A} \cdot \widehat{\mathbf{n}} = -A_1$. The integral is

$$-\int_0^3 \int_0^2 A_1 dy\, dz = -6A_1. \text{ [\textbf{2 marks}]}$$

(iii) The plane $z = 0$. Here $\widehat{\mathbf{n}} = -\mathbf{k} \Rightarrow \mathbf{A} \cdot \widehat{\mathbf{n}} = -z$ and the region of integration is a quarter disc D, say. The integral is

$$-\int\int_D z\, dx\, dy = 0,$$

since $z = 0$ on D.[**1 mark**]

(iv) The plane $z = 3$. This time $\widehat{\mathbf{n}} = \mathbf{k}$ and so $\mathbf{A} \cdot \widehat{\mathbf{n}} = z = 3$. The integral is

$$\int_0^{\pi/2} \int_0^2 3\, r\, dr\, d\theta = 3\pi. \text{ [\textbf{2 marks}]}$$

(It is fine to state that the area of D is one quarter the area of a circle of radius 2 $(= \pi)$).

(e) To get the flux we add the five contributions together:

$$\oint_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \sum_{i=1}^{5} \int_{S_i} \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = 6A_1 + 6A_2 - 6A_2 - 6A_1 + 0 + 3\pi = 3\pi. \text{ [\textbf{1 mark}]}$$

(f) Applying the divergence theorem to the closed surface S (enclosing a volume V):

$$\oint_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_V \text{div}\mathbf{A}\, dV = \int_V dV = 3\pi,$$

since div$\mathbf{A} = \mathbf{1}$ [**1 mark**] and the volume V is one quarter the volume of a cylinder of radius 2 and height 3 [**1 mark**].

Question Two Solution

(a) Let

$$L = 4y^2 - (y')^2.$$

Then

$$\frac{\partial L}{\partial y} = 8y, \quad \frac{\partial L}{\partial y'} = -2y'. \text{ [1 mark]}$$

The Euler-Lagrange equation is

$$\frac{\partial L}{\partial y} - \frac{d}{\partial x}\left(\frac{\partial L}{\partial y'}\right) = 0.$$

Substituting for L we have

$$8y + 2y'' = 0 \text{ [1 mark]}$$

which has the general solution

$$y = A\cos 2x + B\sin 2x. \text{ [2 marks]}$$

Applying the end conditions $y(0) = 1, y'(\pi) = 0$ we find

$$A = 1, \ B = 0$$

and hence

$$y = \cos 2x,$$

as required [**2 marks**].

(b) Now let

$$g = y\cos 2x$$

and apply the Euler-Lagrange equation to $L + \lambda g$:

$$\frac{\partial}{\partial y}\left(4y^2 - y'^2 + \lambda y\cos 2x\right) - \frac{d}{dx}\left(\frac{\partial}{\partial y'}(4y^2 - y'^2 + \lambda y\cos 2x)\right) = 0$$

$$\Rightarrow 8y + \lambda\cos 2x + 2y''$$

$$\Rightarrow y'' + 4y = -\frac{\lambda}{2}\cos 2x. \text{ [2 marks]}$$

The homogeneous solution (as above) is

$$y_H = A\cos 2x + B\sin 2x. \text{ [1 mark]}$$

For the particular solution we observe that the RHS occurs in y_H so we look for

$$y_{PS} = x(C\cos 2x + D\sin 2x).$$

Then, by direct computation

$$y''_{PS} = -4y_{PS} + 2(-2C\sin 2x + 2D\cos 2x)$$

and so we require

$$-4C\sin 2x + 4D\cos 2x = -\frac{\lambda}{2}\cos 2x$$

which can only be satisfied provided

$$C = 0, \ \ D = -\lambda/8.$$

We therefore have

$$y = y_H + y_{PS} = A\cos 2x + B\sin 2x - \frac{1}{8}\lambda x\sin 2x. \ \textbf{[3 marks]}$$

Applying the end conditions $y(0) = 1$ we find

$$A = 1,$$

as before [**1 mark**]. Applying $y'(\pi) = 0$ we have

$$-2\sin 2\pi + 2B\cos 2\pi - \frac{1}{8}\lambda\sin 2\pi - \frac{1}{4}\lambda\pi\cos 2\pi = 0$$

and hence

$$B = \lambda\pi/8. \ \textbf{[1 mark]}$$

To find λ we substitute into the integral constraint

$$\int_0^\pi y\cos 2x\, dx = \pi$$

to get

$$\int_0^\pi \cos^2 2x + \frac{\lambda\pi}{8}\sin 2x\cos 2x - \frac{\lambda}{8}x\sin 2x\cos 2x\, dx = \pi. \ \textbf{[1 mark]}$$

The second term integrates to zero [**1 mark**] while the first term is equal to $\pi/2$ [**1 mark**]. In addition we need to integrate by parts on the final term:

$$\begin{aligned} \int_0^\pi x\sin 2x\cos 2x\, dx &= \int_0^\pi \frac{1}{2}x\sin 4x\, dx \\ &= \left[-\frac{1}{8}x\cos 4x\right]_0^\pi + \int_0^\pi \frac{1}{8}\cos 4x\, dx \\ &= -\frac{1}{8}\pi\cos 4\pi \\ &= -\pi/8. \ \textbf{[2 marks]} \end{aligned}$$

Putting these calculations together:

$$\pi/2 - (\lambda/8)(-\pi/8) = \pi$$
$$\Rightarrow \lambda = 32.$$

The constrained extremal curve is therefore

$$y = \cos 2x + 4\pi\sin 2x - 4x\sin 2x. \ \textbf{[1 mark]}.$$