Imperial College
London

MATH50004

BSc, MSci and MSc EXAMINATIONS (MATHEMATICS)
May-June 2021

This paper is also taken for the relevant examination for the
Associateship of the Royal College of Science

Multi-variable Calculus and Differential Equations

Date: Monday, 10 May 2021

Time: 09:00 to 12:00

Time Allowed: 3 hours

Upload Time Allowed: 45 minutes

This paper has 6 Questions.

Candidates should start their solutions to each question on a new sheet of paper.

Each sheet of paper should have your CID, Question Number and Page Number on the top.

Only use 1 side of the paper.

Allow margins for marking.

Any required additional material(s) will be provided.

Credit will be given for all questions attempted.

Each question carries equal weight.

SUBMIT YOUR ANSWERS SEPARATE PDFs TO THE RELEVANT DROPBOXES ON BLACKBOARD (ONE FOR EACH QUESTION) WITH COMPLETED COVERSHEETS WITH YOUR CID NUMBER, QUESTION NUMBERS ANSWERED AND PAGE NUMBERS PER QUESTION.

1. (a) Consider two vector fields $\mathbf{A}$ and $\mathbf{B}$. The vector field $\mathbf{B}$ is solenoidal. Use subscript notation to simplify

$$(\mathbf{A} \times \nabla) \times \mathbf{B} - \mathbf{A} \times \operatorname{curl} \mathbf{B}.$$

You may assume the relation $\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. (7 marks)

(b) Determine the constants α, β, γ such that the surfaces

$$\alpha x^2 z - xy^2 = \beta, \quad \gamma xz - y^2 = 0,$$

intersect orthogonally at the point $(x, y, z) = (-2, 2, -1)$. (7 marks)

(c) Consider the double integral

$$I = \iint_R (y - x) \, \mathrm{d}x \, \mathrm{d}y$$

where the finite region R is bounded by the lines

$$y = x + 1, \quad y = x - 3, \quad y = 2 - \frac{1}{3}x, \quad y = 4 - \frac{1}{3}x.$$

Use the substitution

$$u = y - x, \quad v = y + \frac{1}{3}x$$

to evaluate I. (6 marks)

(Total: 20 marks)

2. A surface S is described parametrically by

$$x = h\cos\theta, \quad y = h\sin\theta, \quad z = a - h, \quad (0 \le h \le a, \quad 0 \le \theta \le 2\pi),$$

where a is a positive constant.

(a) Express z as a function of x and y. Sketch the surface S. Is this an open or closed surface? Is it convex? (4 marks)

(b) Show that the unit normal $\widehat{\mathbf{n}}$ to S which has $\widehat{\mathbf{n}} \cdot \mathbf{k} > 0$ can be written in the form

$$\widehat{\mathbf{n}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{\sqrt{2}(x^2+y^2)^{1/2}} + \frac{\mathbf{k}}{\sqrt{2}}.$$

(3 marks)

(c) Find the equation of the tangent plane to S at the location where $x = a$ and $y = 0$. (3 marks)

(d) If S is projected onto the $x - y$ plane, what is the shape of the resulting projection? (2 marks)

(e) Suppose $\mathbf{A}$ is the vector field

$$\mathbf{A} = x\,\mathbf{i}\,.$$

Calculate

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\,\mathrm{d}S$$

using the projection theorem. (5 marks)

[You may assume that a small areal element in polar coordinates is given by $r\,\mathrm{d}r\,\mathrm{d}\theta$].

(f) Check your answer to (e) using the divergence theorem. (3 marks)

(Total: 20 marks)

3. (a) The coordinates (u_1, u_2, u_3) are defined in terms of Cartesian coordinates (x, y, z) by

$$x = u_1 u_2 \cos u_3, \quad y = u_1 u_2 \sin u_3, \quad z = \frac{1}{2}(u_1^2 - u_2^2), \quad (u_1 \geq 0, u_2 \geq 0, 0 \leq u_3 \leq 2\pi).$$

(i) By calculating an appropriate Jacobian, find the function $F(u_1, u_2)$ such that an element of the surface $u_3 =$ constant can be expressed as

$$\mathrm{d}S = F(u_1, u_2)\, \mathrm{d}u_1 \mathrm{d}u_2.$$

(3 marks)

(ii) Show that this system is orthogonal, determine the lengthscales h_1, h_2, h_3 and verify that

$$F(u_1, u_2) = h_1 h_2.$$

(6 marks)

(b) A curve $y = y(x)$ joins the points $(-a, 0), (a, 0)$ in the $x - y$ plane where $a > 0$.

(i) What properties of the curve do the integrals

$$I = \int_{-a}^{a} y \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x, \quad J = \int_{-a}^{a} \left(1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{1/2} \mathrm{d}x$$

represent? What does the ratio I/J represent physically? (3 marks)

(ii) Using the Euler-Lagrange equation show that the appropriate form of $y(x)$ which renders I stationary subject to the constraint $J = 2$ is

$$y = C\left(\cosh\left(\frac{x}{C}\right) - \cosh\left(\frac{a}{C}\right)\right),$$

where

$$C \sinh\left(\frac{a}{C}\right) = 1.$$

(6 marks)

(iii) Deduce that solutions can only exist if $a < a_0$ where a_0 is a value to be identified.

(2 marks)

(Total: 20 marks)

4. (a) Consider an initial value problem

$$\dot{x} = f(t, x)\,, \qquad x(t_0) = x_0\,,$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ is continuous and $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^d$ is fixed. Let J be an interval containing t_0 in its interior, and consider the Picard iterates $\{\lambda_n : J \to \mathbb{R}^d\}_{n\in\mathbb{N}_0}$ corresponding to this initial value problem.

(i) Show that $\dot{\lambda}_n(t_0) = f(t_0, \lambda_n(t_0))$ for any $n \in \mathbb{N}$. (3 marks)

(ii) What is the maximal (i.e. largest) interval J on which the functions $\lambda_n : J \to \mathbb{R}^d$ can be defined? Justify your answer. (3 marks)

(iii) Compute λ_0, λ_1 and λ_2 for the one-dimensional initial value problem $\dot{x} = x^2$ with $x(1) = 1$. (6 marks)

(b) Consider an autonomous differential equation

$$\dot{x} = f(x)\,,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous.

(i) Does this differential equation have unique local solutions for every initial condition of the form $x(0) = x_0$, where $x_0 \in \mathbb{R}^d$? Justify your answer. (2 marks)

(ii) Prove that for all $y_0 \in \mathbb{R}^d$, there exist $T > 0$ and $x_0 \in \mathbb{R}^d$ such that there exists a solution $\lambda : I \to \mathbb{R}^d$ to this differential equation with $\lambda(0) = x_0$ and $\lambda(T) = y_0$.
(6 marks)

(Total: 20 marks)

5. Consider the nonlinear differential equation

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \underbrace{\begin{pmatrix} 2 & -2 \\ 1 & 0 \end{pmatrix}}_{=:A} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -x\big((x-y)^2+y^2\big) \\ -y\big((x-y)^2+y^2\big) \end{pmatrix},$$

whose right hand side is written as the sum of the linear part with coefficient matrix A and a nonlinearity.

(i) Show that $(x^*, y^*) := (0,0)$ is the only equilibrium. (3 marks)

(ii) Calculate the real Jordan normal form of the coefficient matrix A using an invertible transformation matrix $T \in \mathbb{R}^{2\times 2}$. (4 marks)

(iii) Explain why the equilibrium $(x^*, y^*) = (0,0)$ is repulsive. (2 marks)

(iv) Show that the set $M_R := \{(x,y) \in \mathbb{R}^2 : (x-y)^2 + y^2 \leq R\}$ is positively invariant for some $R > 0$. (6 marks)
<u>Hint.</u> Consider the orbital derivative of an appropriate scalar-valued function and note that it is helpful to preserve/create terms of the form $((x-y)^2+y^2)$ in your calculations.

(v) Prove that there exists a periodic orbit. (5 marks)

(Total: 20 marks)

6. (a) Decide for each of the following four statements whether it is true or false. All statements involve omega limit sets $\omega(x)$ or alpha limit sets $\alpha(x)$ of a differential equation

$$\dot{x} = f(x)\,,$$

where we require that $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous. Justify your answer by either providing an example (which can also be a picture with short explanation) or an explanation why such an example does not exist.

(i) There exist $f : \mathbb{R}^d \to \mathbb{R}^d$ and $x \in \mathbb{R}^d$ such that $\omega(x)$ is a singleton. (3 marks)

(ii) There exist $f : \mathbb{R}^d \to \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $y \in \omega(x)$ such that $\omega(x)$ is nonempty and compact and $\omega(x) \cap \omega(y) = \emptyset$. (3 marks)

(iii) There exist $f : \mathbb{R}^d \to \mathbb{R}^d$ and $x \in \mathbb{R}^d$ such that $\omega(x) = \alpha(x)$. (3 marks)

(iv) There exist $f : \mathbb{R}^d \to \mathbb{R}^d$ and $x \in \mathbb{R}^d$ such that $\omega(x) = \alpha(x)$ and $x \notin \omega(x)$. (3 marks)

(b) Consider an autonomous differential equation

$$\dot{x} = f(x)\,,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is locally Lipschitz continuous. The flow of this differential equation is denoted by φ, and let $x \in \mathbb{R}^d$ such there exists a $K > 0$ with $\|\varphi(t, x)\| \leq K$ for all $t \geq 0$.

(i) Show that $\omega(x)$ is nonempty. (2 marks)

(ii) Show that for all $\varepsilon > 0$, there exists a $T > 0$ such that

$$\varphi(t, x) \in B_\varepsilon(\omega(x)) \quad \text{for all } t \geq T\,,$$

where $B_\varepsilon(\omega(x)) := \{y \in \mathbb{R}^d : \|y - z\| < \varepsilon \text{ for some } z \in \omega(x)\}$. (6 marks)

(Total: 20 marks)

Imperial College London
MATH 50004 Multivariable Calculus and Differential Equations
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SOLUTIONS

Question One Solution

(a) Considering the ith component:

$$\begin{aligned}
[(\mathbf{A}\times\nabla)\times\mathbf{B}-\mathbf{A}\times\text{curl}\mathbf{B}]_i &= \varepsilon_{ijk}\{(\mathbf{A}\times\nabla)_j B_k - A_j(\text{curl}\mathbf{B})_k\}\\
&= \varepsilon_{ijk}\{\varepsilon_{jlm}A_l\frac{\partial}{\partial x_m}B_k - \varepsilon_{klm}A_j\frac{\partial}{\partial x_l}B_m\} \ \textbf{[2 marks (A)]}\\
&= -\varepsilon_{ikj}\varepsilon_{jlm}A_l\frac{\partial B_k}{\partial x_m} - \varepsilon_{ijk}\varepsilon_{klm}A_j\frac{\partial B_m}{\partial x_l}\\
&= -(\delta_{il}\delta_{km}-\delta_{im}\delta_{kl})A_l\frac{\partial B_k}{\partial x_m} - (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})A_j\frac{\partial B_m}{\partial x_l}\\
&= -A_i\frac{\partial B_k}{\partial x_k} + A_k\frac{\partial B_k}{\partial x_i} - A_j\frac{\partial B_j}{\partial x_i} + A_j\frac{\partial B_i}{\partial x_j} \ \textbf{[2 marks (A)]}\\
&= -A_i\,\text{div}\,\mathbf{B} + (\mathbf{A}\cdot\nabla)B_i \ \textbf{[2 marks (A)]}\\
&= (\mathbf{A}\cdot\nabla)B_i
\end{aligned}$$

since $\mathbf{B}$ is solenoidal (i.e. $\text{div}\,\mathbf{B}=0$) **[1 mark (A)]**. Therefore the answer is $(\mathbf{A}\cdot\nabla)\mathbf{B}$.

(b) Firstly the point $(-2,2,-1)$ needs to be a point on both surfaces: this implies

$$-4\alpha+8=\beta, \quad 2\gamma-4=0$$

and so we have

$$\gamma=2, \quad \beta+4\alpha=8. \ \textbf{[2 marks (D)]}$$

Let $\phi=\alpha x^2z-xy^2$. Then the normal to the surface ϕ constant is

$$\nabla\phi=(2\alpha xz-y^2)\mathbf{i}-2xy\mathbf{j}+\alpha x^2\mathbf{k}=(4\alpha-4)\mathbf{i}+8\mathbf{j}+4\alpha\mathbf{k}$$

at $P(-2,2,-1)$ **[1 mark (C)]**. Let $\psi=\gamma xz-y^2$. The corresponding normal is

$$\nabla\psi=\gamma z\mathbf{i}-2y\mathbf{j}+\gamma x\mathbf{k}=-\gamma\mathbf{i}-4\mathbf{j}-2\gamma\mathbf{k}$$

at $P(-2,2,-1)$ **[1 mark (C)]**. It follows that

$$(\nabla\phi)_P\cdot(\nabla\psi)_P=-\gamma(4\alpha-4)-32-8\alpha\gamma=-24\alpha-24,$$

upon substituting $\gamma=2$ **[1 mark (D)]**.
For the surfaces to intersect orthogonally at P we require $(\nabla\phi)_P\cdot(\nabla\psi)_P=0$ **[1 mark (D)]** and hence $\alpha=-1$. It then follows from above that $\beta=8-4\alpha=12$ **[1 mark (D)]**.

(c) First we see what happens to the boundaries in $u-v$ space. We see that $y=x+1$ becomes $u=1$ and $y=x-3$ becomes $u=-3$ **[1 mark (B)]**. Similarly the lines $y=2-x/3, y=4-x/3$ become $v=2$ and $v=4$ respectively **[1 mark (B)]**.
We have

$$dx\,dy=|J|\,du\,dv, \quad J=\begin{vmatrix}\partial x/\partial u & \partial x/\partial v\\ \partial y/\partial u & \partial y/\partial v\end{vmatrix}=\begin{vmatrix}\partial u/\partial x & \partial v/\partial x\\ \partial u/\partial y & \partial v/\partial y\end{vmatrix}^{-1}=\begin{vmatrix}-1 & 1/3\\ 1 & 1\end{vmatrix}^{-1}=-\frac{3}{4}$$

and hence $|J|=3/4$ **[2 marks (B)]**. We therefore have

$$I=\int\int_R(y-x)\,dx\,dy=\int_{v=2}^{4}\int_{u=-3}^{1}\frac{3}{4}u\ du\ dv=(2)(\frac{3}{4})\left[\frac{u^2}{2}\right]_{-3}^{1}=-6 \ \ \textbf{[2 marks (B)]}.$$

Question Two Solution

(a) We see that $x^2 + y^2 = h^2$ and hence

$$z = a - (x^2 + y^2)^{1/2} \ \ [\mathbf{1\ mark\ (A)}]$$

The surface is a cone with base radius a and height a [**1 mark (A)**].
The surface is open (there is no circular base) [**1 mark (A)**] and it is convex [**1 mark (A)**] (a straight line intersects S at most twice). Justifications not required.

(b) To find $\widehat{\mathbf{n}}$ we first set

$$\phi = z - a + (x^2 + y^2)^{1/2} = 0 \ \ [\mathbf{1\ mark\ (A)}]$$

(other definitions of ϕ possible with $\phi =$constant on S). Then

$$\begin{aligned}\widehat{\mathbf{n}} &= \pm\nabla\phi/\,|\nabla\phi| = \pm(x\mathbf{i}/(x^2+y^2)^{1/2} + y\mathbf{j}/(x^2+y^2)^{1/2} + \mathbf{k})/(x^2/(x^2+y^2) + y^2/(x^2+y^2) + 1)^{1/2} \\ &= \pm(\frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{2(x^2+y^2)}} + \frac{\mathbf{k}}{\sqrt{2}}). \ [\mathbf{1\ mark\ (A)}]\end{aligned}$$

We need to take the plus sign so that $\widehat{\mathbf{n}} \cdot \mathbf{k} > 0$ [**1 mark (A)**].

(c) Let $x_p = a, y_p = 0$, then the corresponding value of z on S is $z_p = 0$ [**1 mark (B)**]. The tangent plane is

$$(\mathbf{r} - \mathbf{r}_p) \cdot (\nabla\phi)_p = 0 \ \ [\mathbf{1\ mark\ (B)}]$$

where $\mathbf{r}_p = (a, 0, 0) = a\,\mathbf{i}$ and

$$(\nabla\phi)_p = \mathbf{i} + \mathbf{k}.$$

Therefore the tangent plane has the equation

$$((x - a)\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \cdot (\mathbf{i} + \mathbf{k}) = 0,$$

i.e.

$$z = a - x. \ [\mathbf{1\ mark\ (B)}]$$

(d) If we project onto the $x - y$ plane then we set the $z-$coordinate to zero, so that

$$\begin{aligned} x &= h\cos\theta, y = h\sin\theta, z = 0, \ \ (0 \le h \le a,\, 0 \le \theta \le 2\pi) \\ &\Rightarrow \ x^2 + y^2 = h^2, \ \ (0 \le h \le a).\end{aligned}$$

The projection is therefore a circular disc of radius a (not a circle!) [**2 marks (B)**: 1 for stating it is a disc, the other mark for some reasoning along the lines above].

(e) If $\mathbf{A} = x\,\mathbf{i}$ then

$$\mathbf{A} \cdot \widehat{\mathbf{n}} = \frac{x^2}{\sqrt{2}(x^2+y^2)^{1/2}} \ \ [\mathbf{1\ mark\ (C)}]$$

Then by the projection theorem

$$\int_S \mathbf{A} \cdot \widehat{\mathbf{n}}\, dS = \int_{\text{disc}} \frac{x^2}{\sqrt{2}(x^2+y^2)^{1/2}} \frac{dx\,dy}{|\widehat{\mathbf{n}} \cdot \mathbf{k}|} \ \ [\mathbf{1\ mark\ (C)}]$$

with $\widehat{\mathbf{n}} \cdot \mathbf{k} = 1/\sqrt{2}$ [**1 mark (C)**]. To evaluate the integral switch to plane polars so that

$$(x^2+y^2)^{1/2} = r, \ \ x = r\cos\theta, \ \ dx\,dy = r\,dr\,d\theta, \ \ 0 \le r \le a,\, 0 \le \theta \le 2\pi, \ [\mathbf{1\ mark\ (C)}]$$

Then

$$
\begin{aligned}
\int_S \mathbf{A}\cdot\widehat{\mathbf{n}}\,dS &= \int_0^{2\pi}\int_0^a \frac{r^2\cos^2\theta}{r}\,r\,drd\theta \\
&= \int_0^{2\pi}\cos^2\theta\,d\theta\int_0^a r^2dr \\
&= \pi a^3/3\ \ [\mathbf{1\ mark\ (C)}]
\end{aligned}
$$

where the trigonometric integral is evaluated using a double angle formula.

(f) To use the divergence theorem we need to include the circular base (D say) with outward normal $\widehat{\mathbf{n}} = -\mathbf{k}$. Then

$$\int_S \mathbf{A}\cdot\widehat{\mathbf{n}}\;dS + \int_D \mathbf{A}\cdot(-\mathbf{k})\,dx\,dy = \int_V \mathrm{div}\mathbf{A}\,dV.\ \ [\mathbf{1\ mark\ (D)}]$$

But div$\mathbf{A} = 1$ and $\mathbf{A}\cdot\mathbf{k} = 0$. [**1 mark (D)**] Thus:

$$\int_S \mathbf{A}\cdot\widehat{\mathbf{n}}\;dS = \text{ volume of cone } = (1/3)\pi a^3,$$

agreeing with (e) [**1 mark (D)**].

Question Three Solution

(a)(i) We consider the vector Jacobian

$$\begin{aligned}\mathbf{J}(u_1,u_2,u_3) &= \frac{\partial \mathbf{r}}{\partial u_1}\times\frac{\partial \mathbf{r}}{\partial u_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_2\cos u_3 & u_2\sin u_3 & u_1 \\ u_1\cos u_3 & u_1\sin u_3 & -u_2 \end{vmatrix} \\ &= -(u_1^2+u_2^2)\sin u_3\mathbf{i} + (u_1^2+u_2^2)\cos u_3\mathbf{j} + 0\mathbf{k}.\ \textbf{[2 marks (A)]}\end{aligned}$$

Therefore

$$F(u_1,u_2) = |\mathbf{J}| = u_1^2+u_2^2.\ \textbf{[1 mark (A)]}$$

(a)(ii) To demonstrate orthogonality we calculate

$$\begin{aligned}\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial u_1} = (u_2\cos u_3, u_2\sin u_3, u_1), \\ \mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial u_2} = (u_1\cos u_3, u_1\sin u_3, -u_2), \\ \mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial u_3} = (-u_1u_2\sin u_3, u_1u_2\cos u_3, 0).\end{aligned}$$

We then see that

$$\begin{aligned}\mathbf{e}_1\cdot\mathbf{e}_2 &= u_1u_2\cos^2 u_3 + u_1u_2\sin^2 u_3 - u_1u_2 = 0, \\ \mathbf{e}_1\cdot\mathbf{e}_3 &= -u_1u_2^2\sin u_3\cos u_3 + u_1u_2^2\sin u_3\cos u_3 = 0, \\ \mathbf{e}_2\cdot\mathbf{e}_3 &= -u_1^2u_2\sin u_3\cos u_3 + u_1^2u_2\sin u_3\cos u_3 = 0,\end{aligned}$$

and hence the system is orthogonal [**3 marks (A)**]. The lengthscales are

$$\begin{aligned}h_1 &= |\mathbf{e}_1| = \sqrt{u_2^2\cos^2 u_3 + u_2^2\sin^2 u_3 + u_1^2} = \sqrt{u_1^2+u_2^2}, \\ h_2 &= |\mathbf{e}_2| = \sqrt{u_1^2\cos^2 u_3 + u_1^2\sin^2 u_3 + u_2^2} = \sqrt{u_1^2+u_2^2}, \\ h_3 &= |\mathbf{e}_3| = \sqrt{u_1^2u_2^2\sin^2 u_3 + u_1^2u_2^2\cos^2 u_3} = u_1u_2.\end{aligned}$$

From this we can see that $F(u_1,u_2) = h_1h_2$ as required [**3 marks (A)**].
(b)(i) $I/2\pi$ represents the surface area generated by revolving the section of the curve $y=y(x)$ between $x=\pm a$ about the x-axis [**1 mark (D)**].
J represents the distance along the curve $y=y(x)$ between $x=-a$ and $x=a$ [**1 mark (A)**].
The ratio I/J gives the $y-$coordinate of the centre of gravity (or centroid) of the curve $y=y(x)$ [**1 mark (D)**].
(b)(ii) We apply the Euler-Lagrange equation to the functional

$$L = y\left(1+(y')^2\right)^{1/2} + \lambda\left(1+(y')^2\right)^{1/2} = (y+\lambda)\left(1+(y')^2\right)^{1/2}.$$

Since L is explicitly independent of x the Euler-Lagrange equation reduces to

$$L - y'\frac{\partial L}{\partial y'} = \text{ constant}$$

and so we have

$$\begin{aligned}(y+\lambda)\left\{\left(1+(y')^2\right)^{1/2} - (y')^2\left(1+(y')^2\right)^{-1/2}\right\} &= \text{ constant} \\ \Rightarrow (y+\lambda)\left(1+(y')^2\right)^{-1/2} &= \text{ constant} \Rightarrow (y+\lambda)^2 = C^2\left(1+(y')^2\right)\ \textbf{[2 marks (B)]}\end{aligned}$$

This can be rearranged to

$$y' = \pm C^{-1}\sqrt{(y+\lambda)^2 - C^2} \Rightarrow x = \pm C \int \frac{dy}{\sqrt{(y+\lambda)^2 - C^2}} = \pm C \cosh^{-1}\left(\frac{y+\lambda}{C}\right) + K,$$

and hence

$$y = -\lambda + C\cosh\left(\frac{x-K}{C}\right).$$

Applying the end conditions $y = 0$ at $x = \pm a$ we see that

$$\lambda = C\cosh\left(\frac{a-K}{C}\right) = C\cosh\left(\frac{a+K}{C}\right)$$

and hence $K = 0$ and $\lambda = C\cosh(a/C)$. Hence we have the extremal curve

$$y = C\cosh(x/C) - C\cosh(a/C).\ [\mathbf{2\ marks\ (B)}]$$

Substituting into the integral constraint we have

$$J = 2 = \int_{-a}^{a} \sqrt{1 + \sinh^2(x/C)}\ dx = [C\sinh(x/C)]_{-a}^{a} = 2C\sinh(a/C) \Rightarrow C\sinh(a/C) = 1,$$

as required [**2 marks (C)**].

(b)(iii) From examination of the integral for J we have $J > 2a$. Thus it is not possible to keep J fixed at the value of 2 if $a \geq 1$. We therefore have $a_0 = 1$ [**2 marks (D)**].

4. (a) (i) We have $\lambda_{n+1}(t) = x_0 + \int_{t_0}^t f(s, \lambda_n(s))\,\mathrm{d}s$ for all $n \in \mathbb{N}_0$, which implies $\lambda_n(t_0) = x_0$ for all $n \in \mathbb{N}$, and using this and the fundamental theorem of calculus, we get $\dot{\lambda}_{n+1}(t_0) = f(t_0, \lambda_n(t_0)) = f(t_0, x_0) = f(t_0, \lambda_{n+1}(t_0))$ for all $n \in \mathbb{N}_0$. sim. seen ⇓ 3, A

(ii) The maximal interval on which the Picard iterates can be defined is $\mathbb{R}$. Iteratively it follows that the functions λ_n, $n \in \mathbb{N}$ are differentiable and thus continuous, and hence, the integrand of the integral $\int_{t_0}^t f(s, \lambda_n(s))\,\mathrm{d}s$ is a continuous function and the integral exists for all $t \in \mathbb{R}$. unseen ⇓ 3, A

(iii) With $f(x) = x^2$, we get for all $t \in J$ that sim. seen ⇓ 6, A

$$\lambda_0(t) = 1\,,$$

$$\lambda_1(t) = 1 + \int_1^t f(\lambda_0(s))\,\mathrm{d}s = t\,,$$

$$\lambda_2(t) = 1 + \int_1^t f(\lambda_1(s))\,\mathrm{d}s = 1 + \int_1^t s^2\,\mathrm{d}s = 1 + \frac{1}{3}s^3\Big|_{s=1}^{s=t} = \frac{2}{3} + \frac{1}{3}t^3\,.$$

(b) (i) The differential equation satisfies the conditions for the local version of the Picard–Lindelöf theorem, and for this reason, all initial value problems have local solution that is unique. seen ⇓ 2, A

(ii) The local version of the Picard–Lindelöf theorem implies that there exist an $h > 0$ and a solution $\mu : [-h, h] \to \mathbb{R}^d$ satisfying $\mu(0) = y_0$. Define $T := \frac{h}{2}$. Translation invariance implies that the function $\lambda : [-T, 3T] \to \mathbb{R}^d$, $\lambda(t) := \mu(t - T)$, is a solution to the differential equation, and we have $\lambda(T) = \mu(0) = y_0$. The function λ satisfies $\lambda(0) = \mu(-\frac{h}{2})$, so the statement is correct with $x_0 = \mu(-\frac{h}{2})$. unseen ⇓ 6, D

5. (i) $\dot{y} = 0$ if and only if $x - y((x-y)^2 + y^2)) = 0$. If $y = 0$, this implies $x = 0$ (clearly $(0,0)$ is a zero of the first equation and thus an equilibrium). To look for more equilibria, we can assume $y \neq 0$. Then $(x-y)^2 + y^2 = \frac{x}{y}$, and we plug this into $\dot{x} = 0$ to obtain $0 = 2x - 2y - \frac{x^2}{y}$, which implies $0 = 2y^2 + x^2 - 2xy = y^2 + (x-y)^2$. Since $y \neq 0$, this is not possible, so there are not more equilibria.

meth seen ⇓

3, B

(ii) The characteristic polynomial reads as $\lambda^2 - 2\lambda + 2$. Its roots, the eigenvalues, are given by $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, and one computes the (complex) eigenvectors

meth seen ⇓

4, B

$$v_1 = \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \quad \text{and} \quad v_2 = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}.$$

Thus, the transformation matrix T and its inverse are given by

$$T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix},$$

so that the real Jordan normal form is given by

$$J = T^{-1} A T = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

(iii) The linearisation in the equilibrium is clearly given by the matrix A. The real parts of the eigenvalues of the matrix A are positive, which implies that the equilibrium (x^*, y^*) is repulsive.

sim. seen ⇓

2, A

(iv) Consider $V(x,y) = (x-y)^2 + y^2$. We obtain

sim. seen ⇓

6, C

$$\begin{aligned}
\dot{V}(x,y) &= (2x - 2y, -2x + 4y) \begin{pmatrix} 2x - 2y - x((x-y)^2 + y^2) \\ x - y((x-y)^2 + y^2) \end{pmatrix} \\
&= 4x^2 - 8xy + 4y^2 - 2x^2 + 4xy \\
&\qquad - ((2x - 2y)x + (4y - 2x)y)((x-y)^2 + y^2)) \\
&= 2x^2 - 4xy + 4y^2 - ((2x - 2y)x + (4y - 2x)y)((x-y)^2 + y^2)) \\
&= 2((x-y)^2 + y^2) - 2(x^2 - xy + 2y^2 - xy)((x-y)^2 + y^2)) \\
&= ((x-y)^2 + y^2))(2 - 2((x-y)^2 + y^2)),
\end{aligned}$$

which is clearly strictly negative whenever $(x-y)^2 + y = R$ for some $R > 1$. Hence M_R is positively invariant whenever $R > 1$.

(v) The compact set M_2 is positively invariant due to (iv). Take some $(x,y) \in M_2 \setminus \{(0,0)\}$. Then $\omega(x,y)$ must be a periodic orbit, since it cannot contain the only equilibrium given by $\{(0,0)\}$ (it is repulsive), and according to the Poincaré–Bendixson theorem, all other possible omega limit sets contain at least an equilibrium.

sim. seen ⇓

5, B

6. (a) (i) The statement is true: $d = 1$, $f(x) = -x$, for all $x \in \mathbb{R}$, $\omega(x) = \{0\}$. sim. seen $\Downarrow$

(ii) The statement is false. Since $\omega(x)$ is compact and invariant, for any $y \in \omega(x)$, $\overline{O^+(y)} \subset \omega(x)$ is compact, and thus $\omega(y)$ is a nonempty subset of $\omega(x)$, giving $\omega(x) \cap \omega(y) = \omega(y) \neq \emptyset$. 3, A

unseen $\Downarrow$

(iii) The statement is true: $d = 1$, $f(x) = -x$, $\omega(0) = \{0\} = \alpha(0)$. 3, C

(iv) The statement is true: for any homoclinic orbit $O(x)$, there exists an equilibrium $y \in \mathbb{R}^d$ such that $\omega(x) = \alpha(x) = \{y\}$. x is not an equilibrium, so $x \notin \omega(x)$. sim. seen $\Downarrow$ 3, A

sim. seen $\Downarrow$

(b) (i) This follows from a result proved in the course, since the assumption implies that $\overline{O^+(x)}$ is a compact subset of the domain $\mathbb{R}^d$. 3, B

(ii) Assume to the contrary that there exists an $\varepsilon > 0$ and a sequence $\{t_n\}_{n\in\mathbb{N}}$ converging to ∞ such that seen $\Downarrow$ 2, A

$$\varphi(t_n, x) \notin B_\varepsilon(\omega(x)) \quad \text{for all } n \in \mathbb{N}\,.$$

unseen $\Downarrow$

Since the sequence $\{\varphi(t_n, x)\}_{n\in\mathbb{N}}$ is bounded, it has an accumulation point. This accumulation point is an omega limit point of x, which is bounded away from $\omega(x)$. This is a contradiction and finishes the proof. 6, D

If your module is taught across multiple year levels, you might have received this form for each level of the module. You are only required to fill this out once for each question.
Please record below, some brief but non-trivial comments for students about how well (or otherwise) the questions were answered. For example, you may wish to comment on common errors and misconceptions, or areas where students have done well. These comments should note any errors in and corrections to the paper. These comments will be made available to students via the MathsCentral Blackboard site and should not contain any information which identifies individual candidates. Any comments which should be kept confidential should be included as confidential comments for the Exam Board and Externals. If you would like to add formulas, please include a sperate pdf file with your email.

ExamModuleCode	QuestionNumber	Comments for Students
MATH50004	1	Part (a) on the use of subscript notation in simplifying a vector calculus expression was surprisingly poorly done. An alarming error made by some students was to assume that the vector product of a vector and a vector operator can be manipulated as if it was the vector product of two vectors. Part (b) on showing 2 surfaces are othogonal was done very well. A significant number of students failed to appreciate that in order to pin down the values of the parameters uniquely one had to impose the condition that the point in question was indeed contained in both surfaces. Part (c) on the use of a Jacobian to solve a 2D integral was also well done. A common error here was not taking the absolute value of the Jacobian. There was also a lack of sufficient detail given in some of the answers and this was penalized even if the final answer was correct.
MATH50004	2	Most students did v well with many getting 20/20

MATH50004	3	Q3, bi: asking properties of the integrals caused most problems, nearly all candidates were not aware that the equation represented center of gravity. While integral I too was hard work for majority, Property of J, an arc length was explained by vast majority. Euler-Lagrange question caused many many problems, as quite a few did not use Beltrami Identity, but proceeded with the direct initial E-L form i.e. dL/du - d/dx [DL/du'] ending up with horrible expression and not getting anywhere : majority of marks were lost on this part. Those that used Beltrami Identity fared much better.
MATH50004	4	This question was not done so well, even though the vast majority got 8 points from the easier parts 4(a)(iii) and 4(b)(i). Many students forgot to mention some arguments in 4(a)(i), and 4(a)(ii) was often confused with the maximal existence interval, but the Picard iterates are independent of this and can be defined on the whole real line. Some students found nice (and different) solutions to the challenging question 4(b)(ii).
MATH50004	5	This question was well answered in most instances. Parts (i,ii,iii,iv) were nearly always done correctly. Part (iv) was the most appealing and insightful part of the question, but most candidates seemed not to notice the geometrically meaningful aspects of this part of the question. I expect that this part of the question should help identify the students who really understood the geometrical aspects of the course.
MATH50004	6	One of the main mistakes was that some students invoked Poincre-Bendixson Theorem (for a system on a plane any bounded orbit converges either to a stationary point, or to a periodic orbit, or to a connection between saddles) in the general multidimensional case. This is a specifically two-dimensional result, which does not hold true in higher dimensions where much more complicated omega-limit sets are possible and Chaos rules!